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ON (α, β)-FUZZY SUBALGEBRAS OF BCK/BCI-ALGEBRAS

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ABSTRACT. Using the *belongs to* relation $(\in)$ and *quasi-coincidence with* relation (q) between fuzzy points and fuzzy sets, the concept of (α, β)-fuzzy subalgebras where α, β are any two of $\{\in, \mathrm{q}, \in\vee\mathrm{q}, \in\wedge\mathrm{q}\}$ with $\alpha \neq\in\wedge\mathrm{q}$ is introduced, and related properties are investigated.

1. Introduction

The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [4], played a vital role to generate some different types of fuzzy subgroups, called (α, β)-fuzzy subgroups, introduced by Bhakat and Das[2]. In particular, $(\in, \in\vee\mathrm{q})$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. It is now natural to investigate similar type of generalizations of the existing fuzzy subsystems of other algebraic structures. With this objective in view, we introduce the concept of (α, β)-fuzzy subalgebra of a BCK/BCI-algebra and investigate related results.

2. Preliminaries

By a *BCI-algebra* we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the axioms:

(i) $(\forall x, y, z \in X)\,(((x * y) * (x * z)) * (z * y) = 0)$,
(ii) $(\forall x, y \in X)\,((x * (x * y)) * y = 0)$,
(iii) $(\forall x \in X)\,(x * x = 0)$,
(iv) $(\forall x, y \in X)\,(x * y = y * x = 0 \Rightarrow x = y)$.

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We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x * y = 0$. If a BCI-algebra X satisfies $0 * x = 0$ for all $x \in X$, then we say that X is a *BCK-algebra*. In what follows let X denote a BCK/BCI-algebra unless otherwise specified. A nonempty subset S of X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. We refer the reader to the book [3] for further information regarding BCK/BCI-algebras.

A fuzzy set μ in a set X of the form

$$\mu(y) := \begin{cases} t \in (0,1] & \text{if } y = x, \\ 0 & \text{if } y \neq x \end{cases}$$

is said to be a *fuzzy point* with support x and value t and is denoted by x_t.

For a fuzzy point x_t and a fuzzy set μ in a set X, Pu and Liu[4] gave meaning to the symbol $x_t \alpha \mu$, where $\alpha \in \{\in, \mathrm{q}, \in\vee\mathrm{q}, \in\wedge\mathrm{q}\}$.

To say that $x_t \in \mu$ (resp. $x_t \mathrm{q} \mu$) means that $\mu(x) \geq t$ (resp. $\mu(x) + t > 1$), and in this case, x_t is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy set μ.

To say that $x_t \in\vee\mathrm{q}\,\mu$ (resp. $x_t \in\wedge\mathrm{q}\,\mu$) means that $x_t \in \mu$ or $x_t \mathrm{q} \mu$ (resp. $x_t \in \mu$ and $x_t \mathrm{q} \mu$). For all $t_1, t_2 \in [0,1]$, $\min\{t_1, t_2\}$ will be denoted by $M(t_1, t_2)$.

A fuzzy set μ in X is called a *fuzzy subalgebra* of X if it satisfies

$$(1) \qquad (\forall x, y \in X)\,(\mu(x * y) \geq M(\mu(x), \mu(y))).$$

PROPOSITION 2.1. *Let μ be a fuzzy set in X. Then μ is a fuzzy subalgebra of X if and only if $U(\mu; t) := \{x \in X \mid \mu(x) \geq t\}$ is a subalgebra of X for all $t \in (0,1]$, for our convenience, the empty set $\emptyset$ is regarded as a subalgebra of X.*

3. (α, β)-fuzzy subalgebras

In what follows let α and β denote any one of $\in$, q, $\in\vee$q, or $\in\wedge$q unless otherwise specified. To say that $x_t \overline{\alpha} \mu$ means that $x_t \alpha \mu$ does not hold.

PROPOSITION 3.1. *For any fuzzy set μ in X, the condition (1) is equivalent to the following condition*

$$(2) \quad (\forall x, y \in X)\,(\forall t_1, t_2 \in (0,1])\,(x_{t_1}, y_{t_2} \in \mu \Rightarrow (x * y)_{M(t_1, t_2)} \in \mu).$$

Proof. Assume that the condition (1) is valid. Let $x, y \in X$ and $t_1, t_2 \in (0,1]$ be such that $x_{t_1}, y_{t_2} \in \mu$. Then $\mu(x) \geq t_1$ and $\mu(y) \geq t_2$, which imply from (1) that

$$\mu(x * y) \geq M(\mu(x), \mu(y)) \geq M(t_1, t_2).$$

Hence $(x * y)_{M(t_1,t_2)} \in \mu$.

Conversely suppose that the condition (2) is valid. Note that $x_{\mu(x)} \in \mu$ and $y_{\mu(y)} \in \mu$ for all $x, y \in X$. Thus $(x * y)_{M(\mu(x),\mu(y))} \in \mu$ by (2), and so $\mu(x * y) \geq M(\mu(x), \mu(y))$.

Note that if μ is a fuzzy set in X defined by $\mu(x) \leq 0.5$ for all $x \in X$, then the set $\{x_t \mid x_t \in\wedge\, \mathrm{q}\, \mu\}$ is empty.

A fuzzy set μ in X is said to be an (α, β)*-fuzzy subalgebra* of X, where $\alpha \neq\, \in\wedge\,\mathrm{q}$, if it satisfies the following conditions:

(3) $(\forall x, y \in X)\,(\forall t_1, t_2 \in (0, 1])\,(x_{t_1} \alpha \mu,\ y_{t_2} \alpha \mu \Rightarrow (x * y)_{M(t_1,t_2)} \beta \mu.)$

EXAMPLE 3.2. Consider a BCI-algebra $X = \{0, a, b, c\}$ with the following Cayley table (see [1]):

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let μ be a fuzzy set in X defined by $\mu(0) = 0.6$, $\mu(a) = 0.7$, and $\mu(b) = \mu(c) = 0.3$. Then μ is an $(\in, \in\vee\,\mathrm{q})$-fuzzy subalgebra of X. But

(1) μ is not an $(\in, \in)$-fuzzy subalgebra of X since $a_{0.62} \in \mu$ and $a_{0.66} \in \mu$, but $(a * a)_{M(0.62,\, 0.66)} = 0_{0.62} \overline{\in}\, \mu$.

(2) μ is not a $(\mathrm{q}, \in\vee\,\mathrm{q})$-fuzzy subalgebra of X since $a_{0.41}\, \mathrm{q}\, \mu$ and $b_{0.77}\, \mathrm{q}\, \mu$, but $(a * b)_{M(0.41,\, 0.77)} = c_{0.41}\, \overline{\in\vee\,\mathrm{q}}\, \mu$.

(3) μ is not an $(\in\vee\,\mathrm{q}, \in\vee\,\mathrm{q})$-fuzzy subalgebra of X since $a_{0.5} \in\vee\,\mathrm{q}\, \mu$ and $c_{0.8} \in\vee\,\mathrm{q}\, \mu$, but $(a * c)_{M(0.5,\, 0.8)} = b_{0.5}\, \overline{\in\vee\,\mathrm{q}}\, \mu$.

THEOREM 3.3. *Every $(\in\vee\,\mathrm{q}, \in\vee\,\mathrm{q})$-fuzzy subalgebra is an $(\in, \in\vee\,\mathrm{q})$-fuzzy subalgebra.*

Proof. Let μ be an $(\in\vee\,\mathrm{q}, \in\vee\,\mathrm{q})$-fuzzy subalgebra of X. Let $x, y \in X$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1} \in \mu$ and $y_{t_2} \in \mu$. Then $x_{t_1} \in\vee\,\mathrm{q}\, \mu$ and $y_{t_2} \in\vee\,\mathrm{q}\, \mu$, which imply that $(x * y)_{M(t_1,\, t_2)} \in\vee\,\mathrm{q}\, \mu$. Hence μ is an $(\in, \in\vee\,\mathrm{q})$-fuzzy subalgebra of X.

THEOREM 3.4. *Every $(\in, \in)$-fuzzy subalgebra is an $(\in, \in\vee\,\mathrm{q})$-fuzzy subalgebra.*

Proof. Straightforward.

Example 3.2 shows that the converse of Theorems 3.3 and 3.4 need not be true.

PROPOSITION 3.5. *If μ is a non-zero (α, β)-fuzzy subalgebra of X, then $\mu(0) > 0$.*

Proof. Assume that $\mu(0) = 0$. Since μ is non-zero, there exists $x \in X$ such that $\mu(x) = t > 0$. If $\alpha = \in$ or $\alpha = \in \vee \mathrm{q}$, then $x_t \alpha \mu$, but $(x * x)_{M(t,t)} = 0_t \overline{\beta} \mu$. This is a contradiction. If $\alpha = \mathrm{q}$, then $x_1 \alpha \mu$ because $\mu(x) + 1 = t + 1 > 1$. But $(x * x)_{M(1,1)} = 0_1 \overline{\beta} \mu$, which is a contradiction. Hence $\mu(0) > 0$.

For a fuzzy set μ in X, we denote $X_0 := \{x \in X \mid \mu(x) > 0\}$.

THEOREM 3.6. *If μ is a nonzero $(\in, \in)$-fuzzy subalgebra of X, then the set X_0 is a subalgebra of X.*

Proof. Let $x, y \in X_0$. Then $\mu(x) > 0$ and $\mu(y) > 0$. Suppose that $\mu(x * y) = 0$. Note that $x_{\mu(x)} \in \mu$ and $y_{\mu(y)} \in \mu$, but $(x * y)_{M(\mu(x), \mu(y))} \overline{\in} \mu$ because $\mu(x * y) = 0 < M(\mu(x), \mu(y))$. This is a contradiction, and thus $\mu(x * y) > 0$, which shows that $x * y \in X_0$. Consequently X_0 is a subalgebra of X.

THEOREM 3.7. *If μ is a nonzero $(\in, \mathrm{q})$-fuzzy subalgebra of X, then the set X_0 is a subalgebra of X.*

Proof. Let $x, y \in X_0$. Then $\mu(x) > 0$ and $\mu(y) > 0$. If $\mu(x * y) = 0$, then

$$\mu(x * y) + M(\mu(x), \mu(y)) = M(\mu(x), \mu(y)) \leq 1.$$

Hence $(x * y)_{M(\mu(x), \mu(y))} \overline{\mathrm{q}} \mu$, which is a contradiction since $x_{\mu(x)} \in \mu$ and $y_{\mu(y)} \in \mu$. Thus $\mu(x * y) > 0$, and so $x * y \in X_0$. Therefore X_0 is a subalgebra of X.

THEOREM 3.8. *If μ is a nonzero $(\mathrm{q}, \in)$-fuzzy subalgebra of X, then the set X_0 is a subalgebra of X.*

Proof. Let $x, y \in X_0$. Then $\mu(x) > 0$ and $\mu(y) > 0$. Thus $\mu(x) + 1 > 1$ and $\mu(y) + 1 > 1$, which imply that $x_1 \, \mathrm{q} \, \mu$ and $y_1 \, \mathrm{q} \, \mu$. If $\mu(x * y) = 0$, then $\mu(x * y) < 1 = M(1, 1)$. Therefore $(x * y)_{M(1,1)} \overline{\in} \mu$, which is a contradiction. It follows that $\mu(x * y) > 0$ so that $x * y \in X_0$. This completes the proof.

THEOREM 3.9. *If μ is a nonzero (q, q)-fuzzy subalgebra of X, then the set X_0 is a subalgebra of X.*

Proof. Let $x, y \in X_0$. Then $\mu(x) > 0$ and $\mu(y) > 0$. Thus $\mu(x)+1 > 1$ and $\mu(y) + 1 > 1$, and therefore $x_1 \,\mathrm{q}\, \mu$ and $y_1 \,\mathrm{q}\, \mu$. If $\mu(x * y) = 0$, then $\mu(x*y)+M(1,1) = 0+1 = 1$, and so $(x*y)_{M(1,1)} \,\overline{\mathrm{q}}\, \mu$. This is impossible, and hence $\mu(x * y) > 0$, i.e., $x * y \in X_0$. This completes the proof.

COROLLARY 3.10. *If μ is one of the following*

(i) *a nonzero $(\in, \in\wedge\,\mathrm{q})$-fuzzy subalgebra of X,*
(ii) *a nonzero $(\in, \in\vee\,\mathrm{q})$-fuzzy subalgebra of X,*
(iii) *a nonzero $(\in\vee\,\mathrm{q}, \mathrm{q})$-fuzzy subalgebra of X,*
(iv) *a nonzero $(\in\vee\,\mathrm{q}, \in)$-fuzzy subalgebra of X,*
(v) *a nonzero $(\in\vee\,\mathrm{q}, \in\wedge\,\mathrm{q})$-fuzzy subalgebra of X,*
(vi) *a nonzero $(\mathrm{q}, \in\wedge\,\mathrm{q})$-fuzzy subalgebra of X,*
(vii) *a nonzero $(\mathrm{q}, \in\vee\,\mathrm{q})$-fuzzy subalgebra of X,*

then the set X_0 is a subalgebra of X.

Proof. The proof is similar to the proof of Theorems 3.6, 3.7, 3.8, and/or 3.9.

THEOREM 3.11. *Every nonzero (q,q)-fuzzy subalgebra of X is constant on X_0.*

Proof. Let μ be a nonzero (q,q)-fuzzy subalgebra of X. Assume that μ is not constant on X_0. Then there exists $y \in X_0$ such that $t_y = \mu(y) \neq \mu(0) = t_0$. Then either $t_y > t_0$ or $t_y < t_0$. Suppose $t_y < t_0$ and choose $t_1, t_2 \in (0, 1]$ such that $1 - t_0 < t_1 < 1 - t_y < t_2$. Then $\mu(0) + t_1 = t_0 + t_1 > 1$ and $\mu(y) + t_2 = t_y + t_2 > 1$, and so $0_{t_1}\mathrm{q}\mu$ and $y_{t_2}\mathrm{q}\mu$. Since

$$\mu(y * 0) + M(t_1, t_2) = \mu(y) + t_1 = t_y + t_1 < 1,$$

we have $(y * 0)_{M(t_1,t_2)}\overline{\mathrm{q}}\mu$, which is a contradiction. Next assume that $t_y > t_0$. Then $\mu(y) + (1 - t_0) = t_y + 1 - t_0 > 1$ and so $y_{1-t_0}\mathrm{q}\mu$. Since

$$\mu(y * y) + (1 - t_0) = \mu(0) + 1 - t_0 = t_0 + 1 - t_0 = 1,$$

we get $(y * y)_{M(1-t_0,1-t_0)}\overline{\mathrm{q}}\mu$. This is impossible. Therefore μ is constant on X_0.

THEOREM 3.12. *Let μ be a non-zero (α,β)-fuzzy subalgebra of X, where (α,β) is one of the following:*

- $(\in, \mathrm{q})$,
- $(\mathrm{q}, \in)$,
- $(\in\vee\,\mathrm{q}, \mathrm{q})$,
- $(\in\vee\,\mathrm{q}, \in)$.
- $(\in, \in\wedge\,\mathrm{q})$,
- $(\mathrm{q}, \in\wedge\,\mathrm{q})$,
- $(\in\vee\,\mathrm{q}, \in\wedge\,\mathrm{q})$,

Then $\mu = \chi_{X_0}$, the characteristic function of X_0.

Proof. Assume that there exists $x \in X_0$ such that $\mu(x) < 1$. For $\alpha = \in$, choose $t \in (0,1]$ such that $t < M(1-\mu(x), \mu(x), \mu(0))$. Then $x_t \alpha\, \mu$ and $0_t \alpha\, \mu$, but $(x * 0)_{M(t,t)} = x_t \overline{\beta}\, \mu$ where $\beta = \mathrm{q}$ or $\beta = \in \wedge\, \mathrm{q}$. This is a contradiction. Now let $\alpha = \mathrm{q}$. Then $x_1 \alpha\, \mu$ and $0_1 \alpha\, \mu$, but $(x * 0)_{M(1,1)} = x_1 \overline{\beta}\, \mu$ for $\beta = \in$ or $\beta = \in \wedge\, \mathrm{q}$, a contradiction. Finally let $\alpha = \in \vee\, \mathrm{q}$ and choose $t \in (0,1]$ such that $x_t \in \mu$ but $x_t \overline{\mathrm{q}}\, \mu$. Then $x_t \alpha\, \mu$ and $0_1 \alpha\, \mu$, but $(x * 0)_{M(t,1)} = x_t \overline{\beta}\, \mu$ for $\beta = \mathrm{q}$ or $\beta = \in \wedge\, \mathrm{q}$. This is impossible. Note that $x_1 \alpha\, \mu$ and $0_1 \alpha\, \mu$ but $(x * 0)_{M(1,1)} = x_1 \overline{\in}\, \mu$, a contradiction. Therefore $\mu = \chi_{X_0}$.

THEOREM 3.13. *Let S be a subalgebra of X and let μ be a fuzzy set in X such that*

(i) $\mu(x) = 0$ *for all* $x \in X \setminus S$,
(ii) $\mu(x) \geq 0.5$ *for all* $x \in S$.

Then μ is a (q, $\in \vee$ q)*-fuzzy subalgebra of X.*

Proof. Let $x, y \in X$ and $t_1, t_2 \in (0,1]$ be such that $x_{t_1} \mathrm{q}\, \mu$ and $y_{t_2} \mathrm{q}\, \mu$, that is, $\mu(x) + t_1 > 1$ and $\mu(y) + t_2 > 1$. Then $x * y \in S$ because if not then $x \in X \setminus S$ or $y \in X \setminus S$. Thus $\mu(x) = 0$ or $\mu(y) = 0$, and so $t_1 > 1$ or $t_2 > 1$. This is a contradiction. If $M(t_1, t_2) > 0.5$, then $\mu(x * y) + M(t_1, t_2) > 1$ and thus $(x * y)_{M(t_1,t_2)} \mathrm{q}\, \mu$. If $M(t_1, t_2) \leq 0.5$, then $\mu(x * y) \geq 0.5 \geq M(t_1, t_2)$ and so $(x * y)_{M(t_1,t_2)} \in \mu$. Therefore $(x * y)_{M(t_1,t_2)} \in \vee\, \mathrm{q}\, \mu$. This completes the proof.

THEOREM 3.14. *Let μ be a* (q, $\in \vee$ q)*-fuzzy subalgebra of X such that μ is not constant on X_0. Then there exists $x \in X$ such that $\mu(x) \geq 0.5$. Moreover, $\mu(x) \geq 0.5$ for all $x \in X_0$.*

Proof. Assume that $\mu(x) < 0.5$ for all $x \in X$. Since μ is not constant on X_0, there exists $x \in X_0$ such that $t_x = \mu(x) \neq \mu(0) = t_0$. Then either $t_0 < t_x$ or $t_0 > t_x$. For the first case, choose $\delta > 0.5$ such that $t_0 + \delta < 1 < t_x + \delta$. It follows that $x_\delta \mathrm{q}\, \mu$, $\mu(x * x) = \mu(0) = y_0 < \delta = M(\delta, \delta)$ and $\mu(x * x) + M(\delta, \delta) = \mu(0) + \delta = t_0 + \delta < 1$ so that $(x * x)_{M(\delta,\delta)} \overline{\in \vee\, \mathrm{q}}\, \mu$. This is a contradiction. Now if $t_0 > t_x$, we can choose $\delta > 0.5$ such that $t_x + \delta < 1 < t_0 + \delta$. Then $0_\delta \mathrm{q}\, \mu$ and $x_1 \mathrm{q}\, \mu$, but $(x * 0)_{M(1,\delta)} = x_\delta \overline{\in \vee\, \mathrm{q}}\, \mu$ since $\mu(x) < 0.5 < \delta$ and $\mu(x) + \delta = t_x + \delta < 1$. This leads a contradiction. Therefore $\mu(x) \geq 0.5$ for some $x \in X$. We now show that $\mu(0) \geq 0.5$. Assume that $\mu(0) = t_0 < 0.5$. Since there exists $x \in X$ such that $\mu(x) = t_x \geq 0.5$, it follows that $t_0 < t_x$. Choose $t_1 > t_0$ such that $t_0 + t_1 < 1 < t_x + t_1$. Then $\mu(x) + t_1 = t_x + t_1 > 1$, and so $x_{t_1} \mathrm{q}\, \mu$. Now we get

$$\mu(x * x) + M(t_1, t_1) = \mu(0) + t_1 = t_0 + t_1 < 1,$$

$$\mu(x * x) = \mu(0) = t_0 < t_1 = M(t_1, t_1).$$

Hence $(x * x)_{M(t_1,t_1)} \overline{\in \vee \mathrm{q}}\, \mu$, a contradiction. Therefore $\mu(0) \geq 0.5$. Finally suppose that $t_x = \mu(x) < 0.5$ for some $x \in X_0$. Take $t > 0$ such that $t_x + t < 0.5$. Then $\mu(x) + 1 = t_x + 1 > 1$ and $\mu(0) + (0.5 + t) > 1$, which imply that $x_1 \mathrm{q}\, \mu$ and $0_{0.5+t} \mathrm{q}\, \mu$. But $(x*0)_{M(1,0.5+t)} = x_{0.5+t} \overline{\in \vee \mathrm{q}}\, \mu$ since $\mu(x * 0) = \mu(x) < 0.5 + t < M(1, 0.5 + t)$ and

$$\mu(x * 0) + M(1, 0.5 + t) = \mu(x) + 0.5 + t = t_x + 0.5 + t < 0.5 + 0.5 = 1.$$

This is a contradiction. Hence $\mu(x) \geq 0.5$ for all $x \in X_0$. This completes the proof.

THEOREM 3.15. *A fuzzy set μ in X is an $(\in, \in \vee \mathrm{q})$-fuzzy subalgebra of X if and only if it satisfies:*

$$(4) \qquad (\forall x, y \in X)\, (\mu(x * y) \geq M(\mu(x), \mu(y), 0.5)).$$

Proof. Suppose that μ is an $(\in, \in \vee \mathrm{q})$-fuzzy subalgebra of X and let $x, y \in X$. If $M(\mu(x), \mu(y)) < 0.5$, then $\mu(x * y) \geq M(\mu(x), \mu(y))$. For, assume that $\mu(x * y) < M(\mu(x), \mu(y))$ and choose t such that $\mu(x * y) < t < M(\mu(x), \mu(y))$. Then $x_t \in \mu$ and $y_t \in \mu$ but $(x * y)_{M(t,t)} = (x * y)_t \overline{\in \vee \mathrm{q}}\, \mu$, a contradiction. Hence $\mu(x * y) \geq M(\mu(x), \mu(y))$ whenever $M(\mu(x), \mu(y)) < 0.5$. Now suppose that $M(\mu(x), \mu(y)) \geq 0.5$. Then $x_{0.5} \in \mu$ and $y_{0.5} \in \mu$, which imply that

$$(x * y)_{M(0.5,0.5)} = (x * y)_{0.5} \in \vee \mathrm{q}\, \mu.$$

Thus $\mu(x * y) \geq 0.5$. Otherwise, $\mu(x * y) + 0.5 < 0.5 + 0.5 = 1$, a contradiction. Consequently, $\mu(x * y) \geq M(\mu(x), \mu(y), 0.5)$ for all $x, y \in X$. Conversely assume that (4) is valid. Let $x, y \in X$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1} \in \mu$ and $y_{t_2} \in \mu$. Then $\mu(x) \geq t_1$ and $\mu(y) \geq t_2$. If $\mu(x * y) < M(t_1, t_2)$, then $M(\mu(x), \mu(y)) \geq 0.5$. Otherwise, we have

$$\mu(x * y) \geq M(\mu(x), \mu(y), 0.5) \geq M(\mu(x), \mu(y)) \geq M(t_1, t_2),$$

a contradiction. It follows that

$$\mu(x * y) + M(t_1, t_2) > 2\mu(x * y) \geq 2M(\mu(x), \mu(y), 0.5) = 1$$

so that $(x * y)_{M(t_1,t_2)} \mathrm{q}\, \mu$. Therefore μ is an $(\in, \in \vee \mathrm{q})$-fuzzy subalgebra of X

THEOREM 3.16. *For any subset S of X, the characteristic function χ_S of S is an $(\in, \in \vee \mathrm{q})$-fuzzy subalgebra of X if and only if S is a subalgebra of X.*

Proof. Assume that χ_S is an $(\in, \in\vee q)$-fuzzy subalgebra of X. Let $x, y \in S$. Then $\chi_S(x) = 1 = \chi_S(y)$, and so $x_1 \in \chi_S$ and $y_1 \in \chi_S$. It follows that $(x*y)_1 = (x*y)_{M(1,1)} \in\vee q\, \chi_S$ which yields $\chi_S(x*y) > 0$. Hence $xy \in S$, and thus S is a subalgebra of X. Conversely if S is a subalgebra of X, then χ_S is an $(\in, \in)$-fuzzy subalgebra of X. It follows from Theorem 3.4 that χ_S is an $(\in, \in\vee q)$-fuzzy subalgebra of X.

THEOREM 3.17. *Let $\{\mu_i \mid i \in \Lambda\}$ be a family of $(\in, \in\vee q)$-fuzzy subalgebras of X. Then $\mu := \bigcap_{i\in\Lambda} \mu_i$ is an $(\in, \in\vee q)$-fuzzy subalgebra of X.*

Proof. Let $x, y \in X$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1} \in \mu$ and $y_{t_2} \in \mu$. Assume that $(x*y)_{M(t_1,t_2)} \overline{\in\vee q}\, \mu$. Then $\mu(x*y) < M(t_1, t_2)$ and $\mu(x*y) + M(t_1, t_2) \leq 1$, which imply that

$$\mu(x*y) < 0.5. \tag{5}$$

Let $\Omega_1 := \{i \in \Lambda \mid (x*y)_{M(t_1,t_2)} \in \mu_i\}$ and

$$\Omega_2 := \{i \in \Lambda \mid (x*y)_{M(t_1,t_2)}\, q\, \mu_i\} \cap \{j \in \Lambda \mid (x*y)_{M(t_1,t_2)} \overline{\in}\, \mu_j\}.$$

Then $\Lambda = \Omega_1 \cup \Omega_2$ and $\Omega_1 \cap \Omega_2 = \emptyset$. If $\Omega_2 = \emptyset$, then $(x*y)_{M(t_1,t_2)} \in \mu_i$ for all $i \in \Lambda$, that is, $\mu_i(x*y) \geq M(t_1, t_2)$ for all $i \in \Lambda$, which yields $\mu(x*y) \geq M(t_1, t_2)$. This is a contradiction. Hence $\Omega_2 \neq \emptyset$, and so for every $i \in \Omega_2$ we have $\mu_i(x*y) < M(t_1, t_2)$ and $\mu_i(x*y) + M(t_1, t_2) > 1$. It follows that $M(t_1, t_2) > 0.5$. Now $x_{t_1} \in \mu$ implies $\mu(x) \geq t_1$ and thus $\mu_i(x) \geq \mu(x) \geq t_1 \geq M(t_1, t_2) > 0.5$ for all $i \in \Lambda$. Similarly we get $\mu_i(y) > 0.5$ for all $i \in \Lambda$. Next suppose that $t := \mu_i(x*y) < 0.5$. Taking $t < r < 0.5$, we get $x_r \in \mu_i$ and $y_r \in \mu_i$, but $(x*y)_{M(r,r)} = (x*y)_r \overline{\in\vee q}\, \mu_i$. This contradicts that μ_i is an $(\in, \in\vee q)$-fuzzy subalgebra of X. Hence $\mu_i(x*y) \geq 0.5$ for all $i \in \Lambda$, and so $\mu(x*y) \geq 0.5$ which contradicts (5). Therefore $(x*y)_{M(t_1,t_2)} \in\vee q\, \mu$ and consequently μ is an $(\in, \in\vee q)$-fuzzy subalgebra of X.

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