Bull. Korean Math. Soc. **42** (2005), No. 4, pp. 691–701

ASYMPTOTIC EQUIVALENCE BETWEEN LINEAR DIFFERENTIAL SYSTEMS

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Abstract. We study the strong stability for linear differential systems in connection with t_∞-similarity, and investigate the asymptotic equivalence between linear differential systems.

1. Introduction

The problem of asymptotic equivalence between differential systems is one of the most important part in the study of asymptotic property for solutions of differential systems, and it shows an asymptotic relationship between solutions of differential systems.

In this paper we study asymptotic equivalence between linear differential systems by using the concepts of t_∞-similarity and strong stability.

The notion of t_∞-similarity in the set of all $n \times n$ continuous matrices defined on $\mathbb{R}_+$ was introduced by Conti [4].

Let $A(t)$ and $B(t)$ be $n \times n$ invertible continuous matrices defined on $\mathbb{R}_+$. They are said to be t_∞-*similar* if there exists an $n \times n$ continuous matrix $F(t)$ with

$$\int_0^\infty |F(t)|dt < \infty$$

and invertible continuous matrix $S(t)$ with bounded $S^{-1}(t)$ such that

$$S'(t) + S(t)B(t) - A(t)S(t) = F(t). \tag{1.1}$$

The notion of t_∞-similarity is an equivalence relation in the set of all $n \times n$ continuous matrices on $\mathbb{R}_+$, and it preserves some stability concepts [4, 6]. Trench [9] introduced the concept of t_∞-quasisimilarity which is not symmetric or transitive, but still preserves stability properties. Choi et al [2, 3] introduced the notion of n_∞-similarity which is the discrete

Received April 1, 2004.
2000 Mathematics Subject Classification: 34D05, 34D23, 34E10.
Key words and phrases: asymptotic equivalence, asymptotic equilibrium, t_∞-similarity, strong stability.

analogue of t_∞-similarity, and studied asymptotic equivalence between linear difference systems.

In section 2, we study the strong stability for linear homogeneous differential systems in connection with t_∞-similarity.

In section 3, we investigate the asymptotic equivalence between linear differential systems.

2. Strong stability for linear differential systems

We consider two linear differential systems

$$x'(t) = A(t)x(t) \tag{2.1}$$

and

$$y'(t) = B(t)y(t), \tag{2.2}$$

where $A(t)$ and $B(t)$ are $n \times n$ continuous matrix functions on $\mathbb{R}_+$.

The notion of strong stability due to G. Ascoli is presented in [5]: System (2.1) is *strongly stable* if for each $\varepsilon > 0$ there exists a corresponding $\delta = \delta(\varepsilon) > 0$ such that any solution $x(t)$ of (2.1) which satisfies the inequality $|x(t_1, t_0, x_0)| < \delta$ for some $t_1 \geq t_0$ exists and satisfies the inequality $|x(t, t_0, x_0)| < \varepsilon$ for all $t \geq t_0$.

REMARK 2.1. Obviously, strong stability implies uniform stability, which in turn implies ordinary stability, and uniform asymptotic stability implies asymptotic stability [5]. But strongly stable system (2.1) cannot be asymptotically stable by the following argument.

Assume that (2.1) is strongly stable. Let the maximal interval definition of the solution $x(t, t_0, x_0)$ is (T, ∞), where $T \geq 0$. First, we prove that $T = 0$.

Suppose that $T > 0$. We can choose t_0', x_0' so that for $T < t_0' < t_0$, $x(t, t_0, x_0) = x(t, t_0', x_0')$. The definition of strong stability yields $|x(t, t_0', x_0')| < \varepsilon$ for $t \geq t_0'$. Since t_0' can be arbitrarily close to T we obtain $|x(t, t_0, x_0)| = |x(t, t_0', x_0')| < \varepsilon$ for all $t \geq T$.

Now, continuity of $f(t, x) = A(t)x$ implies that the solution $x(t, t_0, x_0)$ is Cauchy on the interval $(T, t_0]$ so that $\lim_{t \to T} x(t, t_0, x_0)$ exists. We can prolong the solution to $t < T$ and thus $T = 0$.

From the statement mentioned above we have $|x(t, t_0, x_0)| < \varepsilon$ for all $t \geq 0$. We conclude that if $\liminf_{t \to \infty} |x(t)| = 0$, then $x(t) \equiv 0$.

Also, asymptotically stable system cannot be strongly stable.

The following [5, Theorem 1, p.54] is the characterization of strong stability.

LEMMA 2.2. *(2.1) is strongly stable if and only if there exists a constant $M > 0$ such that*

$$|X(t,t_0)| \le M, \ |X^{-1}(t,t_0)| \le M \text{ for all } t \ge t_0,$$

where $X(t,t_0)$ is a fundamental matrix solution of (2.1).

System (2.1) is said to be *restrictively stable* [1] if it is stable together with the adjoint system

$$y' = -A^T(t)y,$$

where A^T denotes the transpose of the matrix A.

Restrictive stability implies uniform stability, and thus, in turn, implies usual stability. The converses are not true in general [1, p.45].

System (2.1) is said to have *asymptotic equilibrium* [7] if there exists a single $\xi \in \mathbb{R}^n$ and $r > 0$ such that any solution $x(t,t_0,x_0)$ of (2.1) with $|x_0| < r$ satisfies

$$x(t,t_0,x_0) = \xi + o(1) \quad \text{as } t \to \infty$$

and for every $\xi \in \mathbb{R}^n$, there exists a solution of (2.1) such that satisfies the above asymptotic relationship.

Using Lemma 2.2, we obtain the following.

THEOREM 2.3. *Suppose that $X(t,t_0)$ is bounded for each $t \ge t_0$ and $|\det(X(t,t_0))| > \alpha > 0$ for each $t \ge t_0$ and some positive constant α. Then (2.1) is strongly stable.*

Proof. Note that $\frac{1}{|\det(X(t,t_0))|}$ is bounded by $\frac{1}{\alpha}$ and the minor determinants of the bounded matrix $X(t,t_0)$ also are bounded for each $t \ge t_0$. Then we obtain

$$\begin{aligned} |X^{-1}(t,t_0)| &= \left| \frac{1}{\det(X(t,t_0))} \operatorname{adj}(X(t,t_0)) \right| \\ &= \frac{1}{|\det(X(t,t_0))|} \left| [x_{ij}(t,t_0)]^T \right| \le \frac{K}{\alpha} \equiv M, \end{aligned}$$

where $x_{ij}(t,t_0) = (-1)^{i+j} m_{ij}(t,t_0)$ and $m_{ij}(t,t_0)$ is the determinant of the minor matrix of $X(t,t_0)$ for every $i,j = 1,2,\cdots,n$. Thus there exists a positive constant M satisfying $|X^{-1}(t,t_0)| \le M$ for each $t \ge t_0$. Hence (2.1) is strongly stable.

For systems (2.1) and (2.2) strong stability is preserved by t_∞-similarity:

THEOREM 2.4. *Assume that $A(t)$ and $B(t)$ are t_∞-similar. If (2.1) is strongly stable, then (2.2) is also strongly stable.*

Proof. It follows from the strong stability of (2.1) that there exists a positive constant M such that

$$|X(t,t_0)| \le M, \quad |X^{-1}(t,t_0)| \le M, \quad t \ge t_0.$$

Since $A(t)$ and $B(t)$ are t_∞-similar, it is easily seen by differentiating that the solution $S(t)$ of (1.1) is given by

$$\begin{aligned} S(t) &= X(t,t_0)[S(t_0) \\ &\quad + \int_{t_0}^{t} X^{-1}(s,t_0)F(s)Y(s,t_0)ds]Y^{-1}(t,t_0), \quad t \ge t_0. \end{aligned}$$

It follows from the above equality that

$$\begin{aligned} |Y(t,t_0)| &= |S^{-1}(t)[X(t,t_0)S(t_0) \\ &\quad + \int_{t_0}^{t} X(t,s)F(s)Y(s,t_0)ds]| \\ &\le c^2M + cM\int_{t_0}^{t} |F(s)||Y(s,t_0)|ds. \end{aligned} \tag{2.3}$$

By the well-known Gronwall inequality, we have

$$\begin{aligned} |Y(t,t_0)| &\le c^2M\exp(cM\int_{t_0}^{t} |F(s)|ds) \\ &\le c^2M\exp(cM\int_{t_0}^{\infty} |F(s)|ds) \equiv L, \quad t \ge t_0. \end{aligned}$$

Next, we show that $|Y^{-1}(t,t_0)|$ is also bounded for each $t \ge t_0$. We have

$$\begin{aligned} & S^{-1}(t_0)X^{-1}(t,t_0)S(t) \\ &= Y^{-1}(t,t_0) + [S^{-1}(t_0)\int_{t_0}^{t} X^{-1}(s,t_0)F(s)Y(s,t_0)ds]Y^{-1}(t,t_0), \quad t \ge t_0. \end{aligned}$$

Then from the strong stability of (2.1) and boundedness of $S(t)$ and $S^{-1}(t)$, there exist positive constants c and M such that

$$\begin{aligned} & c^2M \\ &\ge |S^{-1}(t_0)X^{-1}(t,t_0)S(t)| \\ &\ge |Y^{-1}(t,t_0)| - [|S^{-1}(t_0)|\int_{t_0}^{t} |X^{-1}(s,t_0)||F(s)||Y(s,t_0)|ds]|Y^{-1}(t,t_0)| \end{aligned}$$

$$\geq |Y^{-1}(t,t_0)| - cML \int_{t_0}^{t} |F(s)|ds|Y^{-1}(t,t_0)|$$
$$\geq [1 - cML \int_{t_0}^{t} |F(s)|ds]|Y^{-1}(t,t_0)|, \quad t \geq t_0.$$

Since $\int_{t_0}^{\infty} |F(s)|ds < \infty$, we can choose $T \geq t_0$ so large that

$$cML \int_{T}^{\infty} |F(s)|ds < \frac{1}{2}$$

for each $t \geq T$. Then have

$$|Y^{-1}(t,T)| \leq \frac{c^2 M}{1 - cML \int_T^t |F(s)|ds} < \infty, \quad t \geq T \geq t_0. \tag{2.4}$$

Thus there exists a constant $\hat{M} > 0$ such that

$$\begin{aligned} |Y^{-1}(t,t_0)| &= |Y^{-1}(T,t_0)Y^{-1}(t,T)| \\ &\leq |Y^{-1}(T,t_0)| \frac{c^2 M}{1 - cML \int_T^{\infty} |F(s)|ds} \\ &\leq \hat{M}, \quad t \geq T \geq t_0. \end{aligned}$$

This completes the proof.

COROLLARY 2.5. *Assume that* $\int_{t_0}^{\infty} |A(t) - B(t)|dt < \infty$ *and* (2.1) *is strongly stable. Then* (2.2) *is also strongly stable.*

Proof. Putting $S(t) = I$, $A(t)$ and $B(t)$ are t_∞-similar. It follows from Theorem 2.4 that (2.2) is strongly stable.

REMARK 2.6. For linear homogeneous systems, restrictive stability and strong stability are equivalent by (3.9. *iii*) in [1, p.46]. Also the linear differential system is restrictively stable if and only if it is reducible to zero [1, p.46]. Theorem 2.4 can be easily proved by using the notion of reducibility in [1] : (2.1) is *reducible(reducible to zero)* if there exists an $n \times n$ matrix $L(t)$ whose elements are absolutely continuous functions which is bounded together with its inverse $L^{-1}(t)$ in $[t_0, \infty)$, such that $L^{-1}(t)A(t)L(t) - L^{-1}(t)L'(t)$ is a constant (zero) matrix in $[t_0, \infty)$.

Next result follows easily from Theorem 2.4. Also, we can prove this result by using the notion of reducibility in [1].

COROLLARY 2.7. *Assume that* $A(t)$ *and* $B(t)$ *are* t_∞*-similar with* $F(t) = 0$ *in* (1.1). *If* (2.1) *is strongly stable, then* (2.2) *is also strongly stable.*

Proof. Since (2.1) is restrictively stable, there exists an $n \times n$ matrix solution $L(t)$ of the following equation (2.5) whose elements are absolutely continuous functions, which is bounded together with $L^{-1}(t)$ in $[t_0, \infty)$, such that

$$L^{-1}(t)A(t)L(t) - L^{-1}(t)L'(t) = 0. \tag{2.5}$$

Letting $T(t) = S^{-1}(t)L(t)$, we have

$$\begin{aligned} T^{-1}(t)B(t)T(t) - T^{-1}(t)T'(t) &= L^{-1}(t)S(t)B(t)S^{-1}(t)L(t) \\ &\quad + L^{-1}(t)S'(t)S^{-1}(t)L(t) - L^{-1}(t)L'(t) \\ &= L^{-1}(t)[S(t)B(t)S^{-1}(t) \\ &\quad + S'(t)S^{-1}(t)]L(t) - L^{-1}(t)L'(t) \\ &= L^{-1}(t)A(t)L(t) - L^{-1}(t)L'(t) \\ &= 0 \end{aligned}$$

by the notion of t_∞-similarity of $A(t)$ and $B(t)$, and

$$T'(t) = -S^{-1}(t)S'(t)S^{-1}(t)L(t) + S^{-1}(t)L'(t).$$

This completes the proof.

3. Asymptotic equivalences

In this section we study asymptotic equivalence between two linear differential systems.

Systems (2.1) and (2.2) are said to be *asymptotically equivalent* [7] if, for every solution $x(t)$ of (2.1), there exists a solution $y(t)$ of (2.2) such that

$$x(t) = y(t) + o(1) \text{ as } t \to \infty,$$

and conversely, for every solution $y(t)$ of (2.2), there exists a solution $x(t)$ of (2.1) such that the above asymptotic relation holds.

REMARK 3.1. As a consequence of Theorem 2.3 we obtain the following : Assume that (2.1) is strongly stable and $\lim_{t\to\infty} X(t, t_0) = X_\infty(t_0)$ exists, where $X(t, t_0)$ is a fundamental matrix solution of (2.1). Then (2.1) has asymptotic equilibrium.

THEOREM 3.2. *System* (2.1) *has asymptotic equilibrium if and only if* $\lim_{t\to\infty} X(t, t_0)$ *exists and is invertible, where* $X(t, t_0)$ *is a fundamental matrix solution of* (2.1).

Proof. It is easy to show the existence of $\lim_{t\to\infty} X(t,t_0) = X_\infty(t_0)$. Let $E_i = (0,\cdots,1,\cdots,0)^T$ be the i-th unit vector in $\mathbb{R}^n$ for each $i = 1,2,\cdots,n$. Then there exist the solutions $x(t,t_0,x_{0i})$ of (2.1) such that

$$\lim_{t\to\infty} x(t,t_0,x_{0i}) = \lim_{t\to\infty} X(t,0)X^{-1}(t_0,0)x_{0i} = E_i,\ i = 1,2,\cdots,n.$$

It follows that

$$\lim_{t\to\infty} X(t,0)X^{-1}(t_0,0)[x_{01}\cdots x_{0n}] = X_\infty(t_0)[x_{01}\cdots x_{0n}] = I,$$

where I is the identity matrix. Thus $X_\infty(t_0)$ is invertible. This completes the proof.

THEOREM 3.3. *If* (2.1) *has asymptotic equilibrium, then*

$$\lim_{t\to\infty}\int_{t_0}^{t} \mathrm{tr}(\mathrm{A(s)})\mathrm{ds}$$

exists.

Proof. Since $\det(X(t,t_0))$ satisfies the first order equation

$$\frac{d}{dt}\det(X(t,t_0)) = \mathrm{tr}(A(t))\det(X(t,t_0)),$$

we have the well-known Jacobi-Liouville formula

$$\det(X(t,t_0)) = \det(X(t_0,t_0))\exp\left(\int_{t_0}^{t}\mathrm{tr}(A(s))ds\right),\ t\geq t_0\geq 0.$$

Also, we have

$$\begin{aligned}0\neq\det(X_\infty(t_0)) &= \lim_{t\to\infty}\det(X(t,t_0))\\ &= \det(X(t_0,t_0))\exp\left(\int_{t_0}^{\infty}\mathrm{tr}(A(s))ds\right).\end{aligned}$$

Thus $\int_{t_0}^{\infty}\mathrm{tr}(A(s))ds$ exists.

COROLLARY 3.4. [1] *If* $\int_{t_0}^{\infty}|A(t)|dt$ *exists, then* (2.1) *has asymptotic equilibrium.*

The following example shows that the converse of Theorem 3.3 does not hold in general :

EXAMPLE 3.5. We consider the linear differential system

$$x'(t) = A(t)x = \begin{pmatrix} t & 0\\ 0 & -t\end{pmatrix}x. \tag{3.1}$$

A fundamental matrix solution $X(t,0)$ of (3.1) is given by

$$X(t,0) = \begin{pmatrix} e^{t^2/2} & 0 \\ 0 & e^{-t^2/2} \end{pmatrix}.$$

Then we easily see that $\int_{t_0}^{\infty} \operatorname{tr}(A(t))dt = 0$ but $\lim_{t\to\infty} X(t,0)$ does not exist.

The following theorem means that asymptotic equilibrium for linear differential systems is preserved by the notion of t_∞-similarity.

THEOREM 3.6. *Suppose that $A(t)$ and $B(t)$ are t_∞-similar with*

$$\lim_{t\to\infty} S(t) = S_\infty.$$

If (2.1) has asymptotic equilibrium, then (2.2) also has asymptotic equilibrium.

Proof. In view of Theorem 2.4, (2.2) is strongly stable. Also, our assumption on $S(t)$ implies that $\lim_{t\to\infty} S(t) = S_\infty$ is invertible and $\lim_{t\to\infty} S^{-1}(t) = S_\infty^{-1}$. Since $\int_{t_0}^{\infty} |F(t)|dt < \infty$, and $X(t,s)$ and $Y(t,t_0)$ are bounded, we easily see from the Cauchy property that

$$\lim_{t\to\infty} \int_{t_0}^{t} X(t,s)F(s)Y(s,t_0)ds$$

exists. It follows from (2.3) and (2.4) that $\lim_{t\to\infty} Y(t,t_0) = Y_\infty(t_0)$ exists and is also invertible. Therefore (2.2) has asymptotic equilibrium by Theorem 3.2.

Now, we obtain the following result about asymptotic equivalence by using the concepts of t_∞-similarity and asymptotic equilibrium.

THEOREM 3.7. *Assume that*

(i) *there exists a positive constant α with $|\det(X(t,t_0))| > \alpha > 0$ for each $t \geq t_0$ and $\lim_{t\to\infty} X(t,t_0) = X_\infty(t_0)$ exists,*
(ii) *$A(t)$ and $B(t)$ are t_∞-similar with $\lim_{t\to\infty} S(t) = S_\infty$.*

Then (2.1) and (2.2) are asymptotically equivalent.

Proof. We easily see that (2.1) has asymptotic equilibrium by the fact that $|\det(X_\infty(t_0))| \geq \alpha > 0$ and Theorem 3.2. It follows from the assumption (ii) and Theorem 3.6 that (2.1) and (2.2) has asymptotic equilibrium, respectively. Let $x(t,t_0,x_0)$ be any solution of (2.1). Then $\lim_{t\to\infty} x(t) = x_\infty$ exists. For $x_\infty \in \mathbb{R}^n$, the condition on asymptotic equilibrium for (2.2) implies that there exists a solution $y(t) = y(t, t_0, y_0)$ of (2.2) such that $\lim_{t\to\infty} y(t) = x_\infty$. This implies that

$$y(t) = x(t) + o(1) \text{ as } t \to \infty.$$

By the same manner, we can obtain the converse asymptotic relationship.

Finally, we study the asymptotic equivalence between homogeneous and nonhomogeneous linear systems by means of asymptotic equilibrium of homogeneous system. So we consider the perturbation of (2.1)

$$y'(t) = A(t)y(t) + g(t), \tag{3.2}$$

where $g(t)$ is a continuous function on $\mathbb{R}_+$.

LEMMA 3.8. *Assume that* (2.1) *has asymptotic equilibrium and the perturbed term* g *in* (3.2) *is absolutely integrable on* $[t_0, \infty)$, *i.e.,*

$$\int_{t_0}^{\infty} |g(s)|ds < \infty.$$

Then (3.2) *also has asymptotic equilibrium.*

Proof. Let $y(t, t_0, y_0)$ be any solution of (3.2). Then the solution $y(t)$ of (3.2) is given by

$$y(t) = X(t, t_0)y_0 + X(t, t_0) \int_{t_0}^{t} X^{-1}(s, t_0)g(s)ds,$$

where $X(t, t_0)$ is a fundamental matrix solution of (2.1). Putting $p(t) = \int_{t_0}^{t} X^{-1}(s, t_0)g(s)ds$, we easily see that $p(t)$ has a finite limit as $t \to \infty$ because $\int_{t_0}^{\infty} |g(s)|ds < \infty$ and $X^{-1}(t, t_0)$ is bounded. Thus $y(t)$ converges to a vector $\xi \in \mathbb{R}^n$ as $t \to \infty$.

Conversely, let ξ be any vector in $\mathbb{R}^n$. Then there exists a solution $y(t) = y(t, t_0, y_0)$ of (3.2) with the initial point $y_0 = X_\infty^{-1}(t_0)\xi - p_\infty$ such that

$$\begin{aligned}
\lim_{t\to\infty} y(t) &= \lim_{t\to\infty} [X(t, t_0)y_0 + X(t, t_0) \int_{t_0}^{t} X^{-1}(s, t_0)g(s)ds] \\
&= X_\infty(t_0)[y_0 + p_\infty] \\
&= X_\infty(t_0)[X_\infty^{-1}(t_0)\xi - p_\infty + p_\infty] \\
&= \xi,
\end{aligned}$$

where $\lim_{t\to\infty} p(t) = p_\infty$ and $\lim_{t\to\infty} X(t, t_0) = X_\infty(t_0)$ is invertible. This completes the proof.

As a consequence of Lemma 3.8 we easily obtain the following result.

THEOREM 3.9. *Suppose that* (2.1) *has asymptotic equilibrium and* $\int_{t_0}^{\infty}|g(s)|ds < \infty$. *Then* (2.1) *and* (3.2) *are asymptotically equivalent.*

Proof. Let $x(t)$ be any solution of (2.1). Then we have $\lim_{t\to\infty} x(t) = x_\infty$ since (2.1) has asymptotic equilibrium. Setting $y_0 = X_\infty^{-1}(t_0)x_\infty - p_\infty$ as in Lemma 3.8, there exists a solution $y(t, t_0, y_0)$ of (3.2) such that

$$\begin{aligned}\lim_{t\to\infty}[y(t) - x(t)] &= X_\infty(t_0)[y_0 + p_\infty] - x_\infty \\ &= X_\infty(t_0)[X_\infty^{-1}(t_0)x_\infty - p_\infty + p_\infty] - x_\infty \\ &= 0.\end{aligned}$$

Conversely, we easily see that the asymptotic relationship also holds by setting $x_0 = y_0 + p_\infty$. This completes the proof.

EXAMPLE 3.10. To illustrate Theorem 3.9 we consider the homogeneous differential system

$$x'(t) = A(t)x(t) = \begin{pmatrix} \frac{-e^{-t}}{2+e^{-t}} & 0 \\ 0 & 0 \end{pmatrix} x(t) \tag{3.3}$$

and nonhomogeneous differential system

$$y'(t) = A(t)y(t) + g(t) = \begin{pmatrix} \frac{-e^{-t}}{2+e^{-t}} & 0 \\ 0 & 0 \end{pmatrix} y(t) + \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}. \tag{3.4}$$

Then (3.3) and (3.4) are asymptotically equivalent by the simple calculation.

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