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A ONE-SIDED VERSION OF POSNER'S SECOND THEOREM ON MULTILINEAR POLYNOMIALS

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ABSTRACT. Let K be a commutative ring with unity, R a prime K-algebra of characteristic different from 2, d a non-zero derivation of R, I a non-zero right ideal of R, $f(x_1,\ldots,x_n)$ a multilinear polynomial in n non-commuting variables over K, $a \in R$. Supppose that, for any $x_1,\ldots,x_n \in I$, $a[d(f(x_1,\ldots,x_n)), f(x_1,\ldots,x_n)] = 0$. If $[f(x_1,\ldots,x_n), x_{n+1}]x_{n+2}$ is not an identity for I and

$$S_4(I,I,I,I)I \neq 0,$$

then $aI = ad(I) = 0$.

Let K be a commutative ring with unity. In all that follows R will be a prime K-algebra, with extended centroid C and Q its two-sided Martindale quotient ring, and $f(x_1,\ldots,x_n)$ is a multilinear polynomials in n non-commuting variables over K. For $x,y \in R$, the commutator of x and y, denoted by $[x,y]$, is defined to be $xy - yx$ and $[x,y]_2 = [[x,y],y] = [x,y]y - y[x,y]$.

Recently, in [6], T.K. Lee explored the relationship between the structure of a prime ring R and the behaviour of its derivations satisfying certain constraints on a polynomial $f(x_1,\ldots,x_n)$. In particular he proved that if $f(x_1,\ldots,x_n)$ is multilinear and I is a non-zero right ideal of R such that $[d(f(x_1,\ldots,x_n)), f(x_1,\ldots,x_n)]_k = 0$ for all $x_1,\ldots,x_n \in I$ and $k \geq 1$ a fixed integer, then there exists an idempotent element e in the socle of RC such that $I = eCR$ and $f(x_1,\ldots,x_n)$ is central-valued on $eCRe$ except when $\text{char}(R) = 2$ and $\dim_C(eCRe) = 4$. We continue this line of investigation by studying the subset of R:

$$T = \{[d(f(x_1,\ldots,x_n)), f(x_1,\ldots,x_n)], x_1,\ldots,x_n \in I\}.$$

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An approach that can be used in studying T is to examine its size. If T is large, we would expect that its left annihilator $l(T) = \{x \in R,\ xt = 0, \forall t \in T\}$ annihilates on the left both I and $d(I)$. In fact we will prove:

THEOREM 1. *Let K be a commutative ring with unity, R a prime K-algebra of characteristic different from 2, d a non-zero derivation of R, I a non-zero right ideal of R, $f(x_1, \dots, x_n)$ a multilinear polynomial in n non-commuting variables over K, $a \in R$. Supppose that, for any $x_1, \dots, x_n \in I$, $a[d(f(x_1, \dots, x_n)), f(x_1, \dots, x_n)] = 0$. If $[f(x_1, \dots, x_n), x_{n+1}]x_{n+2}$ is not an identity for I and $S_4(I, I, I, I)I \neq 0$, then $aI = ad(I) = 0$.*

We first fix the following facts:

FACT 1. Denote by $T = Q *_C C\{X\}$ the free product over C of the C-algebra Q and the free C-algebra $C\{X\}$, with X a countable set consisting of non-commuting indeterminates $\{x_1, \dots, x_n, \dots\}$. The elements of T are called generalized polynomial with coefficients in Q. I, IR and IQ satisfy the same generalized polynomial identities with coefficients in Q. For more details about these objects we refer the reader to [1] and [2].

FACT 2. Any derivation of R can be uniquely extended to a derivation of Q, and so any derivation of R can be defined on the whole of Q [1, proposition 2.5.1]. Moreover Q is a prime ring as well as R and the extended centroid C of R coincides with the center of Q [1, Proposition 2.1.7, Remark 2.3.1].

FACT 3. (V.K. Kharchenko[5]) Let $f(x_1, \dots, x_n, d(x_1), \dots, d(x_n))$ be a differential identity of R. One of the following holds:

1) either d is an inner derivation in Q, in the sense that there exists $q \in Q$ such that $d(x) = [q, x]$, for all $x \in Q$ and Q satisfies the generalized polynomial identity $f(x_1, \dots, x_n, [q, x_1], \dots, [q, x_n])$;

2) or R satisfies the generalized polynomial identity $f(x_1, \dots, x_n, y_1, \dots, y_n)$.

FACT 4. (T.K. Lee[7]) I, IR and IQ satisfy the same differential identities with coefficients in Q.

FACT 5. Since R is a prime ring, we may assume $K \subseteq C$ and so for any $\alpha \in K$ one has $d(\alpha) \in C$.

We will use the following notation:

$$f(x_1, \dots, x_n) = x_1 x_2 .. x_n + \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} x_{\sigma(2)} ... x_{\sigma(n)}$$

for some $\alpha_\sigma \in C$ and moreover we denote by $f^d(x_1, \dots, x_n)$ the polynomial obtained from $f(x_1, \dots, x_n)$ by replacing each coefficient α_σ with

$d(\alpha_\sigma)$. Thus we write $d(f(r_1,\ldots,r_n)) = f^d(r_1,\ldots,r_n) + \sum_i f(r_1,\ldots, d(r_i),\ldots,r_n)$, for all $r_1,r_2,\ldots,r_n$ in R. Hence if $a \in \mathrm{Ann}_R(\{[d(f(r_1,\ldots,r_n)), f(r_1,\ldots,r_n)], r_i \in R\})$ then R satisfies the generalized differential identity

$$a[d(f(x_1,\ldots,x_n)), f(x_1,\ldots,x_n)] = a\biggl(\Bigl[f^d(x_1,\ldots,x_n) + \sum_i f(x_1,\ldots,d(x_i),\ldots,x_n), f(x_1,\ldots,x_n)\Bigr]\biggr).$$

REMARK. The assumption $S_4(I,I,I,I)I \neq 0$ is essential to the main result. For example consider $R = \mathrm{End}_F(V)$, for F a field and $(V:F) \geq 3$ (possibly infinite), and let e_{ij} be the usual matrix unit in R. Let $I = (e_{11}+e_{22})R$, $a = e_{22}$, $f(x_1,x_2) = [x_1,x_2]$, d the inner derivation induced by the element e_{13}, that is $d(x) = [e_{13},x] = e_{13}x - xe_{13}$, for all $x \in R$. In this case, notice that $S_4(x_1,x_2,x_3,x_4)x_5$ is an identity for I, moreover

$$e_{22}\,[e_{13},[(e_{11}+e_{22})x_1,(e_{11}+e_{22})x_2]]_2 = 0$$

for any $x_1,x_2 \in R$, but clearly $e_{22}I \neq 0 \neq e_{22}d(I)$.

For the remainder of the paper we now assume the conclusion of the Theorem to be false; our goal is to ultimately arrive at a contradiction. Thus we shall assume henceforth that either $aI \neq 0$ or $ad(I) \neq 0$.

LEMMA 1. *R is a ring satisfying a non-trivial generalized polynomial identity* (GPI).

Proof. Suppose by contradiction that R does not satisfy any non-trivial generalized polynomial identity. We divide the proof into three cases:

CASE 1. Suppose that d is an inner derivation induced by an element $q \in Q$ and $aI \neq 0$.

Pick $b \in I$ such that $ab \neq 0$. Since R does not satisfy any non-trivial generalized polynomial identity, $a[q, f(bx_1,\ldots,bx_n)]_2$ is the zero element in the free product $T = Q *_C C\{x_1,x_2,...,x_n\}$ ([2]). Then

$$(1)\quad \begin{aligned} &a(qf(bx_1,\ldots,bx_n)^2 + f(bx_1,\ldots,bx_n)^2q \\ &\qquad - 2f(bx_1,\ldots,bx_n)qf(bx_1,\ldots,bx_n)) = 0 \in T. \end{aligned}$$

Since $1, q$ are C-linearly independent, we conclude that $af(bx_1,\ldots, bx_n)^2q = 0 \in T$ that is $af(br_1,\ldots,br_n)^2q = 0$, for all $r_1,\ldots,r_n \in R$. If $\{q, ab\}$ are linearly C-independent then $af(bx_1,\ldots,bx_n)^2q$ is a non-trivial generalizad polynomial identity for R, a contradiction. On the other hand, if there exists $\beta \in C$ such that $q = \beta ab$, then R satisfies $\beta a f(bx_1, \dots, bx_n)^2 ab$, which is non-trivial because $ab \neq 0$, again a contradiction.

<u>CASE 2.</u> Suppose now that d is an outer derivation and $aI \neq 0$.

Pick again $b \in I$ such that $ab \neq 0$ and $r_1, \dots, r_n \in R$. By our assumption we have that $a[d(f(br_1, \dots, br_n)), f(br_1, \dots, br_n)] = 0$ and so

$$0 = a\Big[f^d(br_1, \dots, br_n) + \sum_i f(br_1, \dots, d(b)r_i + bd(r_i), \dots, br_n), f(br_1, \dots, br_n)\Big]. \tag{2}$$

By Fact 3 it follows that, for all $s_i \in R$,

$$a\Big[\sum_i f(br_1, \dots, d(b)r_i + bs_i, \dots, br_n), f(br_1, \dots, br_n)\Big] = 0.$$

Thus

$$a\Big[\sum_i f(br_1, \dots, bs_i, \dots, br_n), f(br_1, \dots, br_n)\Big] = 0$$

and since $ab \neq 0$, we have the contradiction that R satisfies the non-trivial generalized polynomial identity

$$a\Big[\sum_i f(bx_1, \dots, by_i, \dots, bx_n), f(bx_1, \dots, bx_n)\Big] = 0.$$

<u>CASE 3.</u> Finally suppose that $aI = 0$ but $ad(I) \neq 0$.

Let now $b \in I$ such that $ad(b) \neq 0$. We start again with our equation (2).

$$0 = a\Big[f^d(br_1, \dots, br_n) + \sum_i f(br_1, \dots, d(b)r_i + bd(r_i), \dots, br_n), f(br_1, \dots, br_n)\Big]$$

Write $f(x_1, \dots, x_n) = \sum_i x_i h_i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$, where any h_i is a multilinear polynomial of degree $n-1$ and the variable x_i never appears in h_i. In this case, since $ab = 0$, we get

$$\begin{aligned} 0 &= a\sum_i d(b)r_i h_i(br_1, \dots, br_{i-1}, br_{i+1}, \dots, br_n) f(br_1, \dots, br_n) \\ &= ad(b)\sum_i r_i h_i(br_1, \dots, br_{i-1}, br_{i+1}, \dots, br_n) f(br_1, \dots, br_n) \end{aligned}$$

which is a non-trivial GPI for R, because $ad(b) \neq 0$, a contradiction.

LEMMA 2. *Without loss of generality R is simple and equal to its own socle, $IR = I$ and $a \in I$.*

Proof. By Fact 1, R is GPI and so Q has non-zero socle H with non-zero right ideal $J = IH$ ([9]). Note that H is simple, $J = JH$ and J satisfies the same basic conditions as I, in view of Fact 4. Since $Ja \neq 0$, we may replace a by $0 \neq ca \in J$, for some $c \in J$. Now just replace R by H, I by J, a by ca and we are done.

At this point for sake of completeness we present the following results, which will be useful in the proof of the next Lemmas:

LEMMA 3. *Let R be a simple ring, which is not satisfying $S_4(x_1, x_2, x_3, x_4)$, $f(x_1, \ldots, x_n)$ a polynomial on R and a, b fixed in R such that $af(x_1, \ldots, x_n)^2 b = 0$ for all $x_1, \ldots, x_n \in R$. Then $a[x, y]b = 0$, for all $x, y \in R$.*

Proof. Let $U = \text{Span}\{f(x_1, \ldots, x_n)^2 : x_1, \ldots, x_n \in R\}$. It is easy to see that U is a non-central Lie ideal of R. In this condition it is well known that $[R, R] \subseteq U$ and so $a[R, R]b = 0$.

LEMMA 4. *Let R be a prime K-algebra of characteristic different from 2, $f(x_1, \ldots, x_n)$ a non-central multilinear polynomial over K in n non-commuting variables, $a, b \in R$. If $af(r_1, \ldots, r_n)r_{n+1}bf(r_1, \ldots, r_n) = 0$, for any $r_1, \ldots, r_{n+1} \in R$, then either $a = 0$ or $b = 0$.*

Proof. If R does not satisfy any non-trivial generalized polynomial identity then the required conclusion follows immediately. Thus we may suppose that R satisfies a non-trivial generalized polynomial identity. By Martindale's theorem in [9], R is a primitive ring which is isomorphic to a dense ring of linear transformations of a vector space V over a division ring D. If $\dim_D V = \infty$, then, by [11, Lemma 2], we get $ar_1r_2br_1 = 0$, for all $r_1, r_2 \in R$ and, by the primeness of R, we are done.

Therefore we consider the case $\dim_D(V) = k$, with k finite positive integer. Of course $k \geq 2$, because R is not a domain (otherwise we are trivially done). In this condition R is a simple ring which satisfies a non-trivial generalized polynomial identity. By [10, Theorem 2.3.29] $R \subseteq M_t(F)$, for a suitable field F and $t \geq 2$, moreover $M_t(F)$ satisfies the same generalized identity of R, hence $af(r_1, \ldots, r_n)r_{n+1}bf(r - 1, \ldots, r_n) = 0$, for all $r_1, \ldots, r_{n+1} \in M_t(F)$ and moreover $f(x_1, \ldots, x_n)$ is a non-central polynomial for $M_t(F)$. Since $f(x_1, \ldots, x_n)$ is not central then, by [8], there exist $u_1, \ldots, u_n \in M_t(F)$ and $\alpha \in F - \{0\}$, such that $f(u_1,\dots,u_n)=\alpha e_{kl}$, with $k\neq l$. Here e_{kl} denotes the usual matrix unit with 1 in (k,l)-entry and zero elsewhere. Moreover, since the set $\{f(v_1,\dots,v_n): v_1,\dots,v_n\in M_t(F)\}$ is invariant under the action of all F-automorphisms of $M_t(F)$, then for any $i\neq j$ there exist $r_1,\dots,r_n\in M_t(F)$ such that $f(r_1,\dots,r_n)=\alpha e_{ij}$. Hence, for all $i\neq j$,

$$0=af(r_1,\dots,r_n)r_{n+1}bf(r_1,\dots,r_n)=\alpha^2 a e_{ij} r_{n+1} b e_{ij}.$$

In other words, for all i, either the i-th column of the matrix a is zero or the i-th column of the matrix b is zero. In all follows denote $a=\sum_{ij}a_{ij}e_{ij}$ and $b=\sum_{ij}b_{ij}e_{ij}$.

<u>CASE 1: $t=2$.</u>

Suppose that b is not a diagonal matrix, say $b_{12}\neq 0$. In this case, as we said above, the second column of a is zero. Of course we may assume $b_{21}=b_{11}=0$, otherwise the first column of a is zero too, and we are done. In other words we are in the following situation:

$$b=\begin{bmatrix}0 & b_{12}\\ 0 & b_{22}\end{bmatrix},\quad b_{12}\neq 0,\quad a=\begin{bmatrix}a_{11} & 0\\ a_{21} & 0\end{bmatrix}.$$

Now, since $f(x_1,\dots,x_n)$ is not central for $M_t(F)$, by [8, Lemmas 2 and 9], there exists a sequence of matrices $\underline{r}=(r_1,\dots,r_n)$ such that $f(\underline{r})=\alpha e_{11}+\beta e_{22}$ is not central, that is $\alpha\neq\beta$. Let φ the inner automorphism on $M_t(F)$ defined by $\varphi(x)=(1+e_{12})x(1-e_{12})$. Thus $f(\underline{s})=f(\varphi(\underline{r}))=f(\underline{r})+(\alpha-\beta)e_{12}$ is a valuation of f on R.

By calculation, it follows that, for all $X\in M_t(F)$,

$$af(\underline{s})Xbf(\underline{s})=\begin{bmatrix}\alpha a_{11} & (\alpha-\beta)a_{11}\\ \alpha a_{21} & (\alpha-\beta)a_{21}\end{bmatrix}\cdot X\cdot\begin{bmatrix}0 & \beta b_{12}\\ 0 & \beta b_{22}\end{bmatrix}.$$

By the primeness of $M_t(F)$, it follows that either

$$\begin{bmatrix}\alpha a_{11} & (\alpha-\beta)a_{11}\\ \alpha a_{21} & (\alpha-\beta)a_{21}\end{bmatrix}=0\quad\text{or}\quad\begin{bmatrix}0 & \beta b_{12}\\ 0 & \beta b_{22}\end{bmatrix}=0.$$

Suppose $\begin{bmatrix}\alpha a_{11} & (\alpha-\beta)a_{11}\\ \alpha a_{21} & (\alpha-\beta)a_{21}\end{bmatrix}=0$, then if $a\neq 0$ it follows the contradiction $\alpha=\beta=0$.

Let now $\begin{bmatrix}0 & \beta b_{12}\\ 0 & \beta b_{22}\end{bmatrix}=0$. Since $b_{12}\neq 0$, we get $\beta=0$. This means that $f(\underline{r})=\alpha e_{11}$. Let φ the inner automorphism on $M_t(F)$ defined by $\varphi(x)=(1+e_{21})x(1-e_{21})$. Thus $f(\underline{s})=f(\varphi(\underline{r}))=f(\underline{r})+\alpha e_{21}=\alpha(e_{11}+e_{21})$ is a valuation of f on R.

As above it follows that, for all $X \in M_t(F)$,

$$af(\underline{s})Xbf(\underline{s}) = \begin{bmatrix} \alpha a_{11} & 0 \\ \alpha a_{21} & 0 \end{bmatrix} \cdot X \cdot \begin{bmatrix} \alpha b_{12} & 0 \\ \alpha b_{22} & 0 \end{bmatrix}.$$

If $\begin{bmatrix} \alpha b_{12} & 0 \\ \alpha b_{22} & 0 \end{bmatrix} = 0$, since $\alpha \neq 0$, it follows $b = 0$ a contradiction. Thus $\begin{bmatrix} \alpha a_{11} & 0 \\ \alpha a_{21} & 0 \end{bmatrix} = 0$, that is $a = 0$. In any case, we have that if $a \neq 0$ then b is a diagonal matrix.

CASE 2: $t \geq 3$.

Also in this case we want to prove that b is a diagonal matrix. Suppose that there exists some non-zero entry b_{ji} of b, for $i \neq j$. As we said above the i-th column of a is zero. Let $m \neq i, j$ and $\varphi_{mi}(x) = (1 + e_{mi})x(1 - e_{mi})$. Consider the following valutations of $f(x_1, \ldots, x_n)$:

$$f(\underline{r}) = \gamma e_{ij}, \quad f(\underline{s}) = \varphi_{mi}(f(\underline{r})) = \gamma e_{ij} + \gamma e_{mj}, \quad \gamma \neq 0.$$

Since $af(\underline{s})Xbf(\underline{s}) = 0$ we have that either $a\gamma(e_{ij} + e_{mj}) = 0$ or $b\gamma(e_{ij} + e_{mj}) = 0$. In the first case we have $ae_{mj} = 0$, in the second one $b_{ji} + b_{jm} = 0$, that is $b_{jm} = -b_{ji} \neq 0$. In any case the m-th column of a is zero.

Therefore a has at most one non-zero column, the j-th one.

Let now $i \neq j$ and $\varphi_{ji}(x) = (1 + e_{ji})x(1 - e_{ji})$. As above we consider the following valutations of $f(x_1, \ldots, x_n)$:

$$f(\underline{r}) = \gamma e_{ij}, \quad f(\underline{s}) = \varphi_{ji}(f(\underline{r})) = \gamma(e_{ij} + e_{jj} - e_{ii} - e_{ji}), \quad \gamma \neq 0.$$

Since $af(\underline{s})Xbf(\underline{s}) = 0$, we have that either $a\gamma(e_{ij} + e_{jj} - e_{ii} - e_{ji}) = 0$ or $b\gamma(e_{ij} + e_{jj} - e_{ii} - e_{ji}) = 0$.

Suppose $a\gamma(e_{ij} + e_{jj} - e_{ii} - e_{ji}) = 0$, that is, $a\gamma(e_{jj} - e_{ji}) = 0$. It follows that, for all k

$$0 = e_{kk}a\gamma(e_{jj} - e_{ji}) = a_{kj}(e_{kj} - e_{ki}),$$

which means $a_{kj} = 0$, i.e., also the j-th column of a is zero.

On the other hand, if $b\gamma(e_{ij} + e_{jj} - e_{ii} - e_{ji}) = 0$, left multiplying by e_{jj},

$$0 = e_{jj}b\gamma(e_{ij} + e_{jj} - e_{ii} - e_{ji}) = (b_{ji} + b_{jj})(e_{jj} - e_{ji}).$$

This implies $b_{ji} + b_{jj} = 0$, that is, $b_{jj} = -b_{ji} \neq 0$ and also in this case the j-th column of a is zero.

Therefore, if $a \neq 0$, then $b_{ji} = 0$ for all $j \neq i$. The previous two cases show that b is a diagonal matrix, $b = \sum b_{kk}e_{kk}$. Moreover if φ is an automorphism of $M_t(F)$, the same conclusion holds for $\varphi(b)$, since as above

$$0 = \varphi(a)\varphi(f(r_1, \ldots, r_n))X\varphi(b)\varphi f(r_1, \ldots, r_n)$$

Therefore, for any $i \neq j$, $\varphi(b) = (1+e_{ij})b(1-e_{ij})$ must be a diagonal matrix. Thus $(b_{jj} - b_{ii})e_{ij} = 0$, that is, $b_{jj} = b_{ii}$ and b is a central element. But this implies that $af(x_1, \dots, x_n)Xf(x_1, \dots, x_n) = 0$ in $M_t(F)$, that is, in any case $af(x_1, \dots, x_n) = 0$. Since the additive subgroup generated by a polynomial is a Lie ideal, it's annihilator is trivial, i.e. $a = 0$ unless when $f(x_1, \dots, x_n)$ is an identity in $M_t(F)$. This contradiction implies $a = 0$ and we are done.

LEMMA 5. *aI=0.*

Proof. Since I does not satisfy the identities $S_4(x_1, x_2, x_3, x_4)x_5$ and $[f(x_1, \dots, x_n), x_{n+1}]x_{n+2}$, there exist $a_1, \dots, a_{n+7} \in I$, such that

$$[f(a_1, \dots, a_n), a_{n+1}]a_{n+2} \neq 0 \text{ and } S_4(a_{n+3}, a_{n+4}, a_{n+5}, a_{n+6})a_{n+7} \neq 0.$$

Write $f(x_1, \dots, x_n) = \sum_i h_i(x_1, \dots, x_{i-1}, x_{i+1})x_i$, where any h_i is multilinear polynomial of degree $n - 1$ and x_i never appears in h_i.

Moreover, since $[f(x_1, \dots, x_n), x_{n+1}]x_{n+2}$ is not an identity for I, there exists at least one h_i such that $[h_i(x_1, \dots, x_{n_1}), x_n]x_{n+1}$ is not an identity for I. Without loss of generality let $h_i = h_n$, so there exist $b_1, \dots, b_{n+1} \in I$ such that $[h_n(b_1, \dots, b_{n-1}), b_n]b_{n+1} \neq 0$.

Here we suppose that $aI \neq 0$, that is there exists an element $a_{n+8} \in I$ such that $aa_{n+8} \neq 0$, and we proceed to derive a contradiction. Since R is a reguler ring, there exists an element $e^2 = e \in IR$ such that $eR = \sum_{i=1}^{n+8} a_iR + \sum_{j=1}^{n+1} b_iR$ and $ea_i = a_i$, $eb_j = b_j$, for all $i = 1, \dots, n + 8$, $j = 1, \dots, n + 1$.

In view of Fact 3, we divide the proof into two cases:

CASE 1.

If d is an inner derivation induced by the element $q \in Q$, then I satisfies the identity $a[q, f(x_1, \dots, x_n)]_2 = 0$. Let $e^2 = e \in I$. Thus for all $x_1, \dots, x_n \in R$,

$$\begin{aligned} 0 &= a[q, f(ex_1, \dots, ex_{n-1}, ex_n(1 - e))]]_2 \\ &= -2af(ex_1, \dots, ex_{n-1}, ex_n(1 - e))qf(ex_1, \dots, ex_{n-1}, ex_n(1 - e)) \end{aligned}$$

and since $\mathrm{char}(R) \neq 2$,

$$af(ex_1, \dots, ex_{n-1}, ex_n(1 - e))qf(ex_1, \dots, ex_{n-1}, ex_n(1 - e)) = 0. \quad (3)$$

By equation (3), we have

$$0 = ah_n(ex_1, \dots, ex_{n-1})ex_n(1 - e)qh_n(ex_1, \dots, ex_{n-1})ex_n(1 - e)).$$

For sake of clearness, let

$$A = ah_n(ex_1, \dots, ex_{n-1}), B = (1 - e)qh_n(ex_1, \dots, ex_{n-1}), c = (1 - e).$$

Suppose $c \neq 0$, because if $e = 1$ then $I = R$.

Suppose $A \neq 0$, $B \neq 0$. Linearizing $AxBxc = 0$ we have $(AxB)yc + Ay(Bxc) = 0$, for all $x, y \in R$, so $AxB = \lambda_x A$, with $\lambda_x \in C$. Pick $x \in R$ such that $AxB \neq 0$, thus $\lambda_x \neq 0$. Then $0 = AxBxc = \lambda_x Axc$, that is $Axc = 0$. Therefore $0 \neq AxB = Axcqh_n(ex_1, \ldots, ex_{n-1}) = 0$, a contradiction. This means that $AxB = 0$, for all $x \in R$, i.e.,

$$ah_n(ex_1e, \ldots, ex_{n-1}e)(ex_ne)(ey(1-e)qe)h_n(ex_1e, \ldots, ex_{n-1}e)$$

for all $x_1, \ldots, x_n, y \in R$. Applying the Lemma 4 to the ring eRe we get the conclusion that either $ae = 0$ or $(1-e)qe = 0$, unless when $h_n(x_1, \ldots, x_n)$ is central in eRe. On the other hand

$$0 \neq [h_n(eb_1, \ldots, eb_{n-1}), eb_n]eb_{n+1} = [h_n(b_1, \ldots, b_{n-1}), b_n]b_{n+1}$$

and also $0 \neq aa_{n+8}$, therefore must be $(1-e)qe = 0$. Let $\varrho = eR$, then $q\varrho \subseteq \varrho$ and $d(\varrho) \subseteq \varrho$. Let $\overline{\varrho} = \varrho/(\varrho \cap l_R(\varrho))$; $\overline{\varrho}$ is a prime C-algebra with a derivation $\overline{d}$ such that $\overline{d}(\overline{x}) = \overline{d(x)}$, for all $x \in \varrho$. Therefore we have $0 = \overline{a}[\overline{d(f(r_1, \ldots, r_n))}, \overline{f(r_1, \ldots, r_n)}]$, for all $\overline{r_1}, \ldots, \overline{r_n} \in \overline{\varrho}$. By [4] either $\overline{a} = 0$ modulo $l_R(\varrho)$, or $\overline{f(x_1, \ldots, x_n)}$ is central in $\overline{\varrho}$ moduolo $l_R(\varrho)$ or $\overline{d} = 0$ modulo $l_R(\varrho)$.

In the first case, we have $a\varrho = 0$, in particular $aea_{n+8} = aa_{n+8} = 0$ and in the second one $[f(ex_1, \ldots, ex_n), ex_{n+1}]ex_{n+2} = 0$, in particular $[f(ea_1, \ldots, ea_n), ea_{n+1}]ea_{n+2} = [f(a_1, \ldots, a_n), a_{n+1}]a_{n+2} = 0$. In any case we have a contradiction.

Finally, we consider the case when $\overline{d} = 0$ modulo $l_R(\varrho)$, that is, $d(\varrho)\varrho = 0$.

Let $x, y \in \varrho$, $r \in R$. Then $[q, xr]y = 0$, whence $qxry - xrqy = 0$, that is $qx = \beta_x x$, with $\beta_x \in C$, and analogously $qy = \beta_y y$, $q(x+y) = \beta_{x+y}(x+y)$. From this, it is easy to see that β_x is independent from the choice of $x \in I$, therefore there exists $\beta \in C$ such that $(q-\beta)\varrho = 0$.

Let $p = q - \beta$, then q and p induce the same inner derivation. This means that $a[p, f(x_1, \ldots, x_n)]_2 = 0$, for any $x_1, \ldots, x_n \in \varrho$. Since $p\varrho = 0$, it follows that $af(x_1, \ldots, x_n)^2 p = 0$. Moreover $R = H$ is a regular ring, hence there exists $g = g^2 \in R$ such that $a_1R + a_2R + a_3R + a_4R + a_5R + a_6R = gR$. Then $g \in IR = I$ and $a_i = ga_i$ for each $i = 1, \ldots, 6$. Consider now the simple Artinian ring eRe and notice that $(eRae)f(er_1e, \ldots, er_ne)^2(epRe) = 0$. Moreover $S_4(eR, eR, eR, eR)eR \neq 0$, because $S_4(a_{n+3}, a_{n+4}, a_{n+5}, a_{n+6})a_{n+7} \neq 0$, and $ae \neq 0$, because $aea_{n+8} = aa_{n+8} \neq 0$.

Let A be the subgroup of eRe generated by the polynomial $f(ex_1e, \ldots, ex_ne)^2$, then $(eRae)x(epRe) = 0$, for all $x \in A$.

By Lemma 3, we have that $(eRae)[ex_1e, ex_2e](epRe) = 0$, for all $x_1, x_2 \in R$. Let $U = [ex_1e, ex_2e](epRe)$, so $(eRae)U = 0$. Since

$(eRae)[Uex_1e, ex_2e](epRe) = 0$, then $eRaex_2eUex_1epRe = 0$. Moreover $ae \neq 0$ implies that either $ep = 0$ or $U = 0$. If this last case occurs, it follows clearly that $ep = 0$, because eRe cannot be commutative. Hence in any case we have $ep = 0$.

Let $r \in R$ and set $g = e + er(1-e)$, so $g \in I$, $g^2 = g$, $ge = e$, and $gR = eR$.

Since $S_4(eR, eR, eR, eR)eR \neq 0$, $[f(eR), eR]eR \neq 0$ and $ae \neq 0$, by calculation it follows that also $S_4(gR, gR, gR, gR)gR \neq 0$ and $ag \neq 0$.

Moreover $(gRag)f(gx_1g, \dots, gx_ng)^2(gpRg) = 0$ and, following the same above argument, we have $gp = 0$, that is $0 = (e + er(1-e))p = erp$. By the arbitrarity of $r \in R$, and $e \neq 0$, we get $p = 0$ that is $q = \beta \in C$, a contradiction.

<u>CASE 2.</u>

Let now d be an outer derivation. By our assumption we have that, for all $r_1, \dots, r_n \in R$, $a[d(f(er_1, \dots, er_n)), f(er_1, \dots, er_n)] = 0$ and so

$$0 = a\Bigg[f^d(er_1, \dots, er_n) + \sum_i f(er_1, \dots, d(e)r_i + ed(r_i), \dots, er_n), f(er_1, \dots, er_n)\Bigg].$$

By Fact 3, it follows that, for all $s_i \in R$,

$$a\left[\sum_i f(er_1, \dots, d(e)r_i + es_i, \dots, er_n), f(er_1, \dots, er_n)\right] = 0.$$

In particular

$$a\Big[f(er_1, \dots, es_i, \dots, er_n), f(er_1, \dots, er_n)\Big] = 0 \qquad (4)$$

for all $i = 1, \dots, n$.

As above, let $f(x_1, \dots, x_n) = \sum_i h_i(x_1, \dots, x_{i-1}, x_{i+1})x_i$, where any h_i is multilinear polynomial of degree $n-1$ and x_i never appears in h_i. By equation (4), for all i, R satisfies

$$a\Big[f(ex_1, \dots, ey_i(1-e), \dots, ex_n), f(ex_1, \dots, ex_i, \dots, ex_n)\Big] = 0$$

that is,

$$af(ex_1, \dots, ex_i, \dots, ex_n)h_i(ex_1, \dots, ex_{i-1}, ex_{i+1}, \dots, ex_n)ex_i(1-e) = 0.$$

This implies that R satisfies

$$af(ex_1,\dots,ex_i,\dots,ex_n)\sum_i h_i(ex_1,\dots,ex_{i-1},ex_{i+1},\dots,ex_n)ex_i(1-e)$$
$$= af(ex_1,\dots,ex_n)^2(1-e).$$

Let G the additive subgroup generated by the polynomial $f(eR)^2$. Since $f(x_1,\dots,x_n)^2$ cannot be central in eRe, by [3] either there exists a right ideal $I_0 \subseteq eR$ such that $[I_0, eR] \subseteq G$ (in particular if I satisfies some polynomial identity I_0 coincides with I).

Therefore $a[I_0, eR](1-e) = 0$, that is, $0 = a[I_0, eRe](1-e) = -aeReI_0(1-e)$, and by primeness of R we get $ae = 0$. But this contradicts with $aa_{n+8} = aea_{n+8} \neq 0$.

Following the way of previous Lemmas, to complete the proof of the Theorem it is enough to prove:

LEMMA 6. $ad(I) = 0$.

Proof. Since I does not satisfy the identities $S_4(x_1,x_2,x_3,x_4)x_4$ and $[f(x_1,\dots,x_n),x_{n+1}]x_{n+2}$, there exist $a_1,\dots,a_{n+7} \in I$, such that

$$[f(a_1,\dots,a_n),a_{n+1}]a_{n+2} \neq 0 \text{ and } S_4(a_{n+3},a_{n+4},a_{n+5},a_{n+6})a_{n+7} \neq 0.$$

In the following we rewrite $f(x_1,\dots,x_n) = \sum_i x_i h_i(x_1,\dots,x_{i-1},x_{i+1})$, where any h_i is multilinear polynomial of degree $n-1$ and x_i never appears in h_i.

Here we suppose that $ad(I) \neq 0$, that is there exists an element $a_{n+8} \in I$ such that $ad(a_{n+8}) \neq 0$. Since R is a reguler ring, there exists an element $e^2 = e \in IR$ such that $eR = \sum_{i=1}^{n+8} a_iR$ and $ea_i = a_i$, for all $i = 1,\dots,n+8$.

Since $ad(a_{n+8}) \neq 0$, it follows $ad(e) \neq 0$.

Since $aI = 0$, we have that R satisfies the non-trivial generalized polynomial identity

$$\begin{aligned}
&ad(f(ex_1,\dots,ex_n))f(ex_1,\dots,ex_n)\\
&= a\sum_i f(ex_1,\dots,d(e)x_i+ed(x_i),\dots,ex_n)f(ex_1,\dots,ex_n)\\
&= a\sum_i d(e)x_ih_i(ex_1,\dots,ex_{i-1},ex_{i+1},\dots,ex_n)f(ex_1,\dots,ex_n)\\
&= ad(e)\sum_i x_ih_i(ex_1,\dots,ex_{i-1},ex_{i+1},\dots,ex_n)f(ex_1,\dots,ex_n)\\
&= ad(e)f(ex_1,\dots,ex_n)^2.
\end{aligned}$$

Let $\varrho = eI$ and $\overline{\varrho} = \varrho/(\varrho \cap l_R(\varrho))$, with $l_R(\varrho)$ the left annihilator of ϱ in R. Therefore $\overline{\varrho}$ satisfies the generalized polynomial identity $\overline{ad(e)f(x_1, \ldots, x_n)^2}$.

By Lemma 3, since $f(x_1, \ldots, x_n)$ cannot be central in eRe, we have that $\overline{ad(e)}[x, y]$ is a generalized polynomial identity for $\overline{\varrho}$ and it follows easily that $ad(e)\varrho = 0$, unless $\overline{\varrho}$ is commutative. Therefore we have that either $ad(e)eI = 0$ or $[eI, eI]eI = 0$ and a fortiori $S_4(eI, eI, eI, eI)eI = 0$.

On the other hand we have that

$$\begin{aligned} &S_4(ea_{n+3}, ea_{n+4}, ea_{n+5}, ea_{n+6})ea_{n+7} \\ &= S_4(a_{n+3}, a_{n+4}, a_{n+5}, a_{n+6})a_{n+7} \neq 0 \end{aligned}$$

and also $ad(e)ea_{n+8} = ad(e^2a_{n+8}) = ad(ea_{n+8}) = ad(a_{n+8}) \neq 0$. This contradiction completes the proof of the theorem.

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