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BRILL-NOETHER DIVISORS ON THE MODULI SPACE OF CURVES AND APPLICATIONS

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ABSTRACT. Here we generalize previous work by Eisenbud-Harris and Farkas in order to prove that certain Brill-Noether divisors on the moduli space of curves have distinct supports. From this fact we deduce non-trivial regularity results for a higher codimensional Brill-Noether locus and for the general $\frac{g+1}{2}$-gonal curve of odd genus g.

1. Introduction

Let $\mathcal{M}_g$ denote as usual the moduli space of smooth algebraic curves of genus g. By classical Brill-Noether theory (see [1] and references therein), if C is a general element of $\mathcal{M}_g$ and the Brill-Noether number $\rho(g,r,d) = g - (r+1)(g-d+r)$ is negative, then there are no linear series g^r_d of dimension r and degree d on C, hence the locus $\mathcal{M}^r_d \subset \mathcal{M}_g$ which corresponds to curves carrying a g^r_d is a proper subvariety of $\mathcal{M}_g$. In particular, for $\rho(g,r,d) = -1$ we obtain the so-called Brill-Noether divisor D^r_d, which seems to play a special role in the birational geometry of $\mathcal{M}_g$. The class of the closure of D^r_d in the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$ of $\mathcal{M}_g$ was computed by Eisenbud, Harris, and Mumford in [12], [9], and as a consequence they were able to determine the Kodaira dimension of $\mathcal{M}_g$ for $g \geq 24$. In the case $g = 23$, the canonical divisor $K_{\overline{\mathcal{M}}_g}$ turns out to be linearly equivalent to the effective sum of a Brill-Noether divisor and some boundary divisors; hence in order to prove that $\kappa(\overline{\mathcal{M}}_{23}) \geq 1$ it is sufficient to show that D^1_{12} and D^2_{17} have distinct supports on $\overline{\mathcal{M}}_{23}$. This fact is indeed established in [9], Proposition 3, and it has been recently refined by Farkas, who in [11] proves that $\kappa(\overline{\mathcal{M}}_{23}) \geq 2$. The present work is strongly inspired by Eisenbud-Harris and Farkas computations, which in turn rely on the

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theory of limit linear series developed in [7]. Our main result is the following:

THEOREM 1. *Let g, r, s, d, e be positive integers such that $\rho(g,r,d) = \rho(g,s,e) = -1$. Assume that*
(i) *either $r = 1$, $s \geq 2$, and $g > 2s + 1$;*
(ii) *or $r = 2$, $s \geq 3$, and g is odd with $g > \frac{3s^2-3s-3}{s-2}$.*
Then $\mathcal{M}^r_d \neq \mathcal{M}^s_e$.

Besides being rather interesting in its own, Theorem 1 allows us to draw a couple of non-trivial geometric consequences: namely, Corollary 1 exhibits a locus of curves carrying more than one special linear series which has the expected codimension in $\mathcal{M}_g$, while Corollary 2 estimates the minimal degree of a projective model of the general $\frac{g+1}{2}$-gonal curve of odd genus g.

As suggested by the referee, by applying suitable ramification sequences one could address the following nice question:

QUESTION 1. *Fix positive integers r, s, d, e, with $s > r$. Is there an integer $f(r,s,d,e)$ such that neither $\mathcal{M}^r_d \subseteq \mathcal{M}^s_e$ nor $\mathcal{M}^s_e \subseteq \mathcal{M}^r_d$ for all genera $g \geq f(r,s,d,e)$?*

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2. The results

For the benefit of the reader, first we recall some standard notation in the theory of limit linear series. Let C be a smooth curve of genus g, L a g^r_d on C and $p \in C$ a point. The *vanishing sequence* of L at p

$$0 \leq a^L_0(p) < \ldots < a^L_r(p) \leq d$$

is defined by ordering the finite set $\{ord_p(\sigma)\}$, where σ is a section of L. The *ramification sequence* of L at p

$$0 \leq \alpha^L_0(p) \leq \ldots \leq \alpha^L_r(p) \leq d - r$$

is defined as $\alpha^L_i(p) = a^L_i(p) - i$. The *Brill-Noether number* of L at the points $p_1, \ldots, p_n$ of C is by definition

$$\rho(g, L, p_1, \ldots, p_n) = \rho(g,r,d) - \sum_{i=1}^{n} \sum_{j=0}^{r} \alpha^L_j(p_i).$$

If C is a curve of compact type, a *limit* g^r_d on C is a collection of ordinary linear series:

$$L = \{L_F \in G^r_d(F) : F \subseteq C \text{ is a component}\}$$

satisfying the following compatibility condition: if F, G are components of C and $F \cap G = \{p\}$, then

$$a_i^{L_F}(p) + a_{r-i}^{L_G}(p) \geq d$$

for $i = 0, \dots, r$. The limit g^r_d is called *refined* if equality holds everywhere, while the ordinary linear series L_F is called the *F-aspect* of L.

Proof of Theorem 1. In case (i), fix $t := d = \frac{g+1}{2}$, while in case (ii) let $g+1 = 6m$ and fix $t := 2m+1$. Let $C = F_1 \cup E \cup F_2$, where (F_1, p_1) and (F_2, p_2) are general pointed curves of genus $g(F_1) = g(F_2) = \frac{g-1}{2}$, $F_1 \cap F_2 = \emptyset$, and E is a smooth elliptic curve with $F_1 \cap E = \{p_1\}$, $F_2 \cap E = \{p_2\}$, and p_1, p_2 differ by t-torsion in $\mathrm{Pic}(E)$. We are going to construct L, a limit g^r_d on C, aspect by aspect.

In case (i), we take $L_{F_i} := |dp_i|$ for $i = 1, 2$, and as L_E the pencil spanned by $dp_1 \sim dp_2$ on E. Notice that $a^{L_{F_i}}(p_i) = (0, d)$, $a^{L_E}(p_i) = (0, d)$, hence the compatibility conditions are satisfied, while the smoothability is automatic from [7], Proposition 3.1.

In case (ii), we take $L_{F_i} \in G^2_d(F_i)$ such that $a^{L_{F_i}}(p_i) = (m, 2m+1, 3m+1)$. Since $\sum_{i=0}^{2}(\alpha_i + \frac{g-1}{2} - d + 2)_+ = \frac{g-1}{2}$, where $x_+ := \max\{x, 0\}$, the existence of L_{F_i} follows from [9], Proposition 1.2. Next, we take $L_E \subseteq |\mathcal{O}_E(D)|$ with $D = mp_1 + (3m+1)p_2$ and vanishing sequence $(m, 2m, 3m+1)$ at p_i. This time, the existence of L_E is ensured by [8], Proposition 5.2. These choices make L to be a refined limit g^r_d on C with $\rho(L_{F_i}, p_i) = 0$, $\rho(L_E, p_1, p_2) = -1$. Finally, we have to prove that L is smoothable; by [7], Theorem 3.4, it is sufficient to check that L is dimensionally proper. Let $\pi_i : \Gamma_i \to \Delta_i$, $p_i : \Delta_i \to \Gamma_i$ be the versal deformation of $[(C_i, p_i)] \in \mathscr{C}_{\frac{g-1}{2}}$. Since being general is an open condition and $\rho(L_{F_i}, p_i) = 0$, we have

$$\begin{aligned}\dim \mathscr{G}^2_d(\Gamma_i/\Delta_i, (p_i, (m, 2m, 3m-1))) &= \dim \Delta_i + \rho(L_{F_i}, p_i) \\ &= 3\left(\frac{g-1}{2}\right) - 2.\end{aligned}$$

Similarly, let $\pi : \Gamma \to \Delta$, $p : \Delta \to \Gamma$ be the versal deformation of $[(E, p_1, p_2)]$. From [8], Proposition 5.2, and $\rho(L_E, p_1, p_2) = -1$ it follows that (for further details, see [11], proof of Theorem 2, Step 2)

$$\dim \mathscr{G}^2_d(\Gamma/\Delta, (p_i, (m, 2m-1, 3m-1))) = \dim \Delta + \rho(L_E, p_1, p_2) = 2.$$

Hence in both cases (i) and (ii) L exists and it is smoothable.

On the other hand, we claim that C has no g^s_e. Indeed, suppose that M were a g^s_e. Up to exchange the g^s_e with its residual linear series $K - g^s_e$, we may assume $e \leq g-1$. Since (F_i, p_i) are general, we have $\rho(M_{F_i}, p_i) \geq 0$. Since $\rho(g,s,e) = -1$, by additivity ([7], Proposition 4.6) we obtain $\rho(M_E, p_1, p_2) < 0$. For dimensional reasons, there must be sections σ_i such that $\mathrm{div}(\sigma_i) \geq a_i^{M_E}(p_1)p_1 + a_{s-i}^{M_E}(p_2)p_2$, hence $a_i^{M_E}(p_1) + a_{s-i}^{M_E}(p_2) \leq e$. By adding up all these inequalities, we get $\rho(M_E, p_1, p_2) \geq -s$. Furthermore, $\rho(M_E, p_1, p_2) \leq -1$ precisely when for at least two values $i < j$ we have equalities $a_i^{M_E}(p_1) + a_{s-i}^{M_E}(p_2) = a_j^{M_E}(p_1) + a_{s-j}^{M_E}(p_2) = e$, which means that there are sections σ_i, σ_j such that $\mathrm{div}(\sigma_i) = a_i^{M_E}(p_1)p_1 + a_{s-i}^{M_E}(p_2)p_2$, $\mathrm{div}(\sigma_j) = a_j^{M_E}(p_1)p_1 + a_{s-j}^{M_E}(p_2)p_2$. By setting $a_{i,j} := a_j^{M_E}(p_1) - a_i^{M_E}(p_1)$ and by subtracting $\mathrm{div}(\sigma_i)$ from $\mathrm{div}(\sigma_j)$ we obtain $a_{ij}(p_1 - p_2) = 0$. Since $p_1 - p_2$ has exact order t in $\mathrm{Pic}^0(E)$ and $0 < a_{ij} \leq e \leq g-1$, it follows that either $a_{ij} = t$ or $a_{ij} = 2t$. Thus we may write

$$\mathrm{div}(\sigma_i) = D_{ij} + a_{ij}p_2, \tag{1}$$

$$\mathrm{div}(\sigma_j) = D_{ij} + a_{ij}p_1 \tag{2}$$

for some effective divisor D_{ij} of degree $e - a_{ij}$ supported on p_1 and p_2. If $\rho(M_E, p_1, p_2) \leq -2$, then we have at least another equality $a_k^{M_E}(p_1) + a_{s-k}^{M_E}(p_2) = e$. If $j < k$, define exactly as above the integers a_{ij}, a_{ik}, a_{jk} and write

$$\mathrm{div}(\sigma_i) = D_{ik} + a_{ik}p_2, \tag{3}$$

$$\mathrm{div}(\sigma_j) = D_{jk} + a_{jk}p_2, \tag{4}$$

$$\mathrm{div}(\sigma_k) = D_{ik} + a_{ik}p_1, \tag{5}$$

$$\mathrm{div}(\sigma_j) = D_{jk} + a_{jk}p_1. \tag{6}$$

Notice that, since $a_{ij}, a_{ik}, a_{jk} \in \{t, 2t\}$, at least two out of them must be equal. If $a_{ij} = a_{ik}$ then from (1) and (3) it follows that $D_{ij} = D_{ik}$, hence by (5) and (2) we get the contradiction $\mathrm{div}(\sigma_k) = \mathrm{div}(\sigma_j)$. Next, if $a_{ij} = a_{jk}$ then from (2) and (4) it follows that $D_{jk} - D_{ij} = a_{ij}(p_1 - p_2) = 0$, hence $D_{ij} = D_{jk}$ and by (6) and (2) we get the contradiction $\mathrm{div}(\sigma_k) = \mathrm{div}(\sigma_j)$. Finally, if $a_{ik} = a_{jk}$ then from (5) and (6) it follows that $D_{ik} = D_{jk}$, hence by (3) and (4) we get the contradiction $\mathrm{div}(\sigma_i) = \mathrm{div}(\sigma_j)$. If $k > j$ a completely analogous argument applies. Hence we may assume $\rho(M_E, p_1, p_2) = -1$ and $\rho(M_{F_i}, p_i) = 0$. It follows that $\sum_j (\alpha_j^{M_{F_i}}(p_i) + \frac{g-1}{2} - e + s)_+ \leq \frac{g-1}{2}$ and $\sum_j (\alpha_j^{M_{F_i}}(p_i) + \frac{g-1}{2} - e + s) =$

$\frac{g-1}{2}$, from which we deduce $\alpha_j^{M_{F_i}}(p_i) \geq -\frac{g-1}{2} + e - s$ for every i, j. By the compatibility conditions, $a_j^{M_E}(p_i) \leq \frac{g-1}{2} + j$ for each i, j, so $a_s^{M_E}(p_i) \leq \frac{g-1}{2} + s$. On the other hand, in both cases (i) and (ii) our numerical assumptions imply $\frac{g-1}{2} + s < t + \frac{e-t}{2}$: indeed, just notice that $g - e + s = \frac{g+1}{s+1}$ since $\rho(g, s, e) = -1$, recall the expressions of t in terms of g and isolate g with elementary algebraic manipulations. It follows that $a_s^{M_E}(p_i) < a_{ij} + \frac{e-a_{ij}}{2}$, in contradiction with (1) and (2), so M cannot exist. □

Our first application of Theorem 1 concerns higher codimensional Brill-Noether loci, whose geometry is in general rather messy (see for instance [10], section 2; for analogous partial results, see also [6] and [13]).

COROLLARY 1. *Under the assumptions of Theorem 1, $\mathcal{M}_d^r \cap \mathcal{M}_e^s$ has a component of codimension 2 in $\mathcal{M}_g$.*

Proof. Let D_d^r (resp. D_e^s) the divisorial component of $\mathcal{M}_d^r$ (resp. $\mathcal{M}_e^s$), which by [10], Theorem (1.1) (ii) is unique. The proof of Theorem 1 shows that $D_d^r \neq D_e^s$. Indeed, the curve $C = F_1 \cup E \cup F_2$ carries a finite number of g_d^r's (recall that the curves F_i's are general and use [8], Proposition 5.2), hence its smoothing lies in a component of codimension $\rho(g, r, d) = -1$. Since $\mathcal{M}_g$ has only finite quotient singularities, from $D_d^r \neq D_e^s$ it follows that either $D_d^r \cap D_e^s$ has pure codimension 2 or $D_d^r \cap D_e^s = \emptyset$. In order to exclude the last possibility, consider the closure E_d^r (resp. E_e^s) of D_d^r (resp. D_e^s) in the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$ of $\mathcal{M}_g$. The class of such divisors was computed in [9], Theorem 1 (see also the first few lines of p. 219 in [2] for a related remark). Hence, by applying Cornalba-Harris criterion for ampleness (see [3] Theorem (1.3)), we deduce that $\text{supp}(E_d^r) \cup \partial\overline{\mathcal{M}}_g$ and $\text{supp}(E_e^s) \cup \partial\overline{\mathcal{M}}_g$ support an ample divisor on $\overline{\mathcal{M}}_g$. Therefore if $D_d^r \cap D_e^s = \emptyset$ then $\mathcal{M}_g$ would be the union of two affine open subsets, which is definitely not the case for $g \geq 4$ (as it follows from the well-known cohomological properties of $\mathcal{M}_g$). □

Next we turn to the geometry of the general k-gonal curve of genus g, in the special case in which $\rho(g, 1, k) = -1$. As in [5], Definition 2.1, let $s(r) = s(r, C)$ be the minimal degree of a complete, base point free and simple linear series of dimension $r \geq 2$ on a curve C (s(r) is the minimal degree of a birational model of C in $\mathbb{P}^r$).

COROLLARY 2. *If $r \geq 2$ and C is the general $\frac{g+1}{2}$-gonal curve of odd genus $g > 2r + 1$, then $s(r)$ is the minimal positive integer such that $\rho(g, r, s(r)) \geq 0$.*

Proof. Consider the curve C as a general element of $\mathcal{M}^1_{\frac{g+1}{2}}$. We claim that C carries no g^r_d with $\rho(g,r,d) < 0$. Indeed, if $\rho < -1$ the claim follows from [10], Theorem (1.1) (i), while for $\rho = -1$ it is a direct consequence of Theorem 1 (i). On the other hand, if $\rho(g,r,d) \geq 0$ then [4], Theorem 1, ensures that C carries a base point free and simple g^r_d, hence the proof is over. □

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