GRAPHS WITH ONE HOLE AND COMPETITION NUMBER ONE

SUH-RYUNG KIM

ABSTRACT. Let D be an acyclic digraph. The competition graph of D has the same set of vertices as D and an edge between vertices u and v if and only if there is a vertex x in D such that (u, x) and (v, x) are arcs of D. The competition number of a graph G, denoted by $k(G)$, is the smallest number k such that G together with k isolated vertices is the competition graph of an acyclic digraph. It is known to be difficult to compute the competition number of a graph in general. Even characterizing the graphs with competition number one looks hard. In this paper, we continue the work done by Cho and Kim[3] to characterize the graphs with one hole and competition number one. We give a sufficient condition for a graph with one hole to have competition number one. This generates a huge class of graphs with one hole and competition number one. Then we completely characterize the graphs with one hole and competition number one that do not have a vertex adjacent to all the vertices of the hole. Also we show that deleting pendant vertices from a connected graph does not change the competition number of the original graph as long as the resulting graph is not trivial, and this allows us to construct infinitely many graph having the same competition number. Finally we pose an interesting open problem.

1. Introduction

Suppose D is an acyclic digraph (for all undefined graph-theoretical terms, see [1] and [17]). The *competition graph* G of D, denoted by $C(D)$, has the same set of vertices as D and an edge between vertices u and v if and only if there is a vertex x in D such that (u, x) and (v, x) are arcs of D. Roberts[16] observed that if G is any graph, G together with

sufficiently many isolated vertices is the competition graph of an acyclic digraph. Then he defined the *competition number* $k(G)$ of a graph G to be the smallest number k such that G together with k isolated vertices added is the competition graph of an acyclic digraph. We shall use the notation I_r for the graph consisting of r vertices and no edges, and $G \cup I_r$ for the graph consisting of the disjoint union of G and I_r.

The notion of competition graph was introduced by Cohen[5] as a means of determining the smallest dimension of ecological phase space. Since then, various variations have been defined and studied by many authors (see, for example, [2, 4, 9, 11, 12, 13, 18, 19]). Besides an application to ecology, the concept of competition graph can be applied to the study of communication over noisy channel (see Roberts[16] and Shannon[20]) and to problem of assigning channels to radio or television transmitters (see Cozzens and Roberts[6], Hale[8], or Opsut and Roberts[15]).

Roberts[16] observed that characterization of competition graph is equivalent to computation of competition number. It does not seem to be easy in general to compute $k(G)$ for all graphs G, as Opsut[14] showed that the computation of the competition number of a graph is an NP-complete problem (see [11, 12] for graphs whose competition numbers are known). It has been one of important research problems in the study of competition graphs to characterize a graph by its competition number.

We call a cycle of a graph G a *chordless cycle* of G if it is an induced subgraph of G. A chordless cycle of length at least 4 of a graph is called a *hole* of the graph and a graph without holes is called a *chordal graph*. Since chordal graphs have several nice characterizations, many researchers start with chordal graphs when investing a new graph-theoretical problem. Roberts[16] showed that the competition number of a chordal graph is at most one. On the other hand, given a positive integer $n \geq 4$, the maximum competition number of a graph with n vertices is achieved uniquely by the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$, which has a lot of holes (see [10]). These observations led Cho and Kim[3] to ask: Does the competition number of a connected graph increase with the number of hole it has? In order to answer this question, they studied the family $\mathcal{F}$ of connected graphs with exactly one hole. They showed that the competition number of a graph in $\mathcal{F}$ is at most two. Then they tried to characterize the graphs in $\mathcal{F}$ that have competition number one and gave a sufficient condition for the graphs in $\mathcal{F}$ to have competition number one.

In fact, the competition number of a connected graph can be one even if it has many holes.

PROPOSITION 1. *Given a positive integer n, there is a connected graph G with $k(G) = 1$ and n holes*

Proof. Let G be the graph consisting of n edge-disjoint holes $C_1, \ldots, C_n$ of length 4 having a common vertex together, and a clique of size $n + 2$ which has exactly one vertex w in common with C_1 and no vertices in common with $C_2, \ldots, C_n$ (see Figure 1 for illustration). Let G_1 be the
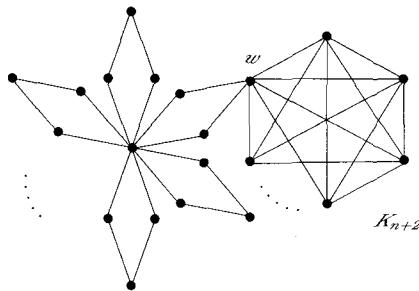

FIGURE 1. A graph G with n holes and $k(G) = 1$

subgraph of G induced by $V(C_1) \cup \cdots \cup V(C_n)$. Since G_1 is triangle-free, its competition number equals $|E(G)| - |V(G)| + 2$ (see Roberts[16]) and so $k(G_1) = 4n - (3n + 1) + 2 = n + 1$. Let D_1 be an acyclic digraph whose competition graph is $G_1 \cup \{i_1, \ldots, i_{n+1}\}$ where i_j $(j = 1, \ldots, n+1)$ are isolated vertices of $C(D_1)$. Let $V(K_{n+2}) = \{v_1, v_2, \ldots, v_{n+2}\}$ and $w = v_{n+2}$. Now we construct a digraph D from D_1 as follows: Let

$$V(D) = V(D_1) \cup V(K_{n+2}) \cup \{a\} - \{i_1, \ldots, i_{n+1}\}.$$

Then we obtain $A(D)$ from $A(D_1)$ in the following way. We replace arc (x, i_j) of D_1 by arc (x, v_j) for $j = 1, \ldots, n + 1$. Then add arcs (v_j, a) for $j = 1, \ldots, n + 21$. It can easily be checked that D is acyclic and $C(D) = G \cup \{a\}$.   □

Proposition 1 implies that characterizing graphs G with $k(G) = 1$ in general is hard. For this reason, Cho and Kim[3] started with $\mathcal{F}$ to characterize the competition graph with competition number one. In this paper, we continue their work to characterize graphs in $\mathcal{F}$ that have competition number one.

2. Main results

We first present a theorem that shows that deleting a vertex of degree one from a graph does not change its competition number as long as the resulting graph is not a trivial graph. Let us call a vertex of degree one a *pendant vertex*.

THEOREM 1. *If G is a connected graph with at least one pendant vertex and G^* is obtained from G by deleting some of the pendant vertices of G, then $k(G) = k(G^*)$ if and only if G^* is not a trivial graph.*

Proof. Suppose that G^* is trivial. Then $k(G^*) = 0$. Since G is connected and has at least one pendant vertex, G has at least one edge, $k(G) \geq 1$, and equality does not hold.

Now we suppose that G^* is not trivial. Then G^* is a connected graph with at least one edge. We take a pendant vertex v and delete it from G. We denote the resulting graph by G'. It is sufficient to show that $k(G') = k(G)$. We first claim that $k(G') \leq k(G)$. Let D be an acyclic digraph whose competition graph is $G \cup I_k$, where $k = k(G)$. Let u be the vertex adjacent to v in G. By the definition of competition graph, there exists a vertex w in $V(D)$ such that (u, w) and (v, w) both are in $A(D)$. Since v is a pendant vertex, vertices u and v are the only in-neighbors of w. Now we define a digraph D' as follows:

$$V(D') = V(D) - \{v\}$$

and

$$A(D') = A(D) - \{(u, w), (v, w)\}$$
$$\cup \{(x, w) \mid (x, v) \in A(D)\} - \{(x, v) \mid (x, v) \in A(D)\}.$$

Then it can be easily checked that D' is still acyclic and $C(D') = G' \cup I_k$. Hence $k(G') \leq k(G)$.

Let D^* be an acyclic digraph whose competition graph is $G' \cup I_{k'}$ where $k' = k(G')$. Since v is a pendant vertex, G' is still connected. By the hypothesis, G' has at least one edge and therefore $k(G') \geq 1$. Thus there exists a vertex a in $V(D^*) - V(G')$. Now define a digraph D as follows:

$$V(D) = V(D^*) \cup \{v\}$$

and

$$A(D) = A(D^*) \cup \{(x, v) \mid (x, a) \in A(D^*)\}$$
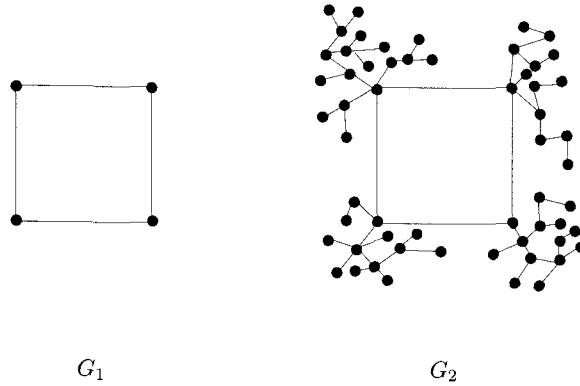$$- \{(x, a) \mid (x, a) \in A(D^*)\} \cup \{(u, a), (v, a)\}.$$

G_1 G_2

FIGURE 2. $k(G_1) = k(G_2) = 2$

Then it is obvious that D is acyclic, $V(C(D)) = V(G) \cup I_{k'}$, and $E(C(D)) = E(G') \cup \{uv\}$. Since u is the only vertex that is adjacent to v in G, $E(G') \cup \{uv\} = E(G)$. Thus $C(D) = G \cup I_{k'}$. Hence $k(G) \le k(G')$ and the theorem follows. $\square$

It follows from Theorem 1 that the graph G_1 given in Figure 2 has the same competition number as G_2.

A graph G with exactly one hole C has the following well-known properties: If v is a vertex adjacent to two non-adjacent vertices of C, then v is adjacent to all the vertices of C and if X is the set of vertices adjacent to all the vertices of C, then X is either a clique or the empty set. In the rest of this section, unless it is stated otherwise, G means a graph with one hole $C = v_0 v_1 \cdots v_{l-1} v_0$ ($l \ge 4$) and X means either the clique of G whose vertices are adjacent to all $v_0, v_1, \ldots, v_{l-1}$ or the empty set if there are no such vertices. In addition, all subscripts are reduced modulo l. By Theorem 1, without loss of generality, we may assume that G does not have a pendant vertex. We call a walk (resp. path) W a *hole-avoiding walk* (resp. *hole-avoiding path*) if none of the internal vertices of W are on C or on X.

The following are some results on the structure of a graph with exactly one hole obtained by Cho and Kim[3] that shall be used to prove our main theorem.

LEMMA 1 (Cho and Kim[3]). *If vertex w is in $V(G) - X - V(C)$ and there is a hole-avoiding path from w to v_i, then there is no hole-avoiding path from w to v_j in G for any j satisfying $|i - j| \ge 2$.*

LEMMA 2 (Cho and Kim[3]). *If there exists a vertex w in $V(G) - X - V(C)$ that is connected to both v_i and v_{i+1} by hole-avoiding paths, then $X \cup \{v_i, v_{i+1}\}$ is a vertex cut. Moreover, w belongs to a component of $G - \{v_i, v_{i+1}\} - X$ different from the one including $V(C) - \{v_i, v_{i+1}\}$.*

Given a vertex v of a graph G, we will denote by $N_G(v)$ the *open neighborhood* of v in G, the set containing v and all vertices adjacent to v in G. In addition, in this paper, we will call a clique of size 3 a *triangle*.

LEMMA 3 (Cho and Kim[3]). *If K is clique of a chordal graph G, then there exists an acyclic digraph D such that $C(D) = G \cup I_1$, and the vertices of K have only outgoing arcs in D.*

LEMMA 4 (Cho and Kim[3]). *Suppose that G is a connected chordal graph with at least one triangle. We take a vertex v of G. Then there is an acyclic digraph D such that $C(D) = G \cup I_1$ and at least three vertices of G including v have only outgoing arcs in D.*

Cho and Kim[3] showed that the competition number of a graph with exactly one hole is at most two.

THEOREM 2 (Cho and Kim[3]). *If G has exactly one hole, then $k(G) \leq 2$.*

Then they identified a large family of graphs with exactly one hole whose competition number is one. An illustration for Theorem 3 is given in Figure 3.
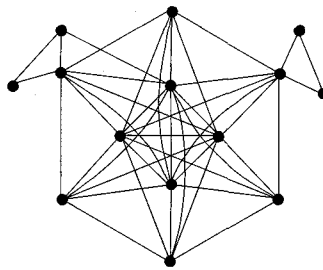

FIGURE 3. A graph satisfying the condition of Theorem 3

THEOREM 3 (Cho and Kim[3]). *Suppose that G is a graph with exactly one hole C. Let X be the set of all the vertices that are adjacent to all the vertices on the hole. Suppose that there is a cut vertex v on*
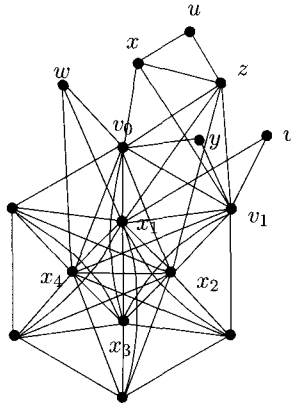
FIGURE 4. A graph not satisfying the condition of Theorem 3 but satisfying the condition of Theorem 4. In this figure, $N = \{x, y, z\}$, Q_1 and Q_2 are the subgraphs induced by $\{x, z, u\}$ and $\{y\}$, respectively, $N^*(x) = N^*(y) = \emptyset$, and $N^*(z) = \{x_1, x_2\}$.

a triangle T such that the component of $G - v$ containing the other two vertices of T does not contain $C - v$. Then $k(G) \leq 1$ and the equality holds if and only if G does not have an isolated vertex.

However, the graph given in Figure 4 does not satisfy the condition of Theorem 3 and yet has competition number one. One of its characteristics is that it has a vertex adjacent to two consecutive vertices of the hole and this observation leads us to construct another huge family of graphs with one hole and competition number one.

Suppose that there exists a vertex in $V(G) - X$ adjacent to two adjacent vertices on C. Without loss of generality, we may assume that there exists a vertex in $V(G) - X$ adjacent to both v_0 and v_1. Let N be the set of the vertices in $V(G) - X$ that are adjacent to both v_0 and v_1. By the assumption, $N \neq \emptyset$. Now by Lemma 2, $X \cup \{v_0, v_1\}$ is a vertex cut of G and no vertex in N belongs to the same component as the one that includes $V(C) - \{v_0, v_1\}$. Let $Q_1, \ldots, Q_s$ be the components of $G - X - \{v_0, v_1\}$ that include at least one vertex in N. For each vertex u in $V(G) - X - V(C)$, we denote by $N^*(u)$ the set of the vertices in X that are adjacent to u, that is, $X \cap N_G(u)$. See Figure 4 for an illustration for these definitions. Now we prove the following lemmas.

LEMMA 5. *If u and v in Q_i are adjacent for some $i \in \{1, 2, \ldots, s\}$, then $N^*(u) \subseteq N^*(v)$ or $N^*(u) \supseteq N^*(v)$.*

Proof. Suppose that there exist vertices x in $N^*(u) - N^*(v)$ and y in $N^*(v) - N^*(u)$. Since X is a clique, x is adjacent to y. Then u, v, y, x, u is a hole of length 4 and we reach a contradiction. Hence the lemma follows. □

LEMMA 6. *If P is a shortest hole-avoiding path from u to v for some u, $v \in N$, then the internal vertices of P also belong to N.*

Proof. If there is an internal vertex w on P that is not adjacent to v_0, then we let w^* be the first vertex on (w, u)-section of P^{-1} that is adjacent to v_0 and w^{**} be the first vertex on (w, v)-section of P that is adjacent to v_0. We denote the (w^*, w^{**})-section of P by P'. Then $v_0 P' v_0$ is a hole of length at least 4 and we reach a contradiction. Similarly we can show that any internal vertex of P is adjacent to v_1. Thus, all the vertices on P belong to N and therefore the lemma follows. □

LEMMA 7. *If a path $u = u_0 u_1 \cdots u_q = v$, denoted by P^*, is a shortest hole-avoiding path from u to v for some u, $v \in N$, then for some $q^* \in \{0, \ldots, q\}$, $N^*(u_0) \cup \cdots \cup N^*(u_q) = N^*(u_{q^*})$.*

Proof. By Lemma 6, u_i is in N for each $i = 0, \ldots, q$. By Lemma 5, $N^*(u_0) \subseteq N^*(u_1)$ or $N^*(u_0) \supseteq N^*(u_1)$ and therefore $N^*(u_0) \cup N^*(u_1) = N^*(u_0)$ or $N^*(u_0) \cup N^*(u_1) = N^*(u_1)$. Now suppose that the lemma is false and u_r, $r < q$, is the last vertex on P^* such that $N^*(u_0) \cup \cdots \cup N^*(u_r) = N^*(u_{r^*})$ for some $r^* \leq r$. If $r^* = r$, then, by Lemma 5, $N^*(u_{r^*}) \subseteq N^*(u_{r+1})$ or $N^*(u_{r^*}) \supseteq N^*(u_{r+1})$ and in any case, we reach a contradiction to our supposition. Thus $r^* < r$. If there exists a vertex x in $N^*(u_{r^*}) - N^*(u_{r+1})$ and y in $N^*(u_{r+1}) - N^*(u_{r^*})$, then y is not contained in $N^*(u_0) \cup N^*(u_1) \cup \cdots \cup N^*(u_r)$, which implies that y is not adjacent to any of u_0, $\ldots$, u_r. Since x and y are in X, x and y are adjacent. Let w^* be the last vertex on the (u_{r^*}, u_{r+1})-section of P^* that is adjacent to x. Denote (w^*, u_{r+1})-section of P^* by P'. Then $x P' y x$ is a hole and we reach a contradiction. Thus $N^*(u_{r^*}) \subseteq N^*(u_{r+1})$ or $N^*(u_{r^*}) \supseteq N^*(u_{r+1})$ and we reach a contradiction. Hence the lemma follows. □

LEMMA 8. *For any $r \in \{1, \ldots, s\}$, there exists a vertex z_r in $N \cap V(Q_r)$ such that $\cup_{w \in N \cap V(Q_r)} N^*(w) = N^*(z_r)$.*

Proof. Let $N \cap V(Q_r) = \{w_1, \ldots, w_{n_r}\}$, where $n_r = |N \cap V(Q_r)|$. Since w_1 and w_2 are in the same component, there is an hole-avoiding

path in Q_r from w_1 to w_2. Let $P_1 = u_0 u_1 \cdots u_{q-1} u_q$ ($u_0 = w_1$, $u_q = w_2$) be the shortest among such paths. By Lemma 7, there is a vertex $w_{\psi(2)} \in \{u_0, u_1, \ldots, u_q\}$ such that $N^*(w_{\psi(2)}) = N^*(u_0) \cup \cdots \cup N^*(u_q)$ where ψ is a mapping on $\{1, 2, \ldots, n_r\}$. By Lemma 6, $\{u_0, u_1, \ldots, u_q\} \subseteq N \cap V(Q_r)$ and so $w_{\psi(2)} \in N \cap V(Q_r)$. Since $\{w_1, w_2\} \subset \{u_0, u_1, \ldots, u_q\}$, it is true that $N^*(w_1) \cup N^*(w_2) \subseteq N^*(w_{\psi(2)})$. By applying a similar argument for vertices $w_{\psi(2)}$ and w_3, we can show that there exists vertex $w_{\psi(3)}$ in $N \cap V(Q_r)$ such that $N^*(w_{\psi(2)}) \cup N^*(w_3) \subseteq N^*(w_{\psi(3)})$. Then clearly $N^*(w_1) \cup N^*(w_2) \cup N^*(w_3) \subseteq N^*(w_{\psi(3)})$. We repeat the argument as above until we find vertex $w_{\psi(n_r)}$ such that $N^*(w_1) \cup \cdots \cup N^*(w_{n_r}) \subseteq N^*(w_{\psi(n_r)})$. Clearly $N^*(w_1) \cup \cdots \cup N^*(w_{n_r}) \supseteq N^*(w_{\psi(n_r)})$ and therefore $N^*(w_1) \cup \cdots \cup N^*(w_{n_r}) = N^*(w_{\psi(n_r)})$. We let $w_{\psi(n_r)} = z_r$ and the lemma follows. □

Now we are ready to show the following theorem that gives a huge family of graph with one hole and competition number one:

THEOREM 4. *Suppose that G is a graph with exactly one hole C. Let X be the set of all the vertices that are adjacent to all the vertices on the hole. Suppose that there exists a vertex in $V(G) - X$ adjacent to two adjacent vertices on C. Then $k(G) \leq 1$ and the equality holds if and only if G does not have an isolated vertex.*

Proof. Without loss of generality, we may assume that there exists a vertex in $V(G) - X$ adjacent to both v_0 and v_1. Recall that N is the set of the vertices in $V(G) - X$ that are adjacent to both v_0 and v_1. Also recall that $Q_1, \ldots, Q_s$ are the components of $G - X - \{v_0, v_1\}$ that include at least one vertex in N. For each vertex u in $V(G) - X - V(C)$, $N^*(u)$ denotes the set of the vertices in X that are adjacent to u, that is, $X \cap N_G(u)$.

For each $r \in \{1, \ldots, s\}$, we let G_r be the graph induced by the vertices in $\{v_0, v_1\} \cup V(Q_r) \cup N^*(z_r)$ where z_r is the vertex obtained in Lemma 8. Then obviously G_r is chordal for any $r \in \{1, \ldots, s\}$. Let G' be the graph induced by the vertices in $V(G) - [V(Q_1) \cup \cdots \cup V(Q_s)]$. We note that $V(C) \subset V(G')$. We denote $G' - v_0 v_1$ by G^*.

Suppose that there exists a vertex in $V(Q_r) - N$ adjacent to some vertex x in $X - N^*(z_r)$. Let u be a vertex closest to z_r among such vertices. We will reach a contradiction. Let Z be a shortest hole-avoiding path from u to z_r. Suppose that an internal vertex z of Z is adjacent to x. If z is in $V(Q_r) - N$, then z is nearer to z_r than u, contradicting to the choice of u. Thus z is in $V(Q_r) \cap N$. Then by Lemma 8, $N^*(z) \subseteq N^*(z_r)$. But $x \in N^*(z)$ and this contradicts the fact that x is in $X - N^*(z_r)$. Thus

no internal vertex of Z is adjacent to x. Since u belongs to $V(Q_r) - N$, u is not adjacent to one of v_0 and v_1. Without loss of generality, we may assume that u is not adjacent to v_0. Let w^* be the first vertex on Z that is adjacent to v_0 and denote the (u, w^*)-section of Z by Z_1. Then $Z_1 v_0 x u$ is a hole, which is a contradiction. Therefore we can conclude that any vertex in $V(Q_r) - N$ is not adjacent to any vertex in $X - N^*(z_r)$. Thus

$$N^*(z_r) = \cup_{u \in V(Q_r)} N^*(u).$$

Hence for each $1 \leq r \leq s$, Q_r is also a component of $G - N^*(z_r) - \{v_0, v_1\}$ and therefore

(1) $$E(G) = E(G_1) \cup \cdots \cup E(G_s) \cup E(G^*).$$

We now claim that G^* is chordal. For otherwise there is a hole C^* of length at least 4 in G^*. Then edge $v_0 v_1$ must be a chord of C^* in G and there must be a vertex v on C^* (but not on C) that is adjacent to both v_0 and v_1 in G. If $V(C^*) = V(C) \cup \{v\}$, then it is obvious that $v \notin X$. If $V(C^*) \neq V(C) \cup \{v\}$, then there is a vertex v' not on C such that $v_0 v v_1 v' v_0$ is the vertex sequence of C^*. Then v and v' are nonadjacent in G, and so one of v, v' cannot be in X. In both cases, we have found a vertex in $V(G) - X - V(C)$. Then by the definition of $Q_1, \ldots, Q_s$, the vertex belongs to Q_r for some $r \in \{1, \ldots, s\}$. However, it is a vertex on C^* and so belongs to to G^*, which is a contradiction.

We note that the vertices in $N^*(z_r)$ together with z_r, v_0, v_1 form a clique in G_r. Then by Lemma 3, for each $1 \leq r \leq s$, there is an acyclic digraph D_r such that $C(D_r) = G_r \cup \{a_r\}$, where a_r is an extra isolated vertex, and the vertices in $N^*(z_r) \cup \{v_0, v_1\} \cup \{z_r\}$ have only outgoing arcs in D_r. Again, since X is a clique of G^*, there is an acyclic digraph D^* such that $C(D^*) = G^* \cup \{a^*\}$ and, where a^* is an extra isolated vertex and the vertices in X have only outgoing arcs in D^* by Lemma 3. Now we construct digraph D as follows:

$$V(D) = V(G_1) \cup \ldots \cup V(G_s) \cup V(G^*) \cup \{a_1\}$$

and

$$A(D) = A(D_1) \cup [A(D_2) \cup \{(x, z_1) \mid (x, a_2) \in A(D_2)\}$$
$$- \{(x, a_2) \mid (x, a_2) \in A(D_2)\}]$$

$$\cup [A(D_s) \cup \{(x, z_{s-1)}) \mid (x, a_s) \in A(D_s)\}$$
$$- \{(x, a_s) \mid (x, a_s) \in A(D_s)\}]$$
$$\cup [A(D^*) \cup \{(x, z_{s)}) \mid (x, a^*) \in A(D^*)\}$$
$$- \{(x, a^*) \mid (x, a^*) \in A(D^*)\}].$$

We note that no vertices other than v_0, v_1, and the vertices in X belong to more than one of D_1, ..., D_s and D^*. But none of v_0, v_1, vertices in X has incoming arcs in each of those digraphs. Thus D is acyclic. Moreover, since z_r does not have incoming arcs in D_r for each $1 \le r \le s$, it follows from (1) that $C(D) = G \cup \{a_1\}$.

If G has no isolated vertices, then $k(G) \ge 1$ and therefore $k(G) = 1$. If G has an isolated vertex z, then we can find an acyclic digraph D' whose competition graph is $(G - z) \cup \{a_1\}$ as above. Then we replace a_1 by z. Therefore, $k(G) = 0$. $\square$

The competition number of a graph with one hole and $X = \emptyset$ is easily determined by the following theorem:

THEOREM 5. *Suppose that a connected graph G has exactly one hole and no vertices of G are adjacent to all the vertices on the hole. Then $k(G) = 1$ if and only if G has at least one triangle.*

Proof. If G has no triangles, then G consists of one hole with a tree attached at each vertex of the cycle (refer to Figure 2). By Theorem 1, $k(G) = 2$.

Now suppose that G has a triangle. If there is a triangle sharing one edge with the cycle, then $k(G) \le 1$ by Theorem 4. Suppose that no triangles share an edge with the cycle. Let T be a triangle of G and $C = v_0 v_1 \cdots v_{l-1}$ the hole. Take two (not necessarily distinct) vertices x and y on T. Suppose that there are hole-avoiding paths P_1 and P_2 from x to v_i and y to v_j, respectively, for distinct i and j in $\{0, 1, \ldots, l-1\}$. We may assume that P_1 and P_2 are the shortest among hole-avoiding (x, v_i)-paths and hole-avoiding (y, v_j)-paths, respectively. Then, by Lemma 1, $j \equiv i + 1$ or $j \equiv i - 1 \pmod{l}$ and so v_i and v_j are adjacent. Our supposition that no triangles share an edge with C tells us that the vertex immediately preceding v_i on P_1 and the vertex immediately preceding v_j on P_2 are nonadjacent to v_j and to v_i, respectively. Now it is not difficult to check that closed walk formed by paths P_1^{-1}, xy, P_2, and $v_j v_i$ includes a hole which is distinct from C. Thus we reach a contradiction and we conclude that there is exactly one vertex on C to which there is an hole-avoiding path from a vertex on T. Let v_i be this vertex. Then clearly v_i is a cut vertex of G and we let Q be the component of $G - v_i$

that contains the vertices of T. Obviously Q contains no vertices of the cycle. Now we consider the graph G' induced by the vertices of Q and v_i. Then G' is chordal. By Lemma 4, there is an acyclic digraph D' such that $C(D') = G' \cup \{a\}$, where a is an extra isolated vertex, and G' has at least three vertices including v_i with only outgoing arcs in D'. We denote by v, w, the two vertices other than v_i. Now we consider the graph G'' induced by the vertices in $V(G) - V(Q)$. Then by Theorem 2, there is an acyclic digraph D'' such that $C(D'') = G'' \cup \{b, c\}$, where b and c are extra isolated vertices. Now we construct an acyclic digraph D as follows: We let

$$V(D) = V(G') \cup V(G'') \cup \{a\}$$

and

$$A(D) = A(D') \cup A(D'')$$
$$\cup \{(x,v) \mid (x,b) \in A(D'')\} \cup \{(x,w) \mid (x,c) \in A(D'')\}$$
$$- \{(x,b) \mid (x,b) \in A(D'')\} - \{(x,c) \mid (x,c) \in A(D'')\}.$$

Then D is acyclic since v_i is the only common vertex of G' and G''' and v_i has only outgoing arcs in D'. Since v_i, v, and w have only incoming arcs in D', $C(D) = G \cup \{a\}$. Hence $k(G) \leq 1$ whether or not there is a triangle sharing one edge with C. Since G is connected, $k(G) \geq 1$ and therefore $k(G) = 1$.   □

3. Closing remarks

In this paper, we gave another sufficient condition for a graph with one hole to have its competition number one. The author has not found any graphs with one hole and competition number one that satisfy neither the sufficient condition given in Theorem 3 nor the sufficient condition given in Theorem 4.

Given a positive integer n and a positive integer k, $k \leq n + 1$, by replacing K_{n+1} with K_{n-k+3} in Figure 1, we can construct a connected graph G with n holes and $k(G) = k$. For example, if $k = n + 1$, then the graph in Figure 1 becomes the one given in Figure 5. Since G is triangle-free, $k(G) = |E(G)| - |V(G)| + 2$ and so its competition number is $n + 1$. Then we may ask: Is $n + 1$ the maximum competition numbers of a graph with n holes? The answer is yes if $n = 0$ or $n = 1$.
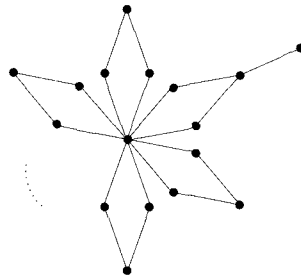
FIGURE 5. A graph G with n holes and $k(G) = n + 1$.

References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, North Holland, New York, 1976.

[2] C. Cable, K. F. Jones, J. R. Lundgren, and S. Seager, *Niche graphs*, Discrete Appl. Math. **23** (1989), 231–241.

[3] H. H. Cho and S-R. Kim, *The Competition Number of a Graph Having Exactly One Hole*, to appear in Discrete Math.

[4] H. H. Cho, S-R. Kim, and Y. Nam, *The m-Step Competition Graph of a Digraph*, Discrete Appl. Math. **105** (2000), 115–127.

[5] J. E. Cohen, *Interval Graphs and Food Webs: A Finding and a Problem*, RAND Corporation Document 17696-PR, Santa Monica, CA, 1968.

[6] M. B. Cozzens and F. S. Roberts, *T-Colorings of Graphs and the Channel Assignment Problem*, Congr. Numer. **25** (1982) 191–208.

[7] G. A. Dirac, *On Rigid Circuit Graphs*, Abh. Math. Sem. Univ. Hamburg **25** (1961), 71–76.

[8] W. K. Hale, *Frequency Assignment: Theory and Application*, Proc. IEEE **68** (1980), 1497–1514.

[9] P. C. Fishburn and W. V. Gehrlein, *Niche numbers*, J. Graph Theory **16** (1992), 131–139.

[10] F. Harary, S-R. Kim, and F. S. Roberts, *Competition numbers as a generalization of Turan's theorem*, J. Ramanujan Math. Soc. **5** (1990), 33–43.

[11] S-R. Kim, *The Competition Number and Its Variants*, in Quo Vadis, Graph Theory, (J. Gimbel, J. W. Kennedy, and L. V. Quintas, eds.), Annals of Discrete Mathematics **55**, North Holland B. V., Amsterdam, the Netherlands, 1993, 313–326.

[12] S-R. Kim and F. S. Roberts, *Competition numbers of graphs with a small number of triangles*, Discrete Appl. Math. **78** (1997), 153–162.

[13] J. R. Lundgren, *Food Webs, Competition Graphs, Competition-Common Enemy Graphs, and Niche Graphs*, in Applications of Combinatorics and Graph Theory to the Biological and Social Sciences, (F. S. Roberts, ed.), IMH Volumes in Mathematics and Its Application, Vol. 17, Springer-Verlag, New York, 1989, 221–243.

[14] R. J. Opsut, *On the Computation of the Competition Number of a Graph*, SIAM J. Discrete Math. **3** (1982), 420–428.

[15] R. J. Opsut and F. S. Roberts, *On the Fleet Maintenance, Mobile Radio frequency, Task Assignment and Traffic phasing Problem*, The Theory and Applications of Graphs, (G. Chartrand, Y. Alavi, D. L. Goldsmith, L. Lesniak-Foster, and D. R. Lick, eds.), Wiley, New York, 1981, 479–492.

[16] F. S. Roberts, *Food Webs, Competition Graphs, and the Boxicity of Ecological Phase Space*, Theory and Applications of Graphs, (Y. Alavi and D. Lick, eds.), Springer Verlag, New York, 1978, 477–490.

[17] ———, *Graph Theory and Its Applications to Problems of Society*, SIAM, Pennsylvania, 1978.

[18] D. Scott, *The competition-common enemy graph of a digraph*, Discrete Appl. Math. **17** (1987), 269–280.

[19] S. Seager, *The Double Competition Number of Some Triangle-Free Graphs*, Discrete Appl. Math. **29** (1990), 265–269.

[20] C. E. Shannon, *The Zero Capacity of a Noisy Channel*, IEEE Trans. Inform. Theory **IT-2** (1956), 8–19.

Department of Mathematics Education
Seoul National University
Seoul 151-742, Korea
E-mail: srkim@snu.ac.kr