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CONSERVATIVE MINIMAL QUANTUM DYNAMICAL SEMIGROUPS GENERATED BY NONCOMMUTATIVE ELLIPTIC OPERATORS

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Abstract. By employing Chebotarev and Fagnola's sufficient conditions for conservativity of minimal quantum dynamical semigroups [7, 8], we construct the conservative minimal quantum dynamical semigroups generated by noncommutative elliptic operators in the sense of [2]. We apply our results to concrete examples.

1. Introduction

Let $\mathcal{M}$ be a von Neumann algebra acting on a separable Hilbert space $\mathfrak{h}$, and let $(\alpha_t)_{t\in\mathbb{R}}$ be a weak*- continuous group of *-automorphisms of $\mathcal{M}$. In [2], using a quantum version of Feynman-Kac formula, the authors constructed the Markov semigroup generated by noncommutative elliptic operator $\mathcal{L}$ on $\mathcal{M}$:

$$(1.1)\qquad \begin{aligned} &D(\mathcal{L}) = D(\delta^2),\\ &\mathcal{L}(X) = \frac{1}{2}\delta^2(X) + a\delta(X) + \delta(X)a - \frac{1}{2}[a,[a,X]],\ \ X \in D(\mathcal{L}),\end{aligned}$$

where a is a self-adjoint element of $\mathcal{M}$, δ is the generator of $(\alpha_t)_{t\in\mathbb{R}}$ and $[A,B] = AB - BA$. See also [15, 17]. This generator $\mathcal{L}$ can be regarded as the quantum version of classical elliptic operator $\frac{1}{2}\Delta + \beta\circ\nabla$, where ∇ and Δ are the gradient and Laplacian operators on $L^2(\mathbb{R}^n)$ respectively, and β is a $\mathbb{R}^n$-valued function on $\mathbb{R}^n$.

Let $\mathcal{M}$ be the Banach space $B(\mathfrak{h})$ of bounded linear operators on $\mathfrak{h}$. Let $(U_t)_{t\in\mathbb{R}}$ be a strongly continuous one parameter group of unitary

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operators, $U_t = e^{itb}$, where b is a self-adjoint generator of U_t. Let the automorphisms $(\alpha_t)_{t\in\mathbf{R}}$ be unitarily implemented by $(U_t)_{t\in\mathbf{R}}$, $\alpha_t(X) = U_t X U_t^*$, $X \in \mathcal{M}$. Then

$$\delta(X) = i[b, X], \ \ X \in D(\delta). \tag{1.2}$$

See Proposition 3.2.55 of [4]. Put

$$L := a - ib, \quad H := \frac{1}{2}(ab + ba). \tag{1.3}$$

A simple algebraic computation shows that $\mathcal{L}$ has the following Lindblad type representation:

$$\mathcal{L}(X) = i[H, X] - \frac{1}{2}L^*LX + L^*XL - \frac{1}{2}XL^*L, \ \ X \in D(\mathcal{L}). \tag{1.4}$$

The purpose of this paper is to extend the construction of the Markov semigroup with generator $\mathcal{L}$ to an unbounded self-adjoint operator a. Let us mention that for an unbounded self-adjoint operator a, the method of the quantum Feynman-Kac formula in [2, 17] can not be applied. So we employ the theory of the minimal quantum dynamical semigroup to construct the Markov semigroup.

A quantum dynamical semigroup(q.d.s.) $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ in $\mathcal{B}(\mathfrak{h})$ is a weak* (or ultraweakly) continuous semigroup of completely positive linear maps on $\mathcal{B}(\mathfrak{h})$. A q.d.s. $\mathcal{T}$ is *conservative* (*or Markov*) if $\mathcal{T}_t(I) = I$ where I is the identity operator on $\mathfrak{h}$. In rather general cases, the infinitesimal generator $\mathcal{L}$ can be written (formally) as

$$\mathcal{L}(X) = i[H, X] - \frac{1}{2}XM + \sum_{l=1}^{\infty} L_l^* X L_l - \frac{1}{2}MX, \quad X \in B(\mathfrak{h}), \tag{1.5}$$

where $M = \sum_{l=1}^{\infty} L_l^* L_l$, L_l is densely defined and H a symmetric operator on $\mathfrak{h}$ [16, 18]. Notice that the generator in (1.4) is the above form with $l = 1$. For the unbounded generator $\mathcal{L}$ in (1.5) with (unbounded) coefficients H and L_l, the solution $\mathcal{T}$ of the quantum master Markov equation

$$\frac{d}{dt}\mathcal{T}_t(X) = \mathcal{L}(\mathcal{T}_t(X)), \quad \mathcal{T}_0(X) = X \tag{1.6}$$

may not be unique and conservative ([3, 5]). Under suitable conditions, the above equation (1.6) has a minimal solution known as the minimal q.d.s.. See Section 2 for the details. Moreover if the minimal q.d.s. is conservative, it is the unique solution of the above equation. Chebotarev gave necessary and sufficient conditions for conservativity ([5]). Some of these conditions are difficult to verify practically. Later simplified forms of sufficient conditions were developed in [6, 7, 8, 9]. For the recent relevant works with the minimal q.d.s., see [1, 13, 14].

In this paper, applying the results of F. Fagnola and A. M. Chebotarev [7, 8], we will obtain the sufficient conditions of operators a, b such that the equation (1.6) has a minimal solution $(\mathcal{T}_t)_{t\geq 0}$ and the minimal q.d.s. $(\mathcal{T}_t)_{t\geq 0}$ is conservative. Indeed, under the boundedness of $[a,b]$ together with appropriate domain conditions, we can construct the minimal q.d.s. $(\mathcal{T}_t)_{t\geq 0}$ generated by $\mathcal{L}$ in (1.4). See Proposition 3.4 and the following argument. In addition, if $[[a,b],a]$ is relatively small with respect to either a or else b, then the minimal q.d.s. is conservative. See Theorem 3.5 and Remark 3.7. We also consider the case $[a,b]$ is unbounded(Theorem 3.9). We apply our results to concrete examples (Example 5.1, 5.2 and 5.3).

The paper is organized as follows: In section 2, we review the theory of the minimal q.d.s. and introduce the sufficient conditions of Fagnola and Chebotarev[7, 8]. In section 3, we state main results. We give the sufficient conditions of a, b such that the equation (1.6) has a minimal solution and the minimal q.d.s. is conservative. Section 4 is devoted to the proofs of Proposition 3.4, Theorem 3.5, Theorem 3.6 and Theorem 3.9. As an application to our results, we give concrete examples in Section 5.

2. Review on the minimal quantum dynamical semigroups

Let $\mathfrak{h}$ be a complex separable Hilbert space with the scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let $\mathcal{B}(\mathfrak{h})$ denote the Banach space of bounded linear operators on $\mathfrak{h}$. The uniform norm in $\mathcal{B}(\mathfrak{h})$ is denoted by $\|\cdot\|_\infty$ and the identity in $\mathfrak{h}$ is denoted by I. We denote by $D(G)$ the domain of operator G in $\mathfrak{h}$.

DEFINITION 2.1. A quantum dynamical semigroup(q.d.s.) on $\mathcal{B}(\mathfrak{h})$ is a family $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ of operators in $\mathcal{B}(\mathfrak{h})$ with the following properties:

(i) $\mathcal{T}_0(X) = X$, for all $X \in \mathcal{B}(\mathfrak{h})$;

(ii) $\mathcal{T}_{t+s}(X) = \mathcal{T}_t(\mathcal{T}_s(X))$, for all $s,t \geq 0$ and all $X \in \mathcal{B}(\mathfrak{h})$;

(iii) $\mathcal{T}_t(I) \leq I$, for all $t \geq 0$;

(iv) (completely positivity) for all $t \geq 0$, all integers n and all finite sequences $(X_j)_{j=1}^n$, $(Y_l)_{l=1}^n$ of elements of $\mathcal{B}(\mathfrak{h})$, we have

$$\sum_{j,\,l=1}^{n} Y_l^*\mathcal{T}_t(X_l^*X_j)Y_j \geq 0;$$

(v) (normality) for every sequence $(X_n)_{n\geq 1}$ of elements of $\mathcal{B}(\mathfrak{h})$ converging weakly to an element X of $\mathcal{B}(\mathfrak{h})$ the sequence $(\mathcal{T}_t(X_n))_{n\geq 1}$ converges weakly to $\mathcal{T}_t(X)$ for all $t \geq 0$;

(vi) (ultraweak continuity) for all trace class operator ρ on $\mathfrak{h}$ and all $X \in \mathcal{B}(\mathfrak{h})$ we have

$$\lim_{t\to 0^+} Tr(\rho\mathcal{T}_t(X)) = Tr(\rho X).$$

We recall that as a consequence of properties (iii), (iv), for each $t \geq 0$ and $X \in \mathcal{B}(\mathfrak{h})$, $\mathcal{T}_t$ is a contraction, i.e.,

$$\|\mathcal{T}_t(X)\|_\infty \leq \|X\|_\infty. \tag{2.1}$$

Also recall that as a consequence of properties (iv), (vi), for all $X \in \mathcal{B}(\mathfrak{h})$, the map $t \mapsto \mathcal{T}_t(X)$ is strongly continuous.

Definition 2.2. A q.d.s. $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ is called to be conservative if $\mathcal{T}_t(I) = I$ for all $t \geq 0$.

As mentioned in Introduction, the natural generator of q.d.s. would be the Lindblad type generator ([16, 18]). Letting

$$G = -iH - \frac{1}{2}M, \quad M = \sum_{l=1}^{\infty} L_l^* L_l, \tag{2.2}$$

the infinitesimal generator in (1.5) can be formally written by

$$\mathcal{L}(X) = XG + G^*X + \sum_{l=1}^{\infty} L_l^* X L_l.$$

A very large class of q.d.s. was constructed by Davies[10] satisfying the following assumption. It is basically corresponding to the condition $\mathcal{L}(I) = 0$.

Assumption 2.3. *The operator G is the infinitesimal generator of a strongly continuous contraction semigroup $P = (P(t))_{t\geq 0}$ in $\mathfrak{h}$. The domain of the operators $(L_l)_{l=1}^{\infty}$ contains the domain $D(G)$ of G. For all $v, u \in D(G)$, we have*

$$\langle v, Gu\rangle + \langle Gv, u\rangle + \sum_{l=1}^{\infty} \langle L_l v, L_l u\rangle = 0. \tag{2.3}$$

As a result of Proposition 2.5 of [7] we can assume only that the domain of the operators L_l contains a subspace D which is a core for G and (2.3) holds for all $v, u \in D$.

For all $X \in \mathcal{B}(\mathfrak{h})$, consider the sesquilinear form $\mathcal{L}(X)$ on $\mathfrak{h}$ with domain $D(G) \times D(G)$ given by

$$\langle v, \mathcal{L}(X)u \rangle = \langle v, XGu \rangle + \langle Gv, Xu \rangle + \sum_{l=1}^{\infty} \langle L_l v, XL_l u \rangle. \tag{2.4}$$

Under the Assumption 2.3, one can construct a q.d.s. $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ satisfying the equation

$$\langle v, \mathcal{T}_t(X)u \rangle = \langle v, Xu \rangle + \int_0^t \langle v, \mathcal{L}(\mathcal{T}_s(X))u \rangle ds \tag{2.5}$$

for all $v, u \in D(G)$ and all $X \in B(\mathfrak{h})$. Indeed, for a strongly continuous family $(\mathcal{T}_t(X))_{t\geq 0}$ of elements of $\mathcal{B}(\mathfrak{h})$ satisfying (2.1), the followings are equivalent:

(i) equation (2.5) holds for all $v, u \in D(G)$,
(ii) for all $v, u \in D(G)$, we have

$$\begin{aligned} \langle v, \mathcal{T}_t(X)u \rangle &= \langle P(t)v, XP(t)u \rangle \\ &\quad + \sum_{l=1}^{\infty} \int_0^t \langle L_l P(t-s)v, \mathcal{T}_s(X) L_l P(t-s)u \rangle ds. \end{aligned} \tag{2.6}$$

We refer to the proof of Proposition 2.3 in [8]. A solution of the equation (2.6) is obtained by the iterations

$$\begin{aligned} \langle u, \mathcal{T}_t^{(0)}(X)u \rangle &= \langle P(t)u, XP(t)u \rangle, \\ \langle u, \mathcal{T}_t^{(n+1)}(X)u \rangle &= \langle P(t)u, XP(t)u \rangle \\ &\quad + \sum_{l=1}^{\infty} \int_0^t \langle L_l P(t-s)u, \mathcal{T}_s^{(n)}(X) L_l P(t-s)u \rangle ds, \end{aligned} \tag{2.7}$$

for all $u \in D(G)$. In fact, for all positive elements $X \in \mathcal{B}(\mathfrak{h})$ and all $t \geq 0$, the sequence of operators $(\mathcal{T}_t^{(n)}(X))_{n\geq 0}$ is non-decreasing. Therefore it is strongly convergent and its limits for $X \in \mathcal{B}(\mathfrak{h})$ and $t \geq 0$ define the *minimal solution* $(\mathcal{T}_t)_{t\geq 0}$ of (2.6) in the sense that, given another solution $(\mathcal{T}_t')_{t\geq 0}$ of (2.5), one can easily check that

$$\mathcal{T}_t(X) \leq \mathcal{T}_t'(X) \leq \|X\|_\infty I$$

for any positive element X and all $t \geq 0$. For details, we refer to [5, 12]. From now on, the minimal solution $(\mathcal{T}_t)_{t\geq 0}$ is called the minimal q.d.s..

Recently Chebotarev and Fagnola have obtained easier criteria to verify the conservativity of minimal q.d.s. $(\mathcal{T}_t)_{t\geq 0}$ obtained under Assumption 2.3. Here we give their results(Theorem 4.4 in [8], Corollary 4.4 in [7]).

THEOREM 2.4. *Suppose that there exists a positive self-adjoint operator C in $\mathfrak{h}$ with the following properties*

(a) *the domain $D(G)$ of G is contained in the domain of the positive square root $C^{1/2}$ and $D(G)$ is a core for $C^{1/2}$,*

(b) *the linear manifolds $L_l(D(G^2))$, $l \geq 1$, are contained in the domain of $C^{1/2}$,*

(c) *there exists a positive self-adjoint operator Φ, with $D(G) \subset D(\Phi^{1/2})$ such that, for all $u \in D(G)$, we have*

$$-2Re\langle u, Gu\rangle = \sum_{l=1}^{\infty} \|L_l u\|^2 = \|\Phi^{1/2}u\|^2,$$

(d) *$D(C) \subset D(\Phi)$ and for all $u \in D(C)$, we have $\|\Phi^{1/2}u\| \leq \|C^{1/2}u\|$,*

(e) *there exists a positive constant k such that*

$$2Re\langle C^{1/2}u, C^{1/2}Gu\rangle + \sum_{l=1}^{\infty} \|C^{1/2}L_l u\|^2 \leq k\|C^{1/2}u\|^2, \tag{2.8}$$

for all $u \in D(G^2)$.

Then the minimal q.d.s. $(\mathcal{T}_t)_{t\geq 0}$ is conservative.

The following is another criteria for conservativity (see Corollary 4.4 and Lemma 5.1 (ii) in [7]).

THEOREM 2.5. *Suppose that there exist a positive self-adjoint operator C and a core D for G in $\mathfrak{h}$ with following properties*

(a) *the domain $D(C)$ contains D and for all $v, u \in D$,*

$$\langle v, u\rangle + \sum_{l=1}^{\infty} \langle L_l v, L_l u\rangle = \langle v, Cu\rangle,$$

(b) *D is a core for G^2,*

(c) *$C(D)$ coincides with D and for all $l \geq 1$, $L_l(D) \subset D(C)$,*

(d) *there exists a constant k such that, for all $u \in D$,*

$$2Re\langle Cu, Gu\rangle + \sum_{l=1}^{\infty} \|C^{1/2}L_l u\|^2 \leq k\|C^{1/2}u\|^2. \tag{2.9}$$

Then the minimal q.d.s. $(\mathcal{T}_t)_{t\geq 0}$ is conservative.

3. Conservative minimal quantum dynamical semigroups: Main results

Let a and b be self-adjoint operators on the Hilbert space $\mathfrak{h}$ with common core $\mathcal{D}$ satisfying $a(\mathcal{D}) \subset \mathcal{D}$, $b(\mathcal{D}) \subset \mathcal{D}$. Let H and L be the operators defined by

$$(3.1) \qquad \begin{aligned} Hu &= \frac{1}{2}\{a(bu) + b(au)\}, \\ Lu &= au - ibu, \end{aligned}$$

for any $u \in \mathcal{D}$. H is a densely defined, symmetric operator. We denote again by H its closure. The adjoint operator L^* of L on $\mathcal{D}$ is given by

$$L^*u = au + ibu, \quad u \in \mathcal{D}.$$

Since $D(L^*)$ is dense, L is closable. Denote again by L its closure.

We consider the elliptic operator $\mathcal{L}$ on $B(\mathfrak{h})$ formally given by

$$(3.2) \qquad \mathcal{L}(X) = i[H, X] - \frac{1}{2}L^*LX + L^*XL - \frac{1}{2}XL^*L, \quad X \in D(\mathcal{L}).$$

As mentioned in Introduction, we will construct the minimal q.d.s. with the formal generator (3.2) under appropriate conditions and study conservativity of the semigroup.

In the rest of this paper, we assume that the operators a and b satisfy the following properties:

Assumption 3.1. *Suppose that a and b are self adjoint operators with common core $\mathcal{D}$ on $\mathfrak{h}$ satisfying*

(i) *$\mathcal{D}$ is an invariant subspace for a, b, i.e., $a(\mathcal{D}) \subset \mathcal{D}$, $b(\mathcal{D}) \subset \mathcal{D}$,*
(ii) *$a^2 + b^2$ is essentially self adjoint with core $\mathcal{D}$.*

Let the operator G on $\mathcal{D}$ defined by

$$(3.3) \qquad Gu = -\frac{1}{2}(a^2 + b^2)u - ibau, \ \forall u \in \mathcal{D}.$$

We rewrite G as the following form

$$(3.4) \qquad Gu = -\frac{1}{2}L^*Lu - iHu, \ \forall u \in \mathcal{D}.$$

Then the adjoint G^* is given by $G^*u = -\frac{1}{2}(a^2+b^2)u + iabu$ for all $u \in \mathcal{D}$. Since $\mathcal{D}$ is dense and $\mathcal{D} \subset D(G^*)$, G is closable. Let us introduce another assumption to endow a strongly continuous contraction semigroup with generator G on $\mathfrak{h}$.

Assumption 3.2. *The closure $\overline{G}$ of G is the infinitesimal generator of a strongly continuous contraction semigroup on $\mathfrak{h}$.*

Recall that H is symmetric on $\mathcal{D}$. Clearly G is dissipative on $\mathcal{D}$, and so is $\overline{G}$. If $\overline{G}^*(=G^*)$ is dissipative then Assumption 3.2 holds. See Corollary 4.4 in p.15 of [19].

From now on we denote again by G instead of $\overline{G}$. Notice that $\mathcal{D}$ is a core for G. Consider the sesquilinear form $\mathcal{L}(X)$ on $\mathfrak{h}$ with domain $\mathcal{D}\times\mathcal{D}$ given by

$$\langle v, \mathcal{L}(X)u\rangle = \langle v, XGu\rangle + \langle Gv, Xu\rangle + \langle Lv, XLu\rangle \tag{3.5}$$

for all $X \in B(\mathfrak{h})$. Clearly we have

$$\langle v, Gu\rangle + \langle Gv, u\rangle + \langle Lv, Lu\rangle = 0 \tag{3.6}$$

for all $u, v \in \mathcal{D}$. Two operators G and L satisfy the condition (2.3) on the domain $\mathcal{D}$, a core for G. As mentioned in Section 2, under Assumption 3.1 and Assumption 3.2, by the iterations, we have the minimal q.d.s. $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ satisfying the equation

$$\langle v, \mathcal{T}_t(X)u\rangle = \langle v, Xu\rangle + \int_0^t \langle v, \mathcal{L}(\mathcal{T}_s(X))u\rangle ds \tag{3.7}$$

for all $u, v \in \mathcal{D}$ and for all $X \in B(\mathfrak{h})$.

We state our main results. Under Assumption 3.1, we are looking for sufficient conditions for existence and conservativity of the minimal q.d.s. $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ generated by (3.2). Put

$$K := -i[a, b].$$

We first start with that K is bounded on $\mathcal{D}$.

Assumption 3.3. *There exists a constant $k_1 \geq 0$ such that*

$$\|Ku\| \leq k_1\|u\|, \ u \in \mathcal{D}. \tag{3.8}$$

We have the following results:

Proposition 3.4. *Under Assumption 3.1 and Assumption 3.3, the closure of G defined as in* (3.3)(or (3.4)) *is the infinitesimal generator of a strongly continuous contraction semigroup on $\mathfrak{h}$.*

Thus under Assumption 3.1 and Assumption 3.3, there exists the minimal q.d.s. $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ satisfying the equation (3.7). In the following theorems we give the conservative conditions of the minimal q.d.s. $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$.

Theorem 3.5. *Let Assumption 3.1 and Assumption 3.3 hold. Suppose that the domain of $C'^{1/2}$ contains the domain $D(G)$ of G and*

$D(G^2) \subset D(C')$, where $C' = a^2 + b^2$. Assume that there exist constants k_2, k_3, k_4 such that

$$\|[K,a]u\| \le k_2\|au\| + k_3\|bu\| + k_4\|u\|, \ u \in \mathcal{D}. \tag{3.9}$$

Then the minimal q.d.s. $(\mathcal{T}_t)_{t\ge 0}$ is conservative.

THEOREM 3.6. *Let Assumption 3.1 and Assumption 3.3 hold. Suppose that $\mathcal{D}$ is a core for G^2 and $(1+L^*L)(\mathcal{D}) = \mathcal{D}$. Assume that there exist constants k_5 and k_6 such that*

$$\|[K,L^*]u\| \le k_5\|Lu\| + k_6\|u\|, \ u \in \mathcal{D}. \tag{3.10}$$

Then the the minimal q.d.s. $(\mathcal{T}_t)_{t\ge 0}$ is conservative.

REMARK 3.7. Since $[K,a]^*$ is densely defined, $[K,a]$ is closable. Denote the closure by $[K,a]$ again. If $[K,a]$ is relatively small with respect to either a or else b, then the inequality (3.9) holds obviously.

REMARK 3.8. One can check easily that the condition (3.10) in Theorem 3.6 can be replaced by the following: there exist constants k_7, k_8 and k_9 such that

$$\|[K,a]u\|, \|[K,b]u\| \le k_7\|au\| + k_8\|bu\| + k_9\|u\|, \quad \forall u \in \mathcal{D}.$$

Let us mention that in Theorem 3.5 we need the domain condition $D(G^2) \subset D(a^2+b^2)$, which is essential to show the inequality (2.8) for all $u \in D(G^2)$, as contrasted with domain condition in Theorem 3.6, $\mathcal{D}$ is a core for G^2 and $(1+L^*L)\mathcal{D} = \mathcal{D}$.

Next, we consider in case K is unbounded on $\mathcal{D}$. In the following theorem, we remove the bounded condition of K in Assumption 3.3.

THEOREM 3.9. *Let Assumption 3.1 and Assumption 3.2 hold. Suppose that the following properties hold:*

(i) *$D(G) \subset D(C'^{1/2})$ and $D(G)$ is a core for $C'^{1/2}$;*
(ii) *for all $u \in D(G^2)$, there exists a convergent sequence $(u_n)_{n\ge 1}$ of elements of $\mathcal{D}$ such that both $(Gu_n)_{n\ge 1}$ and $(C'u_n)_{n\ge 1}$ converge;*
(iii) *there exist constants k_1, k_2, k_3 and k_4 such that, for $u \in \mathcal{D}$,*

$$\begin{aligned} &|\langle u, aKbu\rangle|, \ |\langle u, aKau\rangle| \le k_1\langle u, (C'+1)u\rangle, \\ &\qquad \|Ku\|, \ \|[K,a]u\| \le k_2\|au\| + k_3\|bu\| + k_4\|u\|, \end{aligned} \tag{3.11}$$

where $C' = a^2 + b^2$. Then the minimal q.d.s. $(\mathcal{T}_t)_{t\ge 0}$ is conservative.

4. Proofs of main results

In this section, we produce the proofs of Proposition 3.4, Theorem 3.5, Theorem 3.6 and Theorem 3.9. We first give the the proof of Proposition 3.4.

Proof of Proposition 3.4. Clearly K is symmetric on $\mathcal{D}$. By (3.8) and Theorem X.12 in [20], $L^*L = a^2 + b^2 + K$ is an essentially self adjoint operator with core $\mathcal{D}$. So we denote again by L^*L the self-adjoint operator. It follows from (i) of Assumption 3.1 that $\mathcal{D}$ is invariant for H, L, L^*.

We will show that H is relatively small with respect to $-\frac{1}{2}L^*L$ with relative bound 1, i.e., $G = -iH - \frac{1}{2}L^*L$ is a dissipative perturbation of infinitesimal generator $-\frac{1}{2}L^*L$ of a contraction semigroup. Let $u \in \mathcal{D}$. Using the definition of K, we have the relations that

$$\begin{aligned} ba^2b &= (ba)^2 + ibaK = b^2a^2 + ibKa + ibaK, \\ ab^2a &= a^2b^2 - iaKb - iabK \end{aligned}$$

as bilinear forms on $\mathcal{D}$, which implies

$$\begin{aligned} \|abu\|^2 + \|bau\|^2 &= \langle u, (b^2a^2 + a^2b^2)u\rangle \\ &\quad + i\langle u, (bKa - aKb)u\rangle + \langle u, K^2u\rangle. \end{aligned} \tag{4.1}$$

Applying Schwarz inequality, and by (3.8), we obtain

$$\begin{aligned} &\frac{1}{2}\langle u, (a^2b^2 + b^2a^2)u\rangle \le \frac{1}{2}\langle u, (a^4 + b^4)u\rangle, \\ &i\langle u, (bKa - aKb)u\rangle \le k_1\langle u, (a^2 + b^2)u\rangle. \end{aligned}$$

Substituting the above inequalities into (4.1) and using (3.8), we have

$$\begin{aligned} \|abu\|^2 + \|bau\|^2 &\le \frac{1}{2}\langle u, (a^4 + b^4 + a^2b^2 + b^2a^2)u\rangle \\ &\quad + k_1\langle u, (a^2 + b^2)u\rangle + k_1^2\langle u, u\rangle \\ &= \frac{1}{2}\langle u, (a^2 + b^2 + k_1)^2u\rangle + \frac{1}{2}k_1^2\langle u, u\rangle \end{aligned}$$

and so

$$
\begin{aligned}
\|Hu\| &\le \frac{1}{\sqrt{2}}\sqrt{\|abu\|^2 + \|bau\|^2} \\
&\le \frac{1}{2}\sqrt{\|(a^2+b^2+k_1)u\|^2 + k_1^2\|u\|^2} \\
&\le \left\|\frac{1}{2}(a^2+b^2+k_1)u\right\| + \frac{1}{2}k_1\|u\| \qquad (4.2)\\
&\le \left\|\frac{1}{2}L^*Lu\right\| + \frac{3}{2}k_1\|u\|.
\end{aligned}
$$

We have used (3.8) in the last inequality. Since $\mathcal{D}$ is a core for L^*L, the above inequality implies $D(L^*L) \subset D(H)$. Also we can extend the above inequality to all $u \in D(L^*L)$. Since H is symmetric on $\mathcal{D}$, $-iH$ is dissipative. By Theorem 3.4 in p.83 of [19], the closure of G is the infinitesimal generator of a strongly continuous semigroup of contractions. □

To show the conservativity of the minimal q.d.s., we only check the conditions of Theorem 2.4 or Theorem 2.5.

Proof of Theorem 3.5. To produce the theorem we apply Theorem 2.4 for $C = 2C' + 1$. First we show that the inequality (2.8) holds for $u \in \mathcal{D}$. It follows from (3.1) and (3.3) that we have the relation

$$
\begin{aligned}
&2\mathrm{Re}\langle Cu, Gu\rangle + \langle Lu, CLu\rangle \\
&= 2\langle u, (ab^2a + ba^2b - a^2b^2 - b^2a^2)u\rangle \\
&\quad + 2i\langle u, (aba^2 - a^2ba + ba^3 - a^3b)u\rangle \\
&= 2\langle u, \big(abab - ab[a,b] + baba + ba[a,b] - aabb - bbaa\big)u\rangle \\
&\quad + 2i\langle u, \big(aba^2 - a[a,b]a - [a,b]a^2 - a^2ba - a^2[a,b]\big)u\rangle,
\end{aligned}
$$

and by $K = -i[a,b]$, we can rewrite

$$
\begin{aligned}
&2\mathrm{Re}\langle Cu, Gu\rangle + \langle Lu, CLu\rangle \\
&= 2\langle u, (K^2 - i(aKb - bKa))u\rangle + 2\langle u, (a^2K + Ka^2 + 2aKa)u\rangle \\
&= 2\langle u, K^2u\rangle - 2i\langle u, (aKb - bKa)u\rangle \\
&\quad + 2\langle u, (a[a,K] - [a,K]a + 4aKa)u\rangle \\
&= 2\big(\|Ku\|^2 + 2Im\langle u, aKbu\rangle + 2Re\langle u, a[a,K]u\rangle + 4\langle u, aKau\rangle\big). \qquad (4.3)
\end{aligned}
$$

We get from (3.9) and Schwarz inequality that

$$
\begin{aligned}
\operatorname{Re}\langle u, a[a,K]u\rangle &\le \|au\|\,\|[a,K]u\| \\
&\le k_2\|au\|^2 + k_3\|au\|\|bu\| + k_4\|au\|\,\|u\| \\
&\le k_2\langle u, a^2u\rangle + \frac{k_3}{2}\langle u,(a^2+b^2)u\rangle + \frac{k_4}{2}\langle u,(a^2+1)u\rangle \\
&\le k_5\langle u, C'u\rangle + k_6\langle u,u\rangle, \qquad (4.4)
\end{aligned}
$$

where k_5, k_6 are positive constants. By Schwarz inequality and (3.8), we have

$$
\begin{aligned}
\operatorname{Im}\langle u, aKbu\rangle &\le \frac{k_1}{2}\langle u, C'u\rangle, \qquad (4.5)\\
\langle u, aKau\rangle &\le k_1\langle u, C'u\rangle.
\end{aligned}
$$

Substituting (4.4) and (4.5) into (4.3), we obtain

$$
\begin{aligned}
2\operatorname{Re}\langle Cu, Gu\rangle + \langle Lu, CLu\rangle &\le 2k_7\langle u, C'u\rangle + k_7\langle u,u\rangle \qquad (4.6)\\
&= k_7\langle u, Cu\rangle,
\end{aligned}
$$

where k_7 is a positive constant.

In order to extend the above inequality to $D(G^2)$, we consider $u \in D(G^2)$. By the assumption $D(G^2) \subset D(C')=D(C)$, we have $u \in D(C)$. Recall that $\mathcal{D}$ is a core for C and the inequality (4.2)

$$\|Hu\| \le \frac{1}{4}\|Cu\| + \left(k_1 + \frac{1}{4}\right)\|u\|.$$

Thus there exists a sequence (u_n) of elements of $\mathcal{D}$ such that $\lim_{n\to\infty} u_n = u$, $\lim_{n\to\infty} Cu_n = Cu$ and $\lim_{n\to\infty} Gu_n = Gu$. Inequality (4.6) implies that $(C^{1/2}Lu_n)$ is a Cauchy sequence. Therefore it is convergent and the conditions (b) and (e) in Theorem 2.4 hold.

Since $\mathcal{D}$ is a core for C' and $D(C')$ is a core for $C'^{1/2}$ (see Lemma 2.4 of [7]), $\mathcal{D}$ is a core for $C'^{1/2}$ (see Lemma 2.5 in [11]). Thus $D(G)$ is a core for $C'^{1/2}$, which means that the condition (a) of Theorem 2.4 holds.

By (3.8) and Theorem X.12 of [20], we have $LL^* = a^2 + b^2 - K = C' - K$ and $2C' = L^*L + LL^*$. The conditions (c), (d) of Theorem 2.4 are followed directly. This completes the proof. □

Proof of Theorem 3.6. We choose the operator $C = L^*L + 1$ in order to apply Theorem 2.5. Recall that $\mathcal{D}$ is a core for L^*L and G. The conditions (a)–(c) in Theorem 2.5 are followed from our assumptions.

It remains only to show the inequality (2.8). Let $u \in \mathcal{D}$. Notice $[L^*, L] = 2K$ on $\mathcal{D}$ and

$$\begin{aligned}[(L^*)^2 - L^2, L^*L] &= (L^*)^2[L^*, L] + L^*[L^*, L]L^* + L[L^*, L]L + [L^*, L]L^2 \\ &= 2\big((L^*)^2K + L^*KL^* + LKL + KL^2\big) \\ &= 2\big(L^*[L^*, K] + 2L^*KL^* + 2LKL + [K, L]L\big).\end{aligned}$$

So we can write

$$\begin{aligned} i\langle u, [H, L^*L]u\rangle &= \frac{1}{4}\langle u, [(L^*)^2 - L^2, L^*L]u\rangle \\ &= 2\mathrm{Re}\langle u, LKLu\rangle + \mathrm{Re}\langle u, [K, L]Lu\rangle. \end{aligned} \tag{4.7}$$

By Schwarz inequality, (3.8) and (3.10), we easily obtain

$$i\langle u, [H, L^*L]u\rangle \le k_7\langle u, L^*Lu\rangle + k_8\langle u, u\rangle, \tag{4.8}$$

where k_7, k_8 are constants. It follows from (3.4) and $C = L^*L + 1$ that we have

$$\begin{aligned} 2\mathrm{Re}\langle Cu, Gu\rangle + \langle Lu, CLu\rangle &= i\langle u, [H, L^*L]u\rangle + \langle u, L^*[L^*, L]Lu\rangle \\ &= i\langle u, [H, L^*L]u\rangle + 2\langle u, L^*KLu\rangle. \end{aligned} \tag{4.9}$$

Applying $L(\mathcal{D}) \subset \mathcal{D}$, (4.8) and (3.8) into (4.9) we obtain

$$\begin{aligned} 2\mathrm{Re}\langle Cu, Gu\rangle + \langle Lu, CLu\rangle &\le (k_7 + 2k_1)\langle u, L^*Lu\rangle + k_8\|u\|^2 \\ &\le k\langle u, Cu\rangle, \end{aligned}$$

where $k = \max\{2k_1 + k_7, k_8\}$. The proof is completed. □

Proof of Theorem 3.9. To prove the theorem we apply Theorem 2.4 for $C = 2(a^2 + b^2) + 1$. Clearly the condition (a) of Theorem 2.4 follows from our assumptions.

Let $\Phi = L^*L$. By (3.6), we have for all $u \in \mathcal{D}$

$$-2\mathrm{Re}\langle u, Gu\rangle = \|Lu\|^2 = \|\Phi^{1/2}u\|^2, \tag{4.10}$$

and so

$$\|\Phi^{1/2}u\|^2 \le 2\|u\|\|Gu\|. \tag{4.11}$$

Since $\mathcal{D}$ is a core for G, it is followed from (4.11) that $D(G) \subset D(\Phi^{1/2})$ and the relation (4.10) holds on $D(G)$. Thus the condition (c) of Theorem 2.4 holds.

Notice that $\Phi = C' + K$ on $\mathcal{D}$. We get from the condition of K in (3.11) that K is relatively small with respect to C'. Thus $D(C) \subset D(\Phi)$. By Schwarz inequality, we obtain

$$\langle u, Ku\rangle = -i\langle u, (ab - ba)u\rangle \le \langle u, (a^2 + b^2)u\rangle, \ u \in \mathcal{D}$$

and so

$$\langle u, \Phi u\rangle = \langle u, C'u\rangle + \langle u, Ku\rangle \leq \langle u, Cu\rangle,\ \ u \in \mathcal{D}.$$

Since $\mathcal{D}$ is a core for C, the above inequality holds on $D(C)$, which implies the condition (d) of Theorem 2.4.

Now it remains to check the conditions (b) and (e) of Theorem 2.4. First we show that the inequality (2.8) hold for $u \in \mathcal{D}$. By (4.3), we have

$$\begin{aligned} &2\mathrm{Re}\langle Cu, Gu\rangle + \langle Lu, CLu\rangle \\ &= 2\big(\|Ku\|^2 + 2\mathrm{Im}\langle u, aKbu\rangle + 2\mathrm{Re}\langle u, a[a,K]u\rangle + 4\langle u, aKau\rangle\big). \end{aligned}$$

Using the similar calculations used in the proof of Theorem 3.5, we get from (3.11) and Schwarz inequality that

$$\begin{aligned} &2\mathrm{Re}\langle Cu, Gu\rangle + \langle Lu, CLu\rangle \\ &\leq k\langle u, (2(a^2+b^2)+1)u\rangle = k\langle u, Cu\rangle, \end{aligned}$$

where k is a constant.

Now we consider $u \in D(G^2)$. By the assumption (ii), there exists a sequence (u_n) on $\mathcal{D}$ such that $\lim_{n\to\infty} u_n = u$, $\lim_{n\to\infty} Cu_n = Cu$ and $\lim_{n\to\infty} Gu_n = Gu$. Thus the conditions (b) and (e) of Theorem 2.4 can be checked by the similar method used in the proof of Theorem 3.5. The proof is completed. □

5. Some examples

In this section we apply our results to study conservativity of three minimal q.d.s.. We give two examples in case $[a,b]$ is bounded (Example 5.1 and Example 5.2), and the third example for other case (Example 5.3).

EXAMPLE 5.1. Let $\mathfrak{h} = L^2(\mathbf{R}, dx)$ and D be the subspace $C_0^\infty(\mathbb{R})$, the space of C^∞-functions with compact support. Let $W \in D$.

The multiplication, differential operators $a = W(x)$, $b = i\frac{d}{dx}$ are self adjoint with common core D and D is an invariant subspace for a, b. By Theorem X.28 in [20], $a^2 + b^2 = W(x)^2 - \frac{d^2}{dx^2}$ is a self adjoint operator with core D. Indeed for any $u \in D$, $abu - bau = -iW'u$ and so $\|Ku\| \leq \|W'\|_\infty \|u\|$. Thus we can construct the minimal q.d.s. $(\mathcal{T}_t)_{t\geq 0}$ satisfying (3.7). Since $V(x)\frac{d}{dx}$ is infinitesimally small with respect to $-\frac{d^2}{dx^2}$, for any $V(x) \in C_0^\infty(\mathbf{R})$, we have $D(G) = D(a^2+b^2)(= D(b^2))$. Therefore Theorem 3.5 implies that the minimal q.d.s. $(\mathcal{T}_t)_{t\geq 0}$ is conservative.

REMARK 5.1. In the case a is a bounded self adjoint operator, we would like to mention that one can also use the method of [2] to construct the Markov semigroup generated by in noncommutative elliptic operator $\mathcal{L}$.

In the following example we apply Theorem 3.6.

EXAMPLE 5.2. Let $\mathfrak{h} = l_2(\mathbb{C})$ be the space of sequences $(\alpha_n)_{n=0}^{\infty}$ of complex numbers satisfying $\sum_{n=0}^{\infty} |\alpha_n|^2 < \infty$ and $\{e_n\}_{n=0}^{\infty}$ be the standard orthonormal basis in $\mathfrak{h}$. Denote by N, A, A^* the number, annihilation and creation operators on $\mathfrak{h}$ defined as follows. The number operator N is the self-adjoint multiplication operator $Ne_n = ne_n$, with maximal domain $\{(\alpha_n)_{n=0}^{\infty} \in \mathfrak{h} | \sum_{n\geq 0} |n\alpha_n|^2 < \infty\}$. The annihilation and creation operators are given by $D(A) = D(A^*) = D(\sqrt{N})$ with

$$Ae_n = \begin{cases} \sqrt{n}e_{n-1} & \text{if } n = 1, 2, ..., \\ 0 & \text{if } n = 0 \end{cases}$$

and $A^*e_n = \sqrt{n+1}e_{n+1}$. Let D be the subspace of the finite linear combinations of e_n's. Then $A^*A = AA^* - \mathbf{1} = N$ and D is an invariant core for A, A^* and N.

Let

$$a = A + A^*, \quad b = -i(A - A^*). \tag{5.1}$$

Then D is an invariant core for both a and b, and $a^2 + b^2 = 2(2N + 1)$ is a self-adjoint operator with core D.

We write that for $u \in D$,

$$\begin{aligned} Ku &= -i[a, b]u = 2u, \\ 2Hu &= (ab + ba)u = -2i(A^2 - (A^*)^2)u, \\ L^*Lu &= (a^2 + b^2 + K)u \\ &= 4(N + 1)u = 4(A^*A + 1)u \end{aligned} \tag{5.2}$$

and so

$$\begin{aligned} Gu &= -\frac{1}{2}L^*Lu - iHu \\ &= -2(A^*A + 1)u - (A^2 - (A^*)^2)u. \end{aligned} \tag{5.3}$$

Since Assumption 3.1 and Assumption 3.3 hold, Proposition 3.4 implies that the closure G of $-iH - \frac{1}{2}L^*L$ is the generator of a strongly continuous contraction semigroup. And by (5.2) $[K, L^*] = 0$, which implies that the condition (3.10) is clearly satisfied.

To check that D is a core for G^2, consider $-G^2$. Notice that for each l-particle vector e_l,

$$\|A_1^{\#}A_2^{\#}\cdots A_n^{\#}e_l\| \le (l+1)^{1/2}\cdots(l+n)^{1/2} \le \Big((l+n)!\Big)^{1/2}, \tag{5.4}$$

where $A_i^{\#}$ is either A or A^*. By (5.3), $(-G^2)^n$ has 16^n terms and each of them is a product of less than or equal to $4n$ operators A or A^*. So by tedious but elementary calculation (similar to that of Example 2 in p.204 of [20]), we have

$$\|(-G^2)^n e_l\| \le k^n 16^n \Big((l+4n)!\Big)^{1/2}, \quad n = 1, 2, \cdots,$$

where k is a constant. So e_l is an analytic vector for $-G^2$ and D is a dense subset of invariant, analytic vectors for $-G^2$. We should show the dissipativity of the operator $-G^2$. Let $u \in D$. It is followed from (5.3) that we have

$$\begin{aligned} 2\mathrm{Re}\langle u, -G^2u\rangle &= -\frac{1}{2}\langle u, (L^*L)^2u\rangle + 2\langle u, H^2u\rangle \\ &= -8\langle u, (N+1)^2u\rangle \\ &\quad + 2\langle u, \big(A^2(A^*)^2 + (A^*)^2A^2 - A^4 - (A^*)^4\big)u\rangle. \end{aligned} \tag{5.5}$$

Using $AA^* - A^*A = 1$ and Schwarz inequality, we get

$$\begin{aligned} A^2(A^*)^2 + (A^*)^2A^2 &= 2(N^2+N+1), \\ -(A^4 + (A^*)^4) &\le A^2(A^*)^2 + (A^*)^2A^2 \end{aligned} \tag{5.6}$$

on D. Substituting (5.6) into (5.5), we have

$$2Re\langle u, -G^2u\rangle \le -8\langle u, Nu\rangle \le 0,$$

which implies that $-G^2$ is dissipative. Therefore we get from Theorem 3.1.18. and Corollary 3.1.20 in [4] that D is a core for $-G^2$.

Since $C = L^*L + 1 = 4N + 5$ it is clear that $C(D) = D$. Therefore Theorem 3.6 implies the existence of the conservative minimal q.d.s. $(\mathcal{T}_t)$.

REMARK 5.2. Also we can consider the number N, annihilation A and creation A^* operators defined on $L^2(\mathbf{R}, dx)$ using Hermite functions (see Example 2 p.204 in [20]). Then the relations (5.1) may be written by

$$a = A + A^* = \sqrt{2}x, \quad b = -i(A - A^*) = -\sqrt{2}i\frac{d}{dx},$$

where x and $-i\frac{d}{dx}$ are the position and momentum operator on $L^2(\mathbf{R}, dx)$ respectively.

In the following, we construct the conservative minimal q.d.s. using Theorem 3.9.

EXAMPLE 5.3. Let $\mathfrak{h}$, A, A^* and N as in in Example 5.2. Let

$$a = A + A^*, \quad b = A^*A = N.$$

Let D be the subspace of the finite linear combinations of e_n's. Since D is an invariant domain for a, b and consists of analytic vectors for both operators, it is an invariant core for both a and b. See Theorem X.39 and Example 2 in p.204 in [20]. Similarly one can check that $a^2 + b^2$ is a self-adjoint operator with core D (actually a^2 is infinitesimal small with respect to b^2). Thus Assumption 3.1 are satisfied.

We have that on domain D,

$$K = -i(ab - ba) = i(A^* - A), \tag{5.7}$$

$$\begin{aligned} 2H &= ab + ba \\ &= A + A^* + 2((A^*)^2A + A^*A^2), \end{aligned} \tag{5.8}$$

$$\begin{aligned} L^*L &= a^2 + b^2 + K \\ &= \big(A^2 + (A^*)^2 + 2N + 1\big) + N^2 + i\big(A^* - A\big). \end{aligned} \tag{5.9}$$

By the definition of G (see (3.4)), we have

$$\begin{aligned} Gu &= -\frac{1}{2}L^*Lu - iHu \\ &= -\frac{1}{2}\Big(A^2 + (A^*)^2 + 2N + 1 + N^2 + i(A^* - A)\Big)u \\ &\quad - \frac{i}{2}\Big(A + A^* + 2((A^*)^2A + A^*A^2)\Big)u \end{aligned}$$

for $u \in D$. Using the inequality (5.4), one can check that each l-particle vector e_l is an analytic vector for G. Thus the closure of G generates a strongly continuous contraction semigroup by Theorem 3.1.18. p.179 in [4]. Assumption 3.2 holds. Denote again by G the closure of G.

Applying the inequality

$$\|A_1^{\#} \cdots A_n^{\#}u\| \le k_n\|(2A^*A + 1)^{n/2}u\|, \quad u \in D, \tag{5.10}$$

where $A_i^{\#}$ is either A or A^* and k_n is a constant, we can easily check the condition (3.11) in Theorem 3.9 since aKb, aKa (K and $[K, a]$) are polynomials of degree $n \le 4$ ($n \le 2$, respectively) of the creation and annihilation operators A^*, A. Thus the condition (iii) of Theorem 3.9 is satisfied.

For the domain condition, notice that for $u \in D$,

$$\begin{aligned}\left\|\frac{1}{2}(a^2+b^2)u\right\| &= \left\|\left(G+iH+\frac{1}{2}K\right)u\right\| \\ &\leq \|Gu\|+\|Hu\|+\frac{1}{2}\|Ku\| \\ &\leq \|Gu\|+\epsilon\|(a^2+b^2)u\|+k\|u\|,\end{aligned}$$

where ϵ is any positive constant and and k is a constant depending on ϵ. Here we have used that H, K are infinitesimal small with respect to $b^2 = N^2$. Put $\epsilon = \frac{1}{4}$. Then we have

$$\left\|\frac{1}{4}(a^2+b^2)u\right\| \leq \|Gu\|+k\|u\|, \quad u \in D.$$

Since D is a core for G, we obtain from above inequality that $D(a^2+b^2)$ contains $D(G)$ and for any $u \in D(G)$, there exists a convergent sequence (u_n) of elements of D such that both Gu_n and $(a^2+b^2)u_n$ converge, which implies the condition (ii) of Theorem 3.9.

Since $D \subset D(G)$ and $D(G) \subset D(a^2+b^2)$, $D(G)$ is a core for $C' = a^2+b^2$. Notice that $D(C')$ is a core for $C'^{1/2}$. Thus $D(G)$ is a core for $C'^{1/2}$. The condition (i) of Theorem 3.9 holds. Therefore a and b satisfy all conditions of Theorem 3.9.

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