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ON THE INCREMENTS OF A d-DIMENSIONAL GAUSSIAN PROCESS

ZHENGYAN LIN[1] AND KYO-SHIN HWANG[2]

ABSTRACT. In this paper we establish some results on the increments of a d-dimensional Gaussian process with the usual Euclidean norm. In particular we obtain the law of iterated logarithm and the Book-Shore type theorem for the increments of a d-dimensional Gaussian process, via estimating upper bounds and lower bounds of large deviation probabilities on the suprema of the d-dimensional Gaussian process.

1. Introduction and results

Limit theory on the increments of some kinds of stochastic processes, such as the Wiener process, fractional Brownian motions, multifractional Brownian motions, Ornstein-Ule-nbeck processes, l^2-valued Ornstein-Ulenbeck processes, l^p-valued Gaussian processes, l^∞-valued Gaussian processes and related processes, has been investigated in various direction by many authors, for instance, Choi[5], Choi and Kôno[6], Csáki et al.[7, 8], Csörgő and Révész[10], Kôno[14], Csörgő, Lin and Shao[9], Csörgő and Shao[11], Lin[15], Lin and Lu[17], Lin and Qin[18], Lu[19], Shao[23], Zhang[24, 25].

Furthermore, the law of iterated logarithm and the Book-Shore type theorem [4] for the increments of Gaussian processes have been studied by Arcones[1], He and Chen[13], Monrad and Rootzen[20], Révész[22] and Zhang[24, 25].

In the paper we obtain some results on the increments of a d-dimensional Gaussian process with the usual Euclidean norm $\|\cdot\|$. First of

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all we establish the law of iterated logarithm and the Book-Shore type theorem for the increments of a d-dimensional Gaussian process, via estimating upper bounds and lower bounds of large deviation probabilities on the suprema of the Gaussian process.

Let $\{X_i(t),\ 0 \le t < \infty\}$, $i = 1, \cdots, d$, be real-valued continuous and centered independent Gaussian processes with $X_i(0) = 0$ and $E\{X_i(t) - X_i(s)\}^2 = \sigma_i^2(|t-s|)$, where $\sigma_i(t)$ are positive nondecreasing continuous and regularly varying functions of $t > 0$ with exponents $\alpha_i (0 < \alpha_i < 1)$ at 0 and ∞. Hence $\sigma_i(x)/x$ is non-increasing for large x. Let $\{X^d(t) = (X_1(t), \cdots, X_d(t)),\ 0 \le t < \infty\}$ be a d-dimensional Gaussian process with the norm $\|\cdot\|$. For $0 < T < \infty$, let a_T be a positive continuous function of T with $0 \le a_T \le T$. Denote

$$\beta(a_T, T) = \{2(\log(T/a_T) + \log\log T)\}^{1/2},$$
$$\sigma(d,t) = \max_{1\le i \le d} \sigma_i(t),$$

where $\log x = \ln(\max\{x, 1\})$.

The following Theorem 1.1 was proved by Lin et. al.[16].

THEOREM 1.1. *We have*

$$\limsup_{T\to\infty} \sup_{0\le t\le T} \sup_{0\le s\le a_T} \frac{\|X^d(t+s) - X^d(t)\|}{\sigma(d,a_T)\ \beta(a_T,T)} \le 1 \text{ a.s.}$$

Our main results are as follows:

THEOREM 1.2. *Assume that $\sigma_i^2(x)$ are twice differentiable for $x > 0$ which satisfies*

(i) *both a_T and T/a_T are nondecreasing;*

(ii) $\left|\frac{d\sigma_i^2(x)}{dx}\right| \le c_1 \frac{\sigma_i^2(x)}{x}$ *and* $\left|\frac{d^2\sigma_i^2(x)}{dx^2}\right| \le c_2 \frac{\sigma_i^2(x)}{x^2}$, $i = 1, \cdots, d$

for positive constants c_1 and c_2. Then we have

$$\limsup_{T\to\infty} \frac{\|X^d(T) - X^d(T-a_T)\|}{\sigma(d,a_T)\beta(a_T,T)} \ge 1 \quad \text{a.s.}$$

THEOREM 1.3. *Suppose that*

$$\text{(iii)} \qquad \lim_{T\to\infty} \frac{\log(T/a_T)}{\log\log T} = r, \quad 0 \le r \le \infty.$$

Then

$$\liminf_{T\to\infty} \sup_{0\le t\le T} \sup_{0\le s\le a_T} \frac{\|X^d(t+s) - X^d(t)\|}{\sigma(d,a_T)\beta(a_T,T)} \le \left(\frac{r}{1+r}\right)^{1/2} \text{ a.s.}$$

THEOREM 1.4. *Assume that a_T satisfies condition* (iii) *and there exists a positive constant c_2 such that* $\left|\dfrac{d^2\sigma_i^2(x)}{dx^2}\right| \le c_2 \dfrac{\sigma_i^2(x)}{x^2}$, $i = 1, \cdots, d$. *Then we have*

$$\liminf_{T\to\infty} \sup_{0\le t\le T} \frac{\|X^d(t+a_T) - X^d(t)\|}{\sigma(d, a_T)\ \beta(a_T, T)} \ge \left(\frac{r}{1+r}\right)^{1/2} \quad \text{a.s.}$$

Combining Theorems 1.1 and 1.2, we obtain the following limsup value:

COROLLARY 1.1. *Under the assumptions of Theorem 1.2, we have*

$$\begin{aligned} &\limsup_{T\to\infty} \sup_{0\le t\le T} \sup_{0\le s\le a_T} \frac{\|X^d(t+s) - X^d(t)\|}{\sigma(d, a_T)\ \beta(a_T, T)} \\ &= \limsup_{T\to\infty} \frac{\|X^d(T) - X^d(T-a_T)\|}{\sigma(d, a_T)\beta(a_T, T)} = 1 \quad \text{a.s.} \end{aligned}$$

If, furthermore, $a_T = T$, then we have the law of iterated logarithm for a d-dimensional Gaussian process:

$$\limsup_{T\to\infty} \frac{\|X^d(T)\|}{\sigma(d, T)\ \sqrt{2\log\log T}} = 1 \quad \text{a.s.}$$

From Theorems 1.3 and 1.4, we obtain the following liminf value:

COROLLARY 1.2. *Under the assumptions of Theorem 1.4, we have*

$$\begin{aligned} &\liminf_{T\to\infty} \sup_{0\le t\le T} \sup_{0\le s\le a_T} \frac{\|X^d(t+s) - X^d(t)\|}{\sigma(d, a_T)\ \beta(a_T, T)} \\ &= \liminf_{T\to\infty} \sup_{0\le t\le T} \frac{\|X^d(t+a_T) - X^d(t)\|}{\sigma(d, a_T)\ \beta(a_T, T)} = \left(\frac{r}{1+r}\right)^{1/2} \quad \text{a.s.} \end{aligned}$$

EXAMPLE. Let $\sigma_i^2(t) = t^{2\alpha_i}$, $0 < \alpha_i < 1$ for $i = 1, \cdots, d$, then $X_i(t)$ is a fractional Brownian motion. Hence $X^d(t) = (X_1(t), \cdots, X_d(t))$ is d-dimensional fractional Brownian motion, obviously, which satisfies condition (ii).

2. Proofs

We shall accomplish the proofs of our theorems through several lemmas. The following lemmas 2.1∼2.4 are essential for the proof of Theorem 1.2. Lemma 2.1 is a well-known extension of the second Borel-Cantelli lemma:

LEMMA 2.1. *Let* $\{A_k, k \geq 1\}$ *be a sequence of events. If*

(a) $\sum\limits_{k=1}^{\infty} P(A_k) = \infty$,

(b) $\liminf\limits_{n\to\infty} \sum\limits_{1\leq j<k\leq n} \dfrac{P(A_j \cap A_k) - P(A_j)P(A_k)}{\left(\sum_{j=1}^{n} P(A_j)\right)^2} \leq 0$,

then $P(A_n,$ i.o.$) = 1$.

The proof of Lemma 2.1 can be found, for example, in Theorem 6.4 in Billingsley[3]. The following Lemma 2.2 is a generalization of Slepian's lemma(cf. e.g. Berman[2]).

LEMMA 2.2. *Let* $\{X_j, j = 1, 2, \cdots, n\}$ *be centered stationary Gaussian random variables with* $EX_iX_j = r_{ij}$ *and* $r_{ii} = 1$. *Let* $I_c^{+1} = [c, \infty)$, *and* $I_c^{-1} = (-\infty, c)$. *Denote by* F_j *the event* $\{X_j \in I_{c_j}^{\varepsilon_j}\}$ *for* $c_j \in (-\infty, \infty)$, $j = 1, 2, \cdots, n$, *where* ε_j *is either* $+1$ *or* -1. *Let* $K \subset \{1, 2, \cdots, n\}$ *and* $\{K_l, l = 1, 2, \cdots, s\}$ *is a partition of* K, *then*

$$\left| P\left\{\bigcap_{j\in K} F_j\right\} - \prod_{l=1}^{s} P\left\{\bigcap_{j\in K_l} F_j\right\}\right| \leq \sum_{1\leq l<m\leq s} \sum_{i\in K_l} \sum_{j\in K_m} |r_{ij}|\, \phi(c_i, c_j; r_{ij}^*)$$

where $\phi(x, y; r)$ *is the standard bivariate normal density with correlation* r *and* r_{ij}^* *is a number between* 0 *and* r_{ij}.

LEMMA 2.3. *Assume that a function* $\sigma^2(x)$ *satisfies that* $\left|d^2\sigma^2(x)/dx^2\right| \leq c_2\sigma^2(x)/x^2$ *for some* $c_2 > 0$ *and* $\sigma^2(x)/x^2$ *is non-increasing. Let* P, Q *and* R *be positive real numbers. Then*

$$\left|\int_R^{R+P} d(\sigma^2(x)) - \int_{Q-P}^{Q} d(\sigma^2(x))\right| \leq c_2 \frac{\sigma^2(Q-P)}{(Q-P)^2} P(R+P-Q).$$

Proof. We have

$$\begin{aligned}
&\left|\int_R^{R+P} d(\sigma^2(x)) - \int_{Q-P}^{Q} d(\sigma^2(x))\right| \\
&\le \int_{Q-P}^{Q} \left|\frac{d}{dx}\left(\sigma^2(x-Q+R+P)-\sigma^2(x)\right)\right| dx \\
&\le \int_{Q-P}^{Q}\int_x^{x-Q+R+P} \left|\frac{d^2}{dy^2}\sigma^2(y)\right| dydx \\
&\le \int_{Q-P}^{Q}\int_x^{x-Q+R+P} c_2\frac{\sigma^2(y)}{y^2}dydx \\
&\le c_2\frac{\sigma^2(Q-P)}{(Q-P)^2}P(R+P-Q). \qquad \square
\end{aligned}$$

LEMMA 2.4. *Assume that for $i=1,\cdots,d$, $\sigma_i^2(x)$ is differentiable for $x>0$ and satisfies $\left|\frac{d\sigma_i^2(x)}{dx}\right| \le c_1\frac{\sigma_i^2(x)}{x}$ for a positive constant c_1. Then we have*

$$\limsup_{T\to\infty}\frac{\|X^d(T)\|}{\sigma(d,T)\sqrt{2\log\log T}} \ge 1 \quad \text{a.s.}$$

Proof. Take $i_0=i_0(T)$ such that $\sigma_{i_0}(T)=\sigma(d,T)$. Clearly,

$$\|X^d(T)\|/\sigma(d,T) \ge |X_{i_0}(T)|/\sigma_{i_0}(T).$$

Using the inequality

$$\frac{1}{\sqrt{2\pi}}\left(\frac{1}{x}-\frac{1}{x^3}\right)e^{-x^2/2} < 1-\Phi(x) < \frac{1}{\sqrt{2\pi}}\frac{1}{x}e^{-x^2/2},$$

where $\Phi(x)=P\{N(0,1)\le x\}$, we have

$$\begin{aligned}
P\left\{\frac{\|X^d(T)\|}{\sigma(d,T)} > \sqrt{2(1-\varepsilon)\log\log T}\right\} &\ge \frac{1}{2\sqrt{2\pi}}\frac{\exp\left(-(1-\varepsilon)\log\log T\right)}{\sqrt{2(1-\varepsilon)\log\log T}} \\
&\ge c_3\left(\log T\right)^{-(1-\varepsilon/2)}
\end{aligned}$$

for large T, where $c_3>0$ is a constant. Let $T_k=\theta^k$ with $\theta>1$. Then

$$\begin{aligned}
\sum_{k=1}^{\infty} P\left\{\frac{\|X^d(T)\|}{\sigma(d,T)} > \sqrt{2(1-\varepsilon)\log\log T}\right\} &\ge c_3\sum_{k=1}^{\infty}\left(\log T_k\right)^{-(1-\varepsilon/2)} \\
&= \infty.
\end{aligned}$$

Hence, in order to prove Lemma 2.4, we need to show that (b) of Lemma 2.1 holds. For $j < k$, if $i_0(T_j) \neq i_0(T_k)$, $E\{X_{i_0(T_j)}(T_j)X_{i_0(T_k)}(T_k)\} = 0$. If $i_0(T_j) = i_0(T_k)(=: i_0)$, noting that $\sigma_{i_0}^2(x)/x^2$ is non-increasing for large x, we have

$$\begin{aligned}E\left\{X_{i_0}(T_j)X_{i_0}(T_k)\right\} &= \frac{1}{2}\left(\sigma_{i_0}^2(T_j) + \sigma_{i_0}^2(T_k) - \sigma_{i_0}^2(T_k - T_j)\right)\\ &\le \frac{1}{2}\left(\sigma_{i_0}^2(T_j) + \left(1 - \frac{(T_k - T_j)^2}{T_k^2}\right)\sigma_{i_0}^2(T_k)\right)\\ &< \frac{1}{2}\left(\sigma_{i_0}^2(T_j) + \frac{2T_j}{T_k}\sigma_{i_0}^2(T_k)\right)\end{aligned}$$

and

$$\begin{aligned}&\left|E\left\{\frac{X_{i_0}(T_j)}{\sigma_{i_0}(T_j)}\frac{X_{i_0}(T_k)}{\sigma_{i_0}(T_k)}\right\}\right|\\ &\le \frac{1}{2}\frac{\sigma_{i_0}(T_j)}{\sigma_{i_0}(T_k)} + \frac{T_j}{T_k}\frac{\sigma_{i_0}(T_k)}{\sigma_{i_0}(T_j)}\\ &\le \frac{1}{2}\left(\frac{T_j}{T_k}\right)^{\alpha_{i_0}}\frac{L_{i_0}(T_j)}{L_{i_0}(T_k)} + \left(\frac{T_j}{T_k}\right)^{1-\alpha_{i_0}}\frac{L_{i_0}(T_k)}{L_{i_0}(T_j)}\\ &\le \frac{1}{2}\theta^{-\alpha_0(k-j)} + \theta^{-(1-\alpha_0')(k-j)} =: \eta_{jk}\end{aligned} \tag{1}$$

for large j, where $L_{i_0}(\cdot)$ is a slowly varying function and $0 < \alpha_0 < \min\{\alpha_i, i = 1, \cdots, d\}$, $\max\{\alpha_i, i = 1, \cdots, d\} < \alpha_0' < 1$, and the following fact on a slowly varying function $L(x)$ at the infinite has been used: for any $\varepsilon > 0$

$$\left(\frac{x}{y}\right)^{\varepsilon}\frac{L(y)}{L(x)} \to 0 \quad \text{as } \frac{x}{y} \to 0 \text{ and } x \to \infty. \tag{2}$$

We can now prove that (b) of Lemma 2.1 holds. Set

$$\begin{aligned}&Y_0(T_k) = X_{i_0}(T_k)/\sigma_{i_0}(T_k), \quad x_k = \sqrt{2(1-\varepsilon)\log\log T_k},\\ &r_{jk} = EY_0(T_j)Y_0(T_k) \quad \text{and} \quad A_k = \{Y_0(T_k) > \sqrt{2(1-\varepsilon)\log\log T_k}\}.\end{aligned}$$

By Lemma 2.2 we have, for some fixed m,

$$\begin{aligned}&\sum_{m\le j<k\le n}\left(P(A_jA_k) - P(A_j)P(A_k)\right)\\ &\le \sum_{j=m}^{n-1}\sum_{k=j+1}^{n}|r_{jk}|\phi(x_j, x_k; r_{jk}^*)\end{aligned} \tag{3}$$

$$= \sum_{j=m}^{n-1} \left(\sum_{k=j+1}^{j+\xi_j} + \sum_{k=j+\xi_j+1}^{\xi'_j} + \sum_{k=\xi'_j+1}^{n} \right) \frac{|r_{jk}|}{2\pi\sqrt{1-{r^*_{jk}}^2}}$$
$$\times \exp\left\{ -\frac{x_j^2 + x_k^2 - 2x_j x_k r^*_{jk}}{2(1-{r^*_{jk}}^2)} \right\}$$
$$=: I_1 + I_2 + I_3,$$

where $\xi_j = \left[\frac{2}{\alpha''} \log_\theta j\right]$ with $\alpha'' = \min(\alpha_0, 1-\alpha'_0)$ and $\xi'_j = (2j) \wedge n$. Note that for $j < k$,

$$-\frac{x_j^2 + x_k^2 - 2x_j x_k r}{2(1-r^2)} = -\frac{x_j^2}{2} - \frac{(x_k - x_j r)^2}{2(1-r^2)} \le -\frac{x_j^2}{2} - \frac{(1-r)x_j^2}{2(1+r)}.$$

For any $0 < \varepsilon < 1$, the first sum

$$\begin{aligned} I_1 &\le \sum_{j=m}^{n-1} \sum_{k=j+1}^{j+\xi_j} \frac{r}{2\pi\sqrt{1-r^2}} e^{-x_j^2/2} \exp\left\{ -\frac{1-r}{1+r}\frac{x_j^2}{2} \right\} \\ &\le \sum_{j=m}^{n-1} \frac{r}{2\pi\sqrt{1-r^2}} \xi_j e^{-x_j^2/2} \exp\left\{ -M \frac{2(1-\varepsilon)\log\log\theta^j}{2} \right\} \\ &\le \sum_{j=m}^{n-1} \frac{r}{2\pi\sqrt{1-r^2}} e^{-x_j^2/2} \frac{2}{\alpha''} \left(j \log\theta\right)^{-M(1-\varepsilon)} \log_\theta j \\ &\le \varepsilon \sum_{j=1}^{n} P(A_j) \end{aligned} \tag{4}$$

for large n, where $M = \frac{1-r}{1+r}$ and r is the maximum of the covariances $|r_{jk}|$ for $j, k = 1, \cdots, n$. We have $r < 1$ by noting (1) and taking θ to be large enough. Note that

$$-\frac{x_j^2 + x_k^2 - 2x_j x_k r^*_{jk}}{2(1-{r^*_{jk}}^2)} \le -\frac{x_j^2 + x_k^2 - 2x_j x_k r^*_{jk}}{2}.$$

Choosing m to be large enough, we have

$$\begin{aligned} I_2 &\le \sum_{j=m}^{n-1} \sum_{k=j+\xi_j+1}^{\xi'_j} \frac{|r_{jk}|}{2\pi\sqrt{1-{r^*_{jk}}^2}} e^{-(x_j^2+x_k^2)/2} \exp\left\{ r^*_{jk} x_j x_k \right\} \\ &\le \sum_{j=m}^{n-1} \sum_{k=j+\xi_j+1}^{\xi'_j} \frac{1}{2\pi} 2j^{-2} e^{-(x_j^2+x_k^2)/2} \end{aligned} \tag{5}$$

$$\leq \frac{\varepsilon}{2}\Big(\sum_{j=1}^{n} P(A_j)\Big)^2,$$

where we have used the fact that for $k - j > \xi_j$ and $k \leq 2j$

$$\begin{aligned}|r_{jk}| \leq \eta_{jk} &= \frac{1}{2}\theta^{-\alpha_0(k-j)} + \theta^{-(1-\alpha_0')(k-j)} \\ &\leq \frac{1}{2}j^{-2\alpha_0/\alpha''} + j^{-2(1-\alpha_0')/\alpha''} \\ &\leq \frac{3}{2}j^{-2}\end{aligned}$$

and

$$\begin{aligned}r^*_{jk}x_jx_k &\leq \eta_{jk}x_jx_k \\ &\leq 3j^{-2}\sqrt{(\log\log\theta^j)(\log\log\theta^k)} \\ &\leq 3j^{-2}\log(2j\log\theta) \longrightarrow 0 \qquad \text{as} \quad j\to\infty.\end{aligned}$$

For I_3 we have the same bound by noting that for $k - j > \xi_j$ and $k > 2j$

$$r^*_{jk}x_jx_k \leq \big(\theta^{-\alpha_0 k/2} + 2\theta^{-(1-\alpha_0')k/2}\big)\log(k\log\theta) \longrightarrow 0 \quad \text{as} \quad k\to\infty.$$

Thus, by combining these results, for any $\varepsilon > 0$ there is an m such that

$$\sum_{m\leq j<k\leq n}\big(P(A_jA_k) - P(A_j)P(A_k)\big) \leq \varepsilon\Big(\sum_{j=1}^{n}P(A_j) + \Big(\sum_{j=1}^{n}P(A_j)\Big)^2\Big),$$

which implies (b) of Lemma 2.1. Therefore Lemma 2.4 is proved. □

Proof of Theorem 1.2. Let $i_0 = i_0(a_T)$ and

$$Z_0(T, a_T) = \frac{X_{i_0}(T) - X_{i_0}(T - a_T)}{\sigma_{i_0}(a_T)}.$$

Then, for any $0 < \varepsilon < 1$, we have

$$\begin{aligned}&P\left\{\frac{\|X^d(T) - X^d(T - a_T)\|}{\sigma(d, a_T)} > (1-\varepsilon)\big(2\big(\log(T/a_T) + \log\log T\big)\big)^{1/2}\right\} \\ &\geq P\Big\{|Z_0(T, a_T)| > (1-\varepsilon)\big(2\big(\log(T/a_T) + \log\log T\big)\big)^{1/2}\Big\} \\ &\geq \frac{1}{2\sqrt{2\pi}}\frac{\exp\big\{-(1-\varepsilon)^2\big(\log(T/a_T) + \log\log T\big)\big\}}{\big\{2(1-\varepsilon)\big(\log(T/a_T) + \log\log T\big)\big\}^{1/2}} \\ &\geq \Big(\frac{a_T}{T\log T}\Big)^{1-\varepsilon}\end{aligned}$$

for T large. Let $T_1 = 1$ and define T_{k+1} by $T_{k+1} - a_{T_{k+1}} = T_k$ if $\rho < 1$, $T_k = \theta^k$ if $\rho = 1$, where $\lim_{T\to\infty} a_T/T = \rho$ and $\theta > 1$. In the case of $\rho = 1$, then necessarily $a_T = T$ and $\|X^d(T) - X^d(T - a_T)\| = \|X^d(T)\|$. By Lemma 2.4, the conclusion of the theorem holds. Consider the case of $\rho < 1$. Set

$$B_k = \Big\{Z_0(T_k, a_{T_k}) > \Big(2(1-\varepsilon)\big(\log(T_k/a_{T_k}) + \log\log T_k\big)\Big)^{1/2}\Big\},\ k \geq 2,$$

then

$$\sum_{k=2}^{\infty} P(B_k) \geq \sum_{k=2}^{\infty} \left(\frac{a_{T_k}}{T_k \log T_k}\right)^{1-\varepsilon} = \infty.$$

Hence, in order to prove our result, we need to show that (b) of Lemma 2.1 holds. Without loss of generality, assume that $a_1 < 1$. By condition (i) and the definition of T_k, we have $T_k(1 - a_1) \leq T_{k-1}, a_{T_k} \leq (1 - a_1)^{-1} a_{T_{k-1}}$ and $\sum_{m=j+1}^{k-1} a_{T_m} \geq (k-j-1) a_{T_{j+1}}$. For $k \geq j+2$, if $i_0(a_{T_j}) \neq i_0(a_{T_k})$, $E\{Z_0(T_j, a_{T_j}) Z_0(T_k, a_{T_k})\} = 0$. If $i_0(a_{T_j}) = i_0(a_{T_k}) (:= i_0)$, then by Lemma 2.3, we have

$$\begin{aligned}
r^0_{jk} :&= E\{Z_0(T_j, a_{T_j}) Z_0(T_k, a_{T_k})\} \\
&= \frac{1}{2\sigma_{i_0}(a_{T_j})\sigma_{i_0}(a_{T_k})} E\Big\{\sigma^2_{i_0}(T_k - T_{j-1}) - \sigma^2_{i_0}(T_k - T_j) \\
&\qquad\qquad - \big(\sigma^2_{i_0}(T_{k-1} - T_{j-1}) - \sigma^2_{i_0}(T_{k-1} - T_j)\big)\Big\} \\
&= \frac{1}{2\sigma_{i_0}(a_{T_j})\sigma_{i_0}(a_{T_k})} \left[\int_{T_k - T_j}^{T_k - T_{j-1}} d\sigma^2_{i_0}(x) - \int_{T_{k-1} - T_j}^{T_{k-1} - T_{j-1}} d\sigma^2_{i_0}(x)\right] \\
&= \frac{1}{2\sigma_{i_0}(a_{T_j})\sigma_{i_0}(a_{T_k})} \left[\int_{\sum_{m=j+1}^{k} a_{T_m}}^{\sum_{m=j}^{k} a_{T_m}} d\sigma^2_{i_0}(x) - \int_{\sum_{m=j+1}^{k-1} a_{T_m}}^{\sum_{m=j}^{k-1} a_{T_m}} d\sigma^2_{i_0}(x)\right] \\
&\leq c_2 \frac{a_{T_j} a_{T_k}}{\sigma_{i_0}(a_{T_j})\sigma_{i_0}(a_{T_k})} \frac{\sigma^2_{i_0}\big(\sum_{m=j+1}^{k-1} a_{T_m}\big)}{\big(\sum_{m=j+1}^{k-1} a_{T_m}\big)^2}.
\end{aligned}$$

Note that for large k,

$$\begin{aligned}
\frac{a_{T_k}}{\sigma_{i_0}(a_{T_k})} \frac{\sigma_{i_0}(\sum_{m=j+1}^{k-1} a_{T_m})}{\sum_{m=j+1}^{k-1} a_{T_m}} &= \left(\frac{a_{T_k}}{\sum_{m=j+1}^{k-1} a_{T_m}}\right)^{1-\alpha_{i_0}} \frac{L_{i_0}(\sum_{m=j+1}^{k-1} a_{T_m})}{L_{i_0}(a_{T_k})} \\
&\leq (1 - a_1)^{-(1-\alpha_0)}.
\end{aligned}$$

We have

$$\begin{aligned} r_{jk}^0 &\le c_2 \frac{(1-a_1)^{-(1-\alpha_0)} a_{T_j}}{\sigma_{i_0}(a_{T_j})} \frac{\sigma_{i_0}\big((k-j-1)a_{T_{j+1}}\big)}{(k-j-1)a_{T_{j+1}}} \\ &\le c_3 \frac{a_{T_j}}{(k-j-1)a_{T_{j+1}}} \frac{\sigma_{i_0}\big((1-a_1)^{-1}(k-j-1)a_{T_j}\big)}{\sigma_{i_0}(a_{T_j})} \\ &\le c_4(k-j-2)^{\alpha_0-1} \end{aligned}$$

for k large, where c_3 and $c_4 > 0$ are constants. Then along the lines of proof corresponding to that of Theorem 1 in Ortega[21], we obtain that (b) of Lemma 2.1 holds. Theorem 1.2 is proved. □

Using another version of Fernique's lemma [12] on the d-dimensional Gaussian process, the following lemma estimates an upper bound of the large deviation probability (cf. [16]).

LEMMA 2.5. *For any given $\varepsilon > 0$, there exists a positive constant C_ε depending only on ε such that for all $x > 1$*

$$\begin{aligned} &P\left\{\sup_{0\le t\le T}\ \sup_{0\le s\le a_T} \frac{\|X^d(t+s)-X^d(t)\|}{\sigma(d,a_T)} \ge x\right\} \\ &\le C_\varepsilon\left(\frac{T}{a_T}\right)\Phi_d\left(\frac{2}{2+\varepsilon}x\right), \end{aligned} \tag{6}$$

where $\Phi_d(x) = P\{\|N^d(0,1)\| \ge x\}$ and $N^d(0,1)$ denotes a d-dimensional standard normal random vector.

Proof of Theorem 1.3. First suppose that $0 < r \le \infty$. By condition (iii), for any ε, $0 < \varepsilon < 1$

$$\frac{T}{a_T} \ge (\log T)^{r-\varepsilon}$$

for sufficiently large T. Note that for sufficiently large $x > 0$ we have

$$\Phi_d(x) \le Cx^{d-2}e^{-x^2/2} \le C\exp\Big(-\frac{x^2}{2+\varepsilon}\Big)$$

for some $C > 0$, then by Lemma 2.5, we have

$$\begin{aligned} &P\left\{\sup_{0\le t\le T}\ \sup_{0\le s\le a_T} \frac{\|X^d(t+s)-X^d(t)\|}{\sigma(d,a_T)\{2\log(T/a_T)\}^{1/2}} \ge 1+2\varepsilon\right\} \\ &\le C_\varepsilon \frac{T}{a_T}\Phi_d\left(\frac{2(1+2\varepsilon)}{2+\varepsilon}\sqrt{2\log(T/a_T)}\right) \end{aligned}$$

$$\le C_\varepsilon \frac{T}{a_T} \exp\big(-(1+\varepsilon)\log\big(T/a_T\big)\big)$$
$$\le C_\varepsilon\big(\log T\big)^{-\varepsilon(r-\varepsilon)} \longrightarrow 0 \quad \text{as} \quad T \to \infty.$$

Hence there exists a sequence $\{T_n\}$ such that

$$\liminf_{n\to\infty} \sup_{0\le t\le T_n} \sup_{0\le s\le a_{T_n}} \frac{\|X^d(t+s)-X^d(t)\|}{\sigma(d,a_{T_n})\{2\log\big(T_n/a_{T_n}\big)\}^{1/2}} \le 1+2\varepsilon \quad \text{a.s.}$$

Hence, by condition (iii), we obtain

$$\text{(7)} \qquad \liminf_{n\to\infty} \sup_{0\le t\le T_n} \sup_{0\le s\le a_{T_n}} \frac{\|X^d(t+s)-X^d(t)\|}{\sigma(d,a_{T_n})\ \beta(T_n,a_{T_n})} \le \sqrt{\frac{r}{1+r}} \quad \text{a.s.}$$

Next consider the case of $r=0$. It follows from (iii) that for small $\varepsilon>0$, $T/a_T < \big(\log T\big)^{\varepsilon/(2(2-\varepsilon))}$ for large T. Applying Lemma 2.5, we get

$$P\left\{\sup_{0\le t\le T} \sup_{0\le s\le a_T} \frac{\|X^d(t+s)-X^d(t)\|}{\sigma(d,a_T)\beta(T,a_T)} \ge \epsilon\right\}$$
$$\le C_\varepsilon\Big(\frac{T}{a_T}\Big)\exp\Big(-\frac{\varepsilon}{2}\log\Big(\frac{T}{a_T}\log T\Big)\Big)$$
$$\le C_\varepsilon\big(\log T\big)^{-\varepsilon/4} \longrightarrow 0 \quad \text{as} \quad T\to\infty$$

and hence there exists a sequence $\{T_n\}$ such that

$$\text{(8)} \qquad \liminf_{n\to\infty} \sup_{0\le t\le T_n} \sup_{0\le s\le a_{T_n}} \frac{\|X^d(t+s)-X^d(t)\|}{\sigma(d,a_{T_n})\ \beta(T_n,a_{T_n})} \le 0 \quad \text{a.s.}$$

Combining (7) and (8) completes the proof of Theorem 1.3. □

Proof of Theorem 1.4. When $r=0$, our result is trivial and the result for the case of $r=\infty$ was proved by Lin et al.[16]. We here consider the case of $0<r<\infty$. For $\theta>1$ and integers k and j, let

$$\text{(9)} \qquad A_{kj} = \{T : \theta^{k-1} < T \le \theta^k,\ \theta^{j-1} < a_T \le \theta^j\}.$$

For any $0 < \tau < 1$, by condition (iii), $I_k := k - \left[\frac{r(1+\tau)}{\log\theta}\log k\right] \le j \le k - \left[\frac{r(1-\tau)}{\log\theta}\log k\right] =: I'_k$ provided k is large enough. Clearly

$$\begin{aligned}\inf_{T\in A_{kj}} \beta(T, a_T) &\ge \left(2\log\left(\frac{\theta^{k-1}}{\theta^j}\log\theta^{k-1}\right)\right)^{1/2} \\ &\ge \theta^{-1}\left(2\log\left(\frac{\theta^k}{\theta^j}\log\theta^k\right)\right)^{1/2} \\ &=: \theta^{-1}\beta_{kj}.\end{aligned}$$

For some $M > 0$ set $N_{k,j} = [\theta^{k-1}/(M\theta^j)]$. By (iii) for large k, we have

$$\begin{aligned}\sup_{T\in A_{kj}} \beta(T, a_T) &\le \left\{2\log\left(\frac{\theta^k}{\theta^{j-1}}\log\theta^k\right)\right\}^{1/2} \\ &\le \left\{2\theta^2\left(\frac{1+r}{r}\right)\log N_{k,j}\right\}^{1/2}.\end{aligned}$$

By the regular variation of $\sigma_i(\cdot)$, $i = 1, \cdots, d$, we have

$$\sigma_i(\theta^{j-1}) \ge (\theta-1)^{-\alpha'_0}\sigma_i(\theta^j - \theta^{j-1}).$$

Thus

$$\begin{aligned}&\liminf_{T\to\infty}\sup_{0\le t\le T}\frac{\|X^d(t+a_T) - X^d(t)\|}{\sigma(d, a_T)\,\beta(T, a_T)} \\ &\ge \liminf_{k\to\infty}\inf_{I_k\le j\le I'_k}\inf_{T\in A_{kj}}\sup_{0\le t\le T}\frac{\|X^d(t+a_T) - X^d(t)\|}{\sigma(d, a_T)\,\beta(T, a_T)} \\ &\ge \liminf_{k\to\infty}\inf_{I_k\le j\le I'_k}\sup_{0\le t\le\theta^{k-1}}\frac{\|X^d(t+\theta^j) - X^d(t)\|}{\sigma(d,\theta^j)\theta\{2(1+1/r)\log N_{k,j}\}^{1/2}} \qquad (10)\\ &\quad - \limsup_{k\to\infty}\sup_{I_k\le j\le I'_k}\sup_{0\le t\le\theta^k}\sup_{\theta^{j-1}\le s\le\theta^j}\frac{(\theta-1)^{\alpha'_0}}{\sigma(d,\theta^j-\theta^{j-1})\,\theta^{-1}\beta_{kj}} \\ &\quad \times\|X^d(t+\theta^j) - X^d(t+s)\| \\ &=: H_1/(\theta(1+1/r)^{1/2}) - (\theta-1)^{\alpha'_0}H_2.\end{aligned}$$

At first, we will show that for any $R > 2$,

$$H_2 \le R \quad \text{a.s.} \qquad (11)$$

Take θ being close to 1 such that $\theta^{-1}R \geq 2$. By the same way as the proof of Lemma 2.5, we obtain

$$P\left\{\sup_{I_k\leq j\leq I'_k}\ \sup_{0\leq t\leq\theta^k}\ \sup_{\theta^{j-1}\leq s\leq\theta^j}\frac{\|X^d(t+\theta^j)-X^d(t+s)\|}{\sigma(d,\theta^j-\theta^{j-1})}\geq\theta^{-1}R\beta_{kj}\right\}$$
$$\leq C_\varepsilon\sum_{j=I_k}^{I'_k}\frac{\theta^k}{\theta^j-\theta^{j-1}}\exp\left\{-\frac{8}{2+\varepsilon}\log\left(\theta^{k-j}\log\theta^k\right)\right\}$$
$$\leq C_\varepsilon k^{-2},$$

where $C_\varepsilon > 0$ is a constant.

By the Borel-Cantelli lemma, we obtain (11).

Consider H_1. Let $i_0 = i_0(\theta^j)$ and

$$W_0(j;l)=\frac{X_{i_0}(lM\theta^j+\theta^j)-X_{i_0}(lM\theta^j)}{\sigma_{i_0}(\theta^j)},\quad 1\leq l\leq N_{k,j},$$

then $W_0(j;l)$ is a standard normal random variable. We have

$$H_1\geq\liminf_{k\to\infty}\ \min_{I_k\leq j\leq I'_k}\ \max_{1\leq l\leq N_{k,j}}\frac{\|X^d(lM\theta^j+\theta^j)-X^d(lM\theta^j)\|}{\sigma(d,\theta^j)\left(2\log N_{k,j}\right)^{1/2}}$$
$$\geq\liminf_{k\to\infty}\ \min_{I_k\leq j\leq I'_k}\ \max_{1\leq l\leq N_{k,j}}\ \max_{1\leq i\leq d}\frac{|X_i(lM\theta^j+\theta^j)-X_i(lM\theta^j)|}{\sigma(d,\theta^j)\left(2\log N_{k,j}\right)^{1/2}}\tag{12}$$
$$\geq\liminf_{k\to\infty}\ \min_{I_k\leq j\leq I'_k}\ \max_{1\leq l\leq N_{k,j}}\frac{W_0(j;l)}{\left(2\log N_{k,j}\right)^{1/2}}=:H_3.$$

Let us estimate a lower bound of H_3. Using the elementary relation $ab=(a^2+b^2-(a-b)^2)/2$ and Lemma 2.3, it follows that for $l>l'$

$$|r_{ll'}|:=\left|\mathrm{Cov}\big(W_0(j;l),W_0(j;l')\big)\right|$$
$$=\frac{1}{\sigma_{i_0}^2(\theta^j)}\Big|E\Big\{X_{i_0}(Ml\theta^j+\theta^j)X_{i_0}(Ml'\theta^j+\theta^j)$$
$$-X_{i_0}(Ml\theta^j+\theta^j)X_{i_0}(Ml'\theta^j)$$
$$-X_{i_0}(Ml\theta^j)X_{i_0}(Ml'\theta^j+\theta^j)+X_{i_0}(Ml\theta^j)X_{i_0}(Ml'\theta^j)\Big\}\Big|$$
$$\leq\frac{1}{2\sigma_{i_0}^2(\theta^j)}\Big|\Big(\sigma_{i_0}^2\big(M(l-l')\theta^j)+\theta^j\big)-\sigma_{i_0}^2\big(M(l-l')\theta^j\big)\Big)$$
$$-\Big(\sigma_{i_0}^2\big(M(l-l')\theta^j\big)-\sigma_{i_0}^2\big(M(l-l')\theta^j-\theta^j\big)\Big)\Big|\tag{13}$$

$$
\begin{aligned}
&= \frac{1}{2\sigma_{i_0}^2(\theta^j)} \left| \int_{M(l-l')\theta^j}^{M(l-l')\theta^j+\theta^j} d(\sigma_{i_0}^2(x)) - \int_{M(l-l')\theta^j-\theta^j}^{M(l-l')\theta^j} d(\sigma_{i_0}^2(x)) \right| \\
&\le \frac{c_2}{2\sigma_{i_0}^2(\theta^j)} \frac{\sigma_{i_0}^2(M(l-l')\theta^j-\theta^j)}{(M(l-l')-1)^2} \\
&\le c_5\,|M(l-l')-1|^{2\alpha_0-2} < \delta|l-l'|^{-\nu}
\end{aligned}
$$

for any given $\delta > 0$ provided M is large enough, where $0 < \nu = 2-2\alpha_0$. Let $\eta_l, l = 1, \cdots, N_{k,j}$, and ξ be independent normal variables with $E\eta_l = E\xi = 0$ and $E\eta_l^2 = 1-\delta$ and $E\xi^2 = \delta$. Then by Slepian's lemma, we have

$$
\begin{aligned}
&P\left\{ \max_{1\le l\le N_{k,j}} \frac{W_0(j;l)}{\left(2\log N_{k,j}\right)^{1/2}} \le 1-3\varepsilon \right\} \\
&\le P\left\{ \max_{1\le l\le N_{k,j}} \eta_l + \xi \le (1-3\varepsilon)(2\log N_{k,j})^{1/2} \right\} \\
&\le P\left\{ \max_{1\le l\le N_{k,j}} \eta_l \le (1-2\varepsilon)(2\log N_{k,j})^{1/2} \right\} + P\{\xi \ge \varepsilon(2\log N_{k,j})^{1/2}\} \\
&\le \left\{1-\exp\left(-(1-\varepsilon)\log N_{k,j}\right)\right\}^{N_{k,j}} + \exp\left\{-\frac{\varepsilon^2}{\delta}\log N_{k,j}\right\} \\
&\le \exp\left\{-N_{k,j}^{\varepsilon}\right\} + N_{k,j}^{-\varepsilon^2/\delta} \\
&\le 2\left(\theta^{k-j-1}/M\right)^{-\varepsilon^2/\delta}
\end{aligned}
$$

and taking $\delta \le \varepsilon^2 r(1-\tau)/2$, we obtain

$$
\begin{aligned}
&P\left\{ \min_{I_k\le j\le I'_k} \max_{1\le l\le N_{k,j}} \frac{W_0(j;l)}{\left(2\log N_{k,j}\right)^{1/2}} \le \sqrt{1-\varepsilon} \right\} \\
&\le 2\sum_{I_k\le j\le I'_k} \left(\theta^{k-1-j}/M\right)^{-\varepsilon^2/\delta} \le c_6\theta^{-2(\log\theta)^{-1}\log k} = c_6 k^{-2},
\end{aligned}
$$

which implies

$$
\sum_{k=1}^{\infty} P\left\{ \min_{I_k\le j\le I'_k} \max_{1\le l\le N_{k,j}} \frac{W_0(j;l)}{\left(2\log N_{k,j}\right)^{1/2}} \le 1-3\varepsilon \right\} < \infty
$$

and hence by the Borel-Cantelli lemma, we have

$$
\liminf_{k\to\infty} \min_{I_k\le j\le I'_k} \max_{1\le l\le N_{k,j}} \frac{W_0(j;l)}{\left(2\log\left(N_{k,j}\right)\right)^{1/2}} > 1-3\varepsilon \quad \text{a.s.}
$$

Since ϵ is arbitrary, we obtain

$$H_3 \geq 1 \quad \text{a.s.} \tag{14}$$

Combining (11), (12), (14) with (10), the proof is completed. □

References

[1] M. A. Arcones, *On the law of the iterated logarithm for Gaussian processes*, J. Theoret. Probab. **8** (1995), no. 4, 877–903.

[2] S. M. Berman, *Limit theorems for the maximum term in stationary sequence*, Ann. Math. Statist. **35** (1964), 502–616.

[3] P. Billingsley, *Probability and Measure*, J. Wiley & Sons, New York, 1986.

[4] S. A. Book and T. R. Shore, *On large intervals in the Csőrgö-Révész theorem on increments of a Wiener process*, Z. Wahrsch. verw. Gebiete **46** (1978), 1–11.

[5] Y. K. Choi, *Erdős-Rényi-type laws applied Gaussian processes*, J. Math. Kyoto Univ. **31** (1991), no. 3, 191–217.

[6] Y. K. Choi and N. Kôno, *How big are the increments of a two-parameter Gaussian process?*, J. Theoret. Probab. **12** (1999), no. 1, 105–129.

[7] E. Csáki, M. Csörgő, Z. Y. Lin, and P. Révész, *On infinite series of independent Ornstein-Uhlenbeck processes*, Stochastic Process Appl. **39** (1991), 25–44.

[8] E. Csáki, M. Csörgő, and Q. M. Shao, *Fernique type inequalities and moduli of continuity for l^2-valued Ornstein-Uhlenbeck processes*, Ann. Inst. H. Poincare **28** (1992), no. 4, 479–517.

[9] M. Csörgő, Z. Y. Lin, and Q. M. Shao, *Path properties for l^∞-valued Gaussian processes*, Proc. Amer. Math. Soc. **121** (1994), 225–236.

[10] M. Csörgő and P. Révész, *Strong Approximations in Probability and Statistic*, Academic Press, New York, 1981.

[11] M. Csörgő and Q. M. Shao, *Strong limit theorems for large and small increments of l^p-valued Gaussian processes*, Ann. Probab. **21** (1993), no. 4, 1958–1990.

[12] X. Fernique, *Continuité des processus Gaussiens*, C. R. Math. Acad. Sci. Paris **258** (1964), 6058–6060.

[13] F. X. He and B. Chen, *Some results on increments of the Wiener process*, Chinese J. Appl. Probab. Statist. **5** (1989), 317–326.

[14] N. Kôno, *The exact modulus of continuity for Gaussian processes taking values of a finite dimensional normed space* in: Trends in Probability and Related Analysis, SAP'96, World Scientific, Singapore, 1996, pp. 219–232.

[15] Z. Y. Lin, *How big the increments of a multifractional Brownian motion?*, Sci. China Ser. A **45** (2002), no. 10, 1291–1300.

[16] Z. Y. Lin, K. S. Hwang, S. Lee, and Y. K. Choi, *Path properties of a d-dimensional Gaussian process*, Statist. Probab. Lett. **68** (2004), 383–393.

[17] Z. Y. Lin and C. R. Lu, *Strong Limit Theorems*, Science Press, Kluwer Academic Publishers, Hong Kong, 1992.

[18] Z. Y. Lin and Y. C. Qin, *On the increments of l^∞-valued Gaussian processes*, Asym. Methods in Probab. and Statist.(Ottawa), Elsevier, 1998, 293–302.

[19] C. R. Lu, *Some results on increments of Gaussian processes*, Chinese J. Appl. Probab. Statist. **2** (1986), 59–65.
[20] D. Monrad and H. Rootzen, *Small values of Gaussian processes and functional laws of the iterated logarithm*, Probab. Theory Related Fields **101** (1995), 173–192.
[21] J. Ortega, *On the size of the increments of non-stationary Gaussian processes*, Stochastic Process Appl. **18** (1984), 47–56.
[22] P. Révész, *A generalization of Strassen's funtional law of iterated logarithm*, Z. Wahrsch. verw. Gebiete **50** (1979b), 257–264.
[23] Q. M. Shao, *p-variation of Gaussian processes with stationary increments*, Studia Sci. Math. Hungar. **31** (1996), 237–247.
[24] L. X. Zhang, *A Note on liminfs for increments of a fractional Bwownian motion*, Acta Math. Hungar. **76** (1997), no. 1–2, 145–154.
[25] ———, *Some liminf results on increments of fractional Brownian motion*, Acta Math. Hungar. **17** (1996), 209–234.

Zhengyan Lin
Department of Mathematics
Zhejiang University
Hangzhou 310028, P.R.China
E-mail: zlin@zju.edu.cn

Kyo-Shin Hwang
Department of Mathematics
Zhejiang University
Hangzhou 310028, P.R.China
and
Research Institute of Natural Science
Geongsang National University
Jinju 660-701, Korea
E-mail: hwang0412@naver.com