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EIGENVALUES OF COUNTABLY CONDENSING ADMISSIBLE MAPS

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ABSTRACT. Applying a fixed point theorem for compact admissible maps due to Górniewicz, we prove that under certain conditions each countably condensing admissible maps in Fréchet spaces has a positive eigenvalue. This result has many consequences, including the well-known theorem of Krasnoselskii.

1. Introduction

In the last thirty years the study of condensing or more generally, countably condensing operators has been one of the main objects of research in nonlinear analysis; see [1, 2, 3, 7, 19]. Using some fixed point theorems for countably condensing operators, Mönch[15] found solutions of boundary value problems for nonlinear differential equations in Banach spaces. In this respect, we attach great importance to countably condensing operators.

The possible existence of an eigenvalue plays a decisive role in the study of nonlinear operators. The following eigenvalue theorem originates from Krasnoselskii[11] :

KRASNOSELSKII'S THEOREM. *Let W be an open bounded neighborhood of 0 in a Banach space E, K a closed cone in E, and $f : \partial W \cap K \to K$ a compact continuous single-valued function. Suppose that there is a real number $a > 0$ such that $\| f(x) \| \geq a$ for all $x \in \partial W \cap K$. Then f has a positive eigenvalue.*

There are various ways to extend Krasnoselskii's theorem to noncompact single-valued maps or set-valued maps and to general topological

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vector spaces; see e.g., [3, 7, 8, 12]. For many applications, it is natural to investigate nonlinear eigenvalue problem for condensing set-valued maps that reduces to the Krasnoselskii's eigenvalue theorem. Following the basic idea of Hahn[7], it is shown in [10] that each condensing admissible map has a positive eigenvalue under certain conditions, where the admissibility is due to Górniewicz[4].

The aim of this paper is to prove the following result on eigenvalues of countably condensing admissible maps in Fréchet spaces. The method is to use a fixed point theorem for compact admissible maps due to Górniewicz[5], based on the fact that there is a compact fundamental set for a countably condensing map due to Väth[19, 21].

Let W be a closed neighborhood of 0 in a Fréchet space E and K a closed wedge in E. Let γ be a measure of noncompactness on K and k a positive real number. Suppose that $F : W \cap K \to \kappa(K)$ is a countably (γ, k)-condensing admissible map and there is a real number $a > k$ such that $F(W \cap K) \cap a(\operatorname{int} W) = \emptyset$. Then there exist a point $x_0 \in \partial W \cap K$ and a real number $\lambda_0 \geq a$ such that $\lambda_0 x_0 \in F(x_0)$.

Moreover, we show that an eigenvalue result of Hahn[7] for condensing set-valued maps with compact convex values can be generalized to countably condensing admissible maps; see also [3, 10]. Thus, our results have many consequences, including the well-known theorem of Krasnoselskii[11].

In addition, some results on positive eigenvalues of countably contractions defined on a normal cone in a Banach space can be found in [9], where degree theory due to Väth[20] is used. For the case of contractions, we refer to Massabo and Stuart[14] and Reich[16].

For a subset K of a topological vector space E, the interior, the closure, the boundary, the convex hull, and the closed convex hull of K in E are denoted by $\operatorname{int} K$, $\overline{K}$, ∂K, $\operatorname{co} K$, and $\overline{\operatorname{co}}\, K$, respectively. In what follows, $\kappa(K)$ denotes the collection of all nonempty compact subsets of K and $\kappa c(K)$ the collection of all nonempty compact convex subsets of K, respectively.

A set K in E is called a *wedge* if $ax + by \in K$ whenever $a, b \in [0, \infty)$ and $x, y \in K$. A wedge K is called a *cone* if $K \cap (-K) = \{0\}$; see [3].

For arbitrary topological spaces X and Y, a map $F : X \to \kappa(Y)$ is said to be *upper semicontinuous* if for any open set V in Y, the set $\{x \in X : Fx \subset V\}$ is open in X. A map $F : X \to \kappa(Y)$ is said to be *compact* if its range $F(X)$ is contained in a compact subset of Y.

An upper semicontinuous map $F : X \multimap \kappa(Y)$ is said to be *admissible* if there exist a topological space Z and continuous functions $p : Z \to X$ and $q : Z \to Y$ with the following properties:

(1) $\emptyset \neq q(p^{-1}(x)) \subset F(x)$ for each $x \in X$;
(2) p is proper; that is, the inverse image $p^{-1}(A)$ of any compact set $A \subset X$ is compact; and
(3) for each $x \in X$, the set $p^{-1}(x)$ is acyclic.

Here a nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish.

Every upper semicontinuous map $F : X \multimap \kappa(Y)$ with acyclic values is admissible. In particular, any continuous single-valued function is admissible ; see [4–6].

We introduce the concept of countably condensing maps due to Väth [19, 21] that are main objects of this investigation.

Let K be a closed convex set in a Fréchet space E and Φ a collection of subsets of K containing all precompact sets with the property that for any $M \in \Phi$, the sets $\overline{\text{co}}\, M, M \cup \{x\}(x \in K), tM(t > 0)$, and every subset of M belong to Φ.

A function $\gamma : \Phi \to [0, \infty]$ is called a *measure of noncompactness* on K provided that the following conditions hold for any $M \in \Phi$:

(1) $\gamma(\overline{\text{co}}\, M) = \gamma(M)$;
(2) $\gamma(N) \leq \gamma(M)$ if $N \subset M$;
(3) $\gamma(M \cup \{x\}) = \gamma(M)$ if $x \in K$;
(4) $\gamma(tM) = t\gamma(M)$ for $t > 0$; and
(5) $\gamma(M) = 0$ if and only if M is precompact.

The measure γ of noncompactness on K is said to be *subadditive* if $M + N \in \Phi$ and $\gamma(M + N) \leq \gamma(M) + \gamma(N)$ for each $M, N \in \Phi$; see [1, 2, 7, 8].

Let X be a nonempty subset of K, and γ a measure of noncompactness on K, and $k > 0$. A map $F : X \multimap \kappa(K)$ is said to be (γ, k)-*condensing* if $F(X) \in \Phi$ and $\gamma(F(B)) < k\gamma(B)$ for each set $B \subset X$ with $B \in \Phi$ and $\gamma(B) > 0$. If $k = 1$, we call F γ-*condensing*. More generally, a map $F : X \multimap \kappa(K)$ is said to be *countably* (γ, k)-*condensing* if $F(X) \in \Phi$ and $\gamma(F(C)) < k\gamma(C)$ for each countable but not precompact set $C \subset X$ with $C \in \Phi$. If $k = 1$, we call F *countably* γ-*condensing* ; see [8, 19, 21].

Typical examples of a measure of noncompactness are the Kuratowski and the Hausdorff measures of noncompactness on a Banach space E;

see [1, 2]. Note that every compact map is countably (γ, k)-condensing for any $k > 0$.

2. Eigenvalues of countably condensing admissible maps

Our goal of this section is to prove the existence of a positive eigenvalue for a countably condensing admissible map in Fréchet spaces. This result implies the well-known theorem of Krasnoselskii [11].

Let us begin with the following property of a countably condensing map which is a basic tool for using fixed point theory of compact maps; see [19, Corollary 2.1] and [21, Corollary 3.1].

LEMMA 2.1. *Let K be a closed convex set in a Fréchet space E, X a nonempty closed subset of K, and V a compact subset of K. If $F : X \multimap \kappa(K)$ is a countably γ-condensing map, where γ is a measure of noncompactness on K, there exists a nonempty compact convex subset S of K such that V is a subset of S and $F(X \cap S)$ is a subset of S.*

Motivated by Theorem 3.2 of [10], we give a new eigenvalue result on countably condensing admissible maps which contains many known results as special cases, as we will see later.

THEOREM 2.2. *Let E be a Fréchet space, W a closed neighborhood of the origin 0 in E, and K a closed wedge in E. Let γ be a measure of noncompactness on K, k a positive real number, and $F : W \cap K \multimap \kappa(K)$ a countably (γ, k)-condensing admissible map. Suppose that there is a real number $a > k$ such that*

$$F(W \cap K) \cap a(\operatorname{int} W) = \emptyset.$$

Then there exist a point $x_0 \in \partial W \cap K$ and a real number $\lambda_0 \geq a$ such that $\lambda_0 x_0 \in F(x_0)$.

Proof. Let $G : W \cap K \multimap \kappa(K)$ be a map defined by

$$G(x) := \frac{1}{a} F(x) \quad \text{for } x \in W \cap K.$$

Then G is clearly countably $(\gamma, k/a)$-condensing. From $k/a < 1$, it follows that the map G is countably γ-condensing. Applying Lemma 2.1 with $V = \{0\}$, there exists a compact convex subset S of K with $0 \in S$ such that $G(W \cap S)$ is a subset of S.

Consider the restriction G_0 of G to $W \cap S$. Then $G_0 = G|_{W\cap S} : W\cap S \to \kappa(K)$ is a compact admissible map; see [8, Satz 4.1.6]. Without loss of generality, we may assume that G_0 has no fixed point on $W \cap S$. Otherwise, there exists an $x_0 \in \partial W \cap S$ such that $x_0 \in G_0(x_0)$ because $G(W \cap K) \cap \operatorname{int} W = \emptyset$, what we wanted to prove.

Set $X_1 := \{x \in W \cap S : x \in tG_0(x) \text{ for some } t \in [0,1]\}$. Since the compact map G_0 has closed graph, it is easily verified that X_1 is a nonempty closed subset of E. Now we will claim that

$$X_1 \cap (\partial W \cap S) \neq \emptyset.$$

We suppose to the contrary that $X_1 \cap (\partial W \cap S) = \emptyset$. Since two sets X_1 and $\partial W \cap S$ are disjoint and closed in the metric space E, there is a continuous function $p : E \to [0,1]$ such that $p(x) = 0$ for all $x \in X_1$ and $p(x) = 1$ for all $x \in \partial W \cap S$. Let $T : S \to \kappa(S)$ be a map defined by

$$T(x) := \begin{cases} (1-p(x))G_0(x) & \text{for } x \in W \cap S, \\ \{0\} & \text{for } x \in S \setminus \operatorname{int} W. \end{cases}$$

Notice that every compact admissible self-map defined on an acyclic absolute neighborhood retract has a fixed point; see [5, Corollary V.3.7]. Since the closed convex set S in the Fréchet space E is an acyclic absolute neighborhood retract, the compact admissible map T has a fixed point $x_0 \in S$; that is, $x_0 \in T(x_0)$. Hence we have $x_0 \in X_1$ and so $p(x_0) = 0$ which implies that $x_0 \in G_0(x_0)$. This contradicts our assumption that G_0 has no fixed point on $W \cap S$.

Consequently, we have shown that $X_1 \cap (\partial W \cap S) \neq \emptyset$. This means that there exist an $x_0 \in \partial W \cap S$ and a $t_0 \in [0,1]$ such that $x_0 \in t_0 G(x_0)$. From $x_0 \neq 0$ it follows that $t_0 \neq 0$. Setting $\lambda_0 := a/t_0$, we conclude that $\lambda_0 \geq a$ and $\lambda_0 x_0 \in F(x_0)$. This completes the proof. □

The following result on condensing admissible maps given in [10] generalizes Theorem 2 of [7].

Corollary 2.3. *Let W be a closed neighborhood of 0 in a Fréchet space E and K a closed wedge in E. Let γ be a measure of noncompactness on K, k a positive real number, and $F : W \cap K \to \kappa(K)$ a (γ, k)-condensing admissible map. Let $a > k$ be a real number such that*

$$F(W \cap K) \cap a(\operatorname{int} W) = \emptyset.$$

Then there exist an $x_0 \in \partial W \cap K$ and a $\lambda_0 \geq a$ such that $\lambda_0 x_0 \in F(x_0)$.

COROLLARY 2.4. *Let W be a closed neighborhood of 0 in a Fréchet space E and K a closed wedge in E. Suppose that $F : W \cap K \to \kappa(K)$ is a compact admissible map and there is a real number $a > 0$ such that*

$$F(W \cap K) \cap a(\operatorname{int} W) = \emptyset.$$

Then there exist an $x_0 \in \partial W \cap K$ and a $\lambda_0 \geq a$ such that $\lambda_0 x_0 \in F(x_0)$.

Proof. Recall that the compact map F is (γ, k)-condensing for any $k > 0$, where γ is a measure of noncompactness on K. Applying Corollary 2.3 with some $k \in (0, a)$, the conclusion follows. This completes the proof. □

Now we can give Krasnoselskii's eigenvalue theorem for compact set-valued maps with compact convex values in Fréchet spaces ; see also [7, Folgerung 4].

COROLLARY 2.5. *Let W be a closed neighborhood of 0 in a Fréchet space E and K a closed cone in E. Let $F : \partial W \cap K \to \kappa c(K)$ be a compact upper semicontinuous map such that $0 \notin \overline{F(\partial W \cap K)}$. If $W \cap K$ is bounded, then F has a positive eigenvalue.*

Proof. Suppose that $F : \partial W \cap K \to \kappa c(K)$ is a compact upper semicontinuous map such that $0 \notin \overline{F(\partial W \cap K)}$. By a multi-valued version of Dugundji's extension theorem due to Ma[13, Theorem 2.1], F has a compact upper semicontinuous extension $G : W \cap K \to \kappa c(K)$ such that

$$G(W \cap K) \subset \operatorname{co}(F(\partial W \cap K)).$$

The compactness of G follows from the fact that the closed convex hull of a compact set in a Fréchet space is compact. Since K is a closed cone in E and $\overline{F(\partial W \cap K)}$ is a compact subset of K with $0 \notin \overline{F(\partial W \cap K)}$, we know that $0 \notin \overline{\operatorname{co}}(\overline{F(\partial W \cap K)})$ and so $0 \notin \overline{G(W \cap K)}$; see [17, Hilfssatz 1.3]. Hence there is a neighborhood V of 0 in E such that $V \cap G(W \cap K) = \emptyset$. By the boundedness of $W \cap K$, we can choose a real number $a > 0$ such that $a\,(W \cap K) \subset V$. Therefore, we obtain

$$G(W \cap K) \cap a(\operatorname{int} W) = \emptyset.$$

Since G is a compact admissible map and is an extension of F, Corollary 2.4 implies that there exist an $x_0 \in \partial W \cap K$ and a $\lambda_0 \geq a > 0$ such that $\lambda_0 x_0 \in G(x_0) = F(x_0)$. This completes the proof. □

The following result includes the well-known theorem of Krasnoselskii [11] when E is a Banach space.

COROLLARY 2.6. *Let W be a closed bounded neighborhood of 0 in a Fréchet space E and K a closed cone in E. If $f : \partial W \cap K \to K$ is a compact continuous single-valued function such that $0 \notin \overline{f(\partial W \cap K)}$, then f has a positive eigenvalue.*

In addition, an extension of Krasnoselskii's eigenvalue theorem to general topological vector spaces can be found in [12, Satz 4], where a finite dimensional approximation argument is used in the proof.

3. An application

Applying the main result in the previous section, we show that Satz 5 of Hahn[7] remains true for countably condensing admissible maps; see [3, Theorem 2]. For the case of condensing admissible maps, we refer to [10, Theorem 3.4].

THEOREM 3.1. *Let $(E, \|\cdot\|)$ be a Banach space, K a closed wedge in E, and $B = \{x \in E : \|x\| \le r\}$ the closed ball with center 0 and radius $r > 0$. Let γ be a measure of noncompactness on K, k a positive real number, and $F : B \cap K \to \kappa(K)$ a countably (γ, k)-condensing admissible map. Suppose that there are a point $w \in K$ and a real number $c > 0$ such that*

$$\| y + (r - \|x\|)w \| \ge c, \quad \text{for all } x \in B \cap K \text{ and all } y \in F(x).$$

If γ is subadditive in case $w \neq 0$ and if $rk < c$, then there exist a point $x_0 \in \partial B \cap K$ and a real number $\lambda_0 > k$ such that $\lambda_0 x_0 \in F(x_0)$.

Proof. Let $G : B \cap K \to \kappa(K)$ be a map defined by

$$G(x) := F(x) + (r - \|x\|)w, \quad \text{for } x \in B \cap K.$$

Then G is a countably (γ, k)-condensing admissible map; see [8, Folgerung 4.1.5]. In fact, for each countable set $C \subset B \cap K$ with $\gamma(C) > 0$, the definition and the subadditivity of γ imply that $\gamma(G(C)) \le \gamma(F(C)) + \gamma(\mathrm{co}\{0, rw\}) \le \gamma(F(C)) < k\gamma(C)$. Thus, G is countably (γ, k)-condensing. Set $a := c/r$. Since $\|x\| < c$ for each $x \in a(\operatorname{int} B)$, it follows that $a > k$ and

$$G(B \cap K) \cap a(\operatorname{int} B) = \emptyset.$$

By Theorem 2.2, there exist an $x_0 \in \partial B \cap K$ and a $\lambda_0 \ge a$ such that $\lambda_0 x_0 \in G(x_0) = F(x_0)$. This completes the proof. □

As an immediate consequence of Theorem 3.1, we have the following.

COROLLARY 3.2. *Let B be the closed unit ball in a Banach space E, γ a measure of noncompactness on E, and $k > 0$. Let $f : B \to E$ be an (γ, k)-condensing continuous single-valued function. Suppose that there is a real number $c > k$ such that $\|f(x)\| \geq c$ for all $x \in B$. Then there exist an $x_0 \in \partial B$ and a $\lambda_0 > k$ such that $f(x_0) = \lambda_0 x_0$.*

Proof. Apply Theorem 3.1 with $K = E$ and $w = 0$. □

In addition, some results on positive eigenvalues of countably k-contractions defined on a normal cone in a real Banach space can be found in [9]; see also [14, 16].

Let us close this section with the following question whether Theorem 2.2 does hold for any countably (γ, k)-condensing admissible map $F : \partial W \cap K \to \kappa(K)$.

PROBLEM 3.3. *Let E, W, K, γ, k be as in Theorem 2.2. If $F : \partial W \cap K \to \kappa(K)$ is a countably (γ, k)-condensing admissible map such that $F(\partial W \cap K) \cap a(\operatorname{int} W) = \emptyset$ for some real number $a > k$, does F have a positive eigenvalue ?*

More basically, does an admissible map $F : \partial W \cap K \to \kappa(K)$ with $0 \notin \overline{F(\partial W \cap K)}$ have an admissible extension $G : W \cap K \to \kappa(K)$ with the same property ?

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