

ASYMPTOTIC NUMBER OF GENERAL CUBIC GRAPHS WITH GIVEN CONNECTIVITY

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ABSTRACT. Let $g(2n, l, d)$ be the number of general cubic graphs on $2n$ labeled vertices with l loops and d double edges. We use inclusion and exclusion with two types of properties to determine the asymptotic behavior of $g(2n, l, d)$ and hence that of $g(2n)$, the total number of general cubic graphs of order $2n$. We show that almost all general cubic graphs are connected. Moreover, we determined the asymptotic numbers of general cubic graphs with given connectivity.

1. Introduction

Let $g(2n, l, d)$ be the number of general cubic graphs on $2n$ labeled vertices with l loops and d double edges. In a recent related paper, Palmer, Read, and Robinson found a recurrence relation for the number of labeled claw-free cubic graphs. They derived a linear partial differential equation based on removing a single edge which is satisfied by the exponential generating function (*egf*) of labeled general cubic graphs [8], and mentioned that it could be used to derive a recurrence relation for the number $g(s, d, l)$ of labeled general cubic graphs with s single edges, d double edges and l loops. (The notation $g(s, d, l)$ is used in [3] which need the variable s instead of $2n$ in $g(2n, l, d)$. Note that cubic graphs with $2n$ vertices (*order* $2n$) satisfy the relation $2n = \frac{2s+4d+2l}{3}$.) Chae, Palmer, and Robinson [3] did this by extracting coefficients from their differential equation. Then the recurrence relations for the labeled connected general cubic graphs, and 2-connected general cubic graphs were provided. The 3-connected general cubic graphs are exactly 3-connected cubic graphs and the numbers were already found by Wormald[10]. In

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the paper of McKay, Palmer, Read, Robinson[6], the following asymptotic estimate was stated: For $l, d = o(\sqrt{n})$

$$(1.1) \quad g(2n, l, d) = (1 + o(1)) \frac{e^{-2}}{(3!)^{2n}} \cdot \frac{(6n)!}{2^{3n} \cdot (3n)!} \cdot \frac{2^l \cdot 2^d}{l! \cdot d!}.$$

The authors mentioned that it can be derived directly by the method of inclusion and exclusion but they did not provide any proof. So we want to close this enumeration problems of general cubic graphs exactly and asymptotically by giving the proof and finding the asymptotic behaviors of general cubic graphs with given connectivity in this paper. We will derive an inequality of inclusion and exclusion on two types of properties in the section 2. In the section 3, we use the method of configurations to derive the formula (1.1) in all detail for the number of general cubic graphs with $2n$ vertices, l loops, d double edges, and no triples. In the section 4, the equation (1.1) above can be used to find the total number of general cubic graphs with $2n$ vertices by summing up the terms for loops and double edges from 0 to infinity:

$$(1.2) \quad g(2n) = (1 + o(1)) \frac{e^2}{(3!)^{2n}} \cdot \frac{(6n)!}{2^{3n} \cdot (3n)!}.$$

Wormald first derived (1.2) in [9] by estimating the number of matrices with given row and column sums. It could also be obtained from matrix approximations of Bender and Canfield[1]. Moreover, in the section 5, the asymptotic numbers of general cubic graphs with given connectivity are found as consequential results of the formula (1.1). And the results are summarized in the figure 2. For general graph theoretic terminology and notation we follow [5] and we assume the basic terminology developed in [7] for inclusion and exclusion. Chae also derive an inequality of inclusion and exclusion on finitely many types of properties in [4] which is a generalization of inclusion and exclusion and you can find another way to find the formula (1.1) in the paper.

2. Inclusion and exclusion for two types of properties

Let us start with definition of inclusion and exclusion with one property. Let U be the universal set of S_0 elements, and suppose A_1, \dots, A_s are s subsets of U . The complement of a set C of U is denoted by \bar{C} . For all integer $k > 0$, $[k]$ denote the set $\{1, 2, \dots, k\}$. For $l = 0, \dots, s$, we

define

$$(2.1) \quad S_l = \sum \left| \bigcap_{i \in I} A_i \right|,$$

where the sum is over all l -subsets I of $[s]$. For $l = 0, \dots, s$, let N_l be the number of elements of U that belong to exactly l of the sets $\{A_1, \dots, A_s\}$. That is,

$$(2.2) \quad N_l = \sum \left| \bigcap_{i \in I} A_i \cap \bigcap_{i \notin I} \bar{A}_i \right|,$$

where the sum is over all l -subsets I of $[s]$. Then we can find the following relation between S_l and N_u by considering the contribution of an element to N_u , for $u = l, \dots, s$ which was already multi-counted for S_l .

$$(2.3) \quad S_l = \sum_{l \leq u \leq s} \binom{u}{l} N_u.$$

Or we have a relation:

$$(2.4) \quad N_l = \sum_{0 \leq i \leq s-l} (-1)^i \binom{l+i}{i} S_{l+i}.$$

Then we can have the upper and lower bounds for N_l which can be found in the book [7].

THEOREM 1.

$$(2.5) \quad \sum_{0 \leq i \leq 2\alpha-1} (-1)^i \binom{l+i}{i} S_{l+i} \leq N_l \leq \sum_{0 \leq i \leq 2\alpha} (-1)^i \binom{l+i}{i} S_{l+i}.$$

This is the inequality that can be used to find the bounds for N_l by estimating the sum of S_{l+i} 's.

Now let us derive an inequality of inclusion and exclusion on two types of properties. Let U be the universal set of S_0 elements, and suppose that A_1, \dots, A_s and B_1, \dots, B_t are subsets of U . For $l = 0$ to s and $d = 0$ to t , define

$$(2.6) \quad S_{l,d} = \sum \left| \bigcap_{i \in I} A_i \cap \bigcap_{j \in J} B_j \right|,$$

where the sum is over all l -subsets $I \subset [s]$ and d -subsets $J \subset [t]$. Now for $l = 0$ to s and $d = 0$ to t , let $N_{l,d}$ be the number of elements of U that belong to exactly l of the sets A_i and d of the sets B_i . That is,

$$(2.7) \quad N_{l,d} = \sum \left| \bigcap_{i \in I} A_i \cap \bigcap_{i \notin I} \bar{A}_i \cap \bigcap_{j \in J} B_j \cap \bigcap_{j \notin J} \bar{B}_j \right|,$$

where the sum is again over all l -subsets $I \subset [s]$ and d -subsets $J \subset [t]$. Then clearly, by counting the contribution to $S_{l,d}$ of each elements x of U that contribute to $N_{u,v}$, for $u \geq l, v \geq d$, we have

$$(2.8) \quad S_{l,d} = \sum_{\substack{l \leq u \leq s \\ d \leq v \leq t}} \binom{u}{l} \binom{v}{d} N_{u,v}.$$

The numbers $S_{l,d}$ and $N_{l,d}$ are closely related, and this relation is neatly expressed in terms of ordinary generating functions

$$(2.9) \quad S(x, y) = \sum_{l=0}^s \sum_{d=0}^t S_{l,d} x^l y^d$$

and

$$(2.10) \quad N(x, y) = \sum_{l=0}^s \sum_{d=0}^t N_{l,d} x^l y^d.$$

Then the following proposition can be obtained by the equation (2.8), (2.9) and (2.10).

PROPOSITION 1.

$$(2.11) \quad N(x + 1, y + 1) = S(x, y).$$

If we set $x = y = -1$ in the equation (2.11), we obtain

$$(2.12) \quad N(0, 0) = S(-1, -1).$$

This is the number of elements in U that belong to none of the sets A_1, \dots, A_s and B_1, \dots, B_t . Now let $x - 1$ and $y - 1$ take the place of x, y respectively in the equation (2.11) and compare the coefficients of $x^l y^d$. One finds that

$$(2.13) \quad N_{l,d} = \sum_{\substack{0 \leq i \leq s-l \\ 0 \leq j \leq t-d}} (-1)^{i+j} \binom{l+i}{i} \binom{d+j}{j} S_{l+i, d+j}.$$

It is important to study the upper and lower bounds for $N_{l,d}$. Therefore we consider truncation

$$\begin{aligned}
 & \sum_{\substack{0 \leq i \leq \alpha \\ 0 \leq j \leq \beta}} (-1)^{i+j} \binom{l+i}{i} \binom{d+j}{j} S_{l+i,d+j} \\
 (2.14) \quad &= \sum_{\substack{0 \leq i \leq \alpha \\ 0 \leq j \leq \beta}} (-1)^{i+j} \binom{l+i}{i} \binom{d+j}{j} \sum_{\substack{u > l+i \\ v \geq d+j}} \binom{u}{l+i} \binom{v}{d+j} N_{u,v},
 \end{aligned}$$

where $0 < \alpha \leq s - l$, $0 < \beta \leq t - d$ and the right side has been obtained by substitution of (2.8). Now interchange the order of summation and obtain

$$\begin{aligned}
 & \sum_{\substack{0 \leq i \leq \alpha \\ 0 \leq j \leq \beta}} (-1)^{i+j} \binom{l+i}{i} \binom{d+j}{j} S_{l+i,d+j} \\
 (2.15) \quad &= \sum_{\substack{u \geq l \\ v \geq d}} N_{u,v} \binom{u}{l} \binom{v}{d} \sum_{\substack{0 \leq i \leq \alpha \\ 0 \leq j \leq \beta}} (-1)^{i+j} \binom{u-l}{i} \binom{v-d}{j}.
 \end{aligned}$$

It can be seen that:

$$\begin{aligned}
 & \sum_{\substack{0 \leq i \leq \alpha \\ 0 \leq j \leq \beta}} (-1)^{i+j} \binom{u-l}{i} \binom{v-d}{j} \\
 (2.16) \quad &= \begin{cases} (-1)^{\alpha+\beta} \binom{u-l-1}{\alpha} \binom{v-d-1}{\beta}, & \text{if } u \geq l+1, v \geq d+1; \\ (-1)^\alpha \binom{u-l-1}{\alpha}, & \text{if } u \geq l+1, v = d; \\ (-1)^\beta \binom{v-d-1}{\beta}, & \text{if } u = l, v \geq d+1; \\ 1, & \text{if } u = l, v = d. \end{cases}
 \end{aligned}$$

So

$$\begin{aligned}
 & \sum_{\substack{0 \leq i \leq \alpha \\ 0 \leq j \leq \beta}} (-1)^{i+j} \binom{l+i}{i} \binom{d+j}{j} S_{l+i,d+j} \\
 (2.17) \quad &= N_{l,d} + (-1)^\alpha \sum_{u \geq l+1} N_{u,d} \binom{u}{l} \binom{u-l-1}{\alpha} \\
 & \quad + (-1)^\beta \sum_{v > d+1} N_{l,v} \binom{v}{d} \binom{v-d-1}{\beta}
 \end{aligned}$$

$$+(-1)^{\alpha+\beta} \sum_{\substack{u \geq l+1 \\ v \geq d+1}} N_{u,v} \binom{u}{l} \binom{v}{d} \binom{u-l-1}{\alpha} \binom{v-d-1}{\beta}.$$

Let

$$A_\alpha = \sum_{u \geq l+1} N_{u,d} \binom{u}{l} \binom{u-l-1}{\alpha},$$

$$B_\beta = \sum_{v \geq d+1} N_{l,v} \binom{v}{d} \binom{v-d-1}{\beta},$$

and

$$C_{\alpha,\beta} = \sum_{\substack{u \geq l+1 \\ v \geq d+1}} N_{u,v} \binom{u}{l} \binom{v}{d} \binom{u-l-1}{\alpha} \binom{v-d-1}{\beta}.$$

If α and β are both even then the quantity $(-1)^\alpha A_\alpha + (-1)^\beta B_\beta + (-1)^{\alpha+\beta} C_{\alpha,\beta}$ is positive, so that it gives us an upper bound. For a lower bound, this quantity should be negative. Suppose there are no such α and β which make $(-1)^\alpha A_\alpha + (-1)^\beta B_\beta + (-1)^{\alpha+\beta} C_{\alpha,\beta}$ negative with $\alpha > (s-l-1)/2$ and $\beta > (t-d-1)/2$. If α and β are odd with $\alpha > (s-l-1)/2$ and $\beta > (t-d-1)/2$, then by assumption, we have $(-1)^{\alpha+1} A_{\alpha+1} + (-1)^\beta B_\beta + (-1)^{\alpha+\beta+1} C_{\alpha+1,\beta} \geq 0$. And $(-1)^\alpha A_\alpha + (-1)^{\beta+1} B_{\beta+1} + (-1)^{\alpha+\beta+1} C_{\alpha,\beta+1} \geq 0$. So we have the following

$$B_\beta + C_{\alpha+1,\beta} \leq A_{\alpha+1},$$

and

$$A_\alpha + C_{\alpha,\beta+1} \leq B_{\beta+1}.$$

After adding the inequalities above side by side, we have

$$(2.18) \quad A_\alpha + B_\beta + C_{\alpha+1,\beta} + C_{\alpha,\beta+1} \leq A_{\alpha+1} + B_{\beta+1}.$$

Since $\alpha > (s-l-1)/2$ and $\beta > (t-d-1)/2$, we have

$$\binom{u-l-1}{\alpha} > \binom{u-l-1}{\alpha+1},$$

and

$$\binom{v-d-1}{\beta} > \binom{v-d-1}{\beta+1}$$

for all u with $l+1 \leq u \leq s$ and $d+1 \leq v \leq t$. $C_{\alpha+1,\beta} + C_{\alpha,\beta+1}$ is positive. Hence the inequality (2.18) cannot be true, we get a contradiction.

Therefore we have the following theorem:

THEOREM 2.

$$\begin{aligned}
 \sum_{\substack{0 \leq i \leq \alpha' \\ 0 \leq j \leq \beta'}} (-1)^{i+j} \binom{l+i}{i} \binom{d+j}{j} S_{l+i,d+j} \\
 (2.19) \qquad \qquad \qquad \leq N_{l,d} \\
 \qquad \qquad \qquad \leq \sum_{\substack{0 \leq i \leq 2\alpha \\ 0 \leq j \leq 2\beta}} (-1)^{i+j} \binom{l+i}{i} \binom{d+j}{j} S_{l+i,d+j},
 \end{aligned}$$

where α' and β' are numbers such that $\alpha' > (s - l - 1)/2$ and $\beta' > (t - d - 1)/2$.

This formula will be used to estimate the number of configurations in section 3 which will be used to calculate the asymptotic number of general cubic graphs in section 4.

3. Configurations

Here we use an idea of Bollabás[2] for representing general cubic graphs. Let $V = \bigcup_{1 \leq i \leq 2n} V_i$ be a partition of V into 3-subsets V_i , for $i = 1, \dots, 2n$. A configuration is a perfect matching on this set of vertices. Therefore it is easy to see that the total number of configurations is

$$(3.1) \qquad \qquad \qquad \frac{(6n)!}{2^{3n} \cdot (3n)!}$$

Consider any edge uv in V . If both vertices u and v belong to the same set V_i of the partition, the edge is called a 1-cycle. Otherwise they are contained in two different sets V_i and V_j . If there are exactly two such edges between V_i and V_j , we call this a 2-cycle, or double edge and if there are three, it is a triple.

The next lemma shows that although triples are present, their contribution to all of our asymptotic estimates is negligible.

LEMMA 1. *Configurations almost surely have no triples.*

Proof. The expected number of triples in a configuration is

$$\binom{2n}{2} 3! \frac{6(n-1)!}{2^{3(n-1)} (3(n-1))!} \frac{2^{3n} (3n)!}{(6n)!}$$

Elementary operations using Stirling’s formula in the factorials show that the expectations is $o(1)$, Therefore, for $n \rightarrow \infty$, the probability of a triple in a configuration tends to zero. \square

Triples are also negligible when the number of double edges is restricted. Suppose a configuration has d specific 2-cycles(double edges). Let R be the set of $6n - 6d$ remaining vertices. The same computation used in the above lemma shows that triple edges are negligible in R provided that $n - d$ tends to infinity. Furthermore if $d = o(\sqrt{n})$, there are almost surely no triple edges among the d specified double edges.

Now, for $i = 1, \dots, 2n$, let A_i be the set of configurations which have a 1-cycle in V_i . Assume the $\binom{2n}{2}$ pairs of sets V_i in the partition are ordered from 1 to $\binom{2n}{2}$. Let B_j be the set of configurations which have a 2-cycle in the j^{th} pair for $j = 1, \dots, \binom{2n}{2}$. Let $\hat{c}(2n, l, d)$ be number of configurations with exactly l 1-cycles and d 2-cycles, possibly triples are included. It will be shown later that they are negligible.

Let $S_{l,d}$ be the number of configurations which have l 1-cycles and d 2-cycles and no triple edges(Let us use same notation $S_{l,d}$ as in (2.6), since it is not nebulous). Then $S_{l,d}$ can be found by using the definition (2.6);

$$(3.2) \quad S_{l,d} = \binom{2n}{l, 2d, 2n - l - 2d} 3^l \frac{(2d)!}{2^d \cdot d!} (3^2 \cdot 2)^d (1 + o(1)) \frac{(2(3n - l - 2d))!}{2^{3n-l-2d} \cdot (3n - l - 2d)!},$$

where the term $(1 + o(1))$ allows for a negligible number of triples. From the equation (2.19), we obtain

$$(3.3) \quad \sum_{\substack{0 \leq i \leq \alpha' \\ 0 \leq j \leq \beta'}} (-1)^{i+j} \binom{l+i}{i} \binom{d+j}{j} S_{l+i,d+j} \leq \hat{c}(2n, l, d) \leq \sum_{\substack{0 \leq i \leq 2\alpha \\ 0 \leq j \leq 2\beta}} (-1)^{i+j} \binom{l+i}{i} \binom{d+j}{j} S_{l+i,d+j}.$$

Then on substituting the equation (3.2) in (3.3) and simplifying we have (3.4)

$$\begin{aligned} \frac{1}{l! \cdot d!} \cdot \frac{(6n)!}{2^{3n} \cdot (3n)!} \left[\sum_{\substack{0 \leq i \leq \alpha' \\ 0 \leq j \leq \beta'}} \frac{(-1)^i (-1)^j}{i! \cdot j!} \right] (1 + o(1)) &\leq \hat{c}(2n, l, d) \\ &\leq \frac{1}{l! \cdot d!} \cdot \frac{(6n)!}{2^{3n} \cdot (3n)!} \left[\sum_{\substack{0 \leq i \leq 2\alpha \\ 0 \leq j \leq 2\beta}} \frac{(-1)^i (-1)^j}{i! \cdot j!} \right] (1 + o(1)), \end{aligned}$$

where both l and $d = o(\sqrt{n})$.

Therefore we have:

THEOREM 3. For l and $d = o(\sqrt{n})$,

$$(3.5) \quad \hat{c}(2n, l, d) = (1 + o(1)) \frac{e^{-2}}{l! \cdot d!} \frac{(6n)!}{2^{3n} \cdot (3n)!}.$$

COROLLARY 1. For large l and d ,

$$(3.6) \quad \hat{c}(2n, l, d) = O(1) \frac{1}{l! \cdot d!} \cdot \frac{(6n)!}{2^{3n} \cdot (3n)!}.$$

Proof. When l and d are arbitrary, the upper bound in (3.4) no longer holds but instead we have:

$$\begin{aligned} (3.7) \quad \hat{c}(2n, l, d) &\leq \frac{1}{l! \cdot d!} \cdot \frac{(6n)!}{2^{3n} \cdot (3n)!} \left[\sum_{\substack{0 \leq i \leq 2\alpha \\ 0 \leq j \leq 2\beta}} \frac{(-1)^i (-1)^j}{i! \cdot j!} \right] 6^k \frac{(2n)_k (3n)_k}{(6n)_{2k}} \\ &= O(1) \frac{1}{l! \cdot d!} \cdot \frac{(6n)!}{2^{3n} \cdot (3n)!}, \end{aligned}$$

where $k = (i + 2j) + (l + 2d)$ and $(n)_k = n(n - 1) \dots (n - k + 1)$. In equation (3.7),

$$6^k \cdot \frac{(2n)_k (3n)_k}{(6n)_{2k}} = O(1)$$

can be obtained with simple estimation by using

$$\frac{(n)_k}{n^k} = O(1) \exp\left(\frac{-k^2}{2n} - \frac{k^3}{6n^2}\right), \text{ for all } k.$$

Lower bound can be applied similarly. Here we note that if $l + d$ is sufficiently large such as a constant times n , $6^k \cdot \frac{(2n)_k (3n)_k}{(6n)_{2k}}$ should be $O(1)$, if not, it goes to 1. □

COROLLARY 2. *Triples are negligible in configurations with exactly l 1-cycles and d 2-cycles.*

Proof. The proof is similar to the proof of lemma 1 by using the equation (3.6).

4. Asymptotic number of general cubic graphs

Let $c(2n, l, d)$ be the number of configurations with exactly l 1-cycles and d 2-cycles and no triples. Since the triple edges are negligible, all asymptotic results for \hat{c} hold for c . Let $g(2n, l, d)$ be the number of labeled cubic general graphs G with exactly l loops, d double edges, and no triples. Then we have following relationship between $g(2n, l, d)$ and $c(2n, l, d)$ by shrinking the 3-vertex sets V_i of configurations to single vertices for graphs.

PROPOSITION 2.

$$(4.1) \quad c(2n, l, d) = g(2n, l, d) \cdot 3^l \cdot \left(\binom{3}{2}^2 \cdot 2 \right)^d (3!)^{2n-l-2d}.$$

Then by substituting equations (3.5) and (3.6) in (4.1), we have the following corollaries.

COROLLARY 3. *For $l, d = o(\sqrt{n})$,*

$$(4.2) \quad g(2n, l, d) = (1 + o(1)) \frac{e^{-2}}{(3!)^{2n}} \cdot \frac{(6n)!}{2^{3n} \cdot (3n)!} \cdot \frac{2^l \cdot 2^d}{l! \cdot d!}.$$

It can be shown that for all l, d :

COROLLARY 4.

$$(4.3) \quad g(2n, l, d) = O(1) \frac{1}{(3!)^{2n}} \cdot \frac{(6n)!}{2^{3n} \cdot (3n)!} \cdot \frac{2^l \cdot 2^d}{l! \cdot d!}.$$

If we sum up the values of $g(2n, l, d)$ using the equation (4.2), we have the asymptotic number of general cubic graphs on $2n$ vertices:

COROLLARY 5.

$$(4.4) \quad g(2n) = (1 + o(1)) \frac{e^2}{(3!)^{2n}} \cdot \frac{(6n)!}{2^{3n} \cdot (3n)!}.$$

Proof. When $l, d = o(\sqrt{n})$, by using the equation (4.2) it easily can be seen that

$$\sum_{l, d \geq o} g(2n, l, d) = (1 + o(1)) \frac{e^2}{(3!)^{2n}} \cdot \frac{(6n)!}{2^{3n} \cdot (3n)!}.$$

For $l, d \neq o(\sqrt{n})$, we can have three cases as followings:

- (1) all l , and $d \geq \frac{\sqrt{n}}{w_n}$ for some w_n ,
- (2) all d , and $l \geq \frac{\sqrt{n}}{u_n}$ for some u_n ,
- (3) all $l \geq \frac{\sqrt{n}}{u_n}$, $d \geq \frac{\sqrt{n}}{w_n}$ for some w_n, u_n , where w_n and u_n go to infinity very slowly. Since

$$\sum_{d \geq \frac{\sqrt{n}}{w_n}} \frac{2^d}{d!} = o(1) \text{ and } \sum_{l \geq \frac{\sqrt{n}}{u_n}} \frac{2^l}{l!} = o(1),$$

It can be shown that

$$\sum_{l \geq 0, d \geq \frac{\sqrt{n}}{w_n}} g(2n, l, d) = O(1) \frac{1}{(3!)^{2n}} \cdot \frac{(6n)!}{2^{3n} \cdot (3n)!} \cdot o(1),$$

and, similarly case (2) and (3), from the equation (4.3)

$$\sum_{l \geq \frac{\sqrt{n}}{u_n}, d \geq 0} g(2n, l, d) = \sum_{l \geq \frac{\sqrt{n}}{u_n}, d \geq \frac{\sqrt{n}}{w_n}} g(2n, l, d) = O(1) \frac{1}{(3!)^{2n}} \cdot \frac{(6n)!}{2^{3n} \cdot (3n)!} \cdot o(1).$$

Therefore

$$g(2n) = (1 + o(1)) \frac{e^2}{(3!)^{2n}} \cdot \frac{(6n)!}{2^{3n} \cdot (3n)!} + O(1) \frac{1}{(3!)^{2n}} \cdot \frac{(6n)!}{2^{3n} \cdot (3n)!} \cdot o(1).$$

So we are done. □

So the equation (1.2), the total number of general cubic graphs with $2n$ vertices, is derived.

5. Asymptotic number of general cubic graphs with given connectivity

Let $g_1(2n)$ be the number of connected general cubic graphs of order $2n$. Then $g(2n)$ and $g_1(2n)$ are related by the following sum:

$$(5.1) \quad g(2n) = \sum_{k=1}^n \binom{2n}{2k} \frac{k}{n} g_1(2k) g(2n - 2k),$$

where $g(0) = 1$. Therefore we have

$$(5.2) \quad 1 = \frac{g_1(2n)}{g(2n)} + \sum_{k=1}^{n-1} \binom{2n}{2k} \frac{k}{n} \frac{g_1(2k)g(2n-2k)}{g(2n)}.$$

To show that almost all general cubic graphs are connected i.e., $g(2n) \sim g_1(2n)$, we need to show that

$$(5.3) \quad \sum_{k=1}^{n-1} \binom{2n}{2k} \frac{k}{n} \frac{g_1(2k)g(2n-2k)}{g(2n)} = o(1).$$

Since $k \cdot g_1(2k) < n \cdot g(2k)$, it is enough to show that

$$(5.4) \quad \sum_{k=1}^{n-1} \binom{2n}{2k} \frac{g(2k)g(2n-2k)}{g(2n)} = o(1).$$

By using the equation (4.4), Stirling's formula, i.e., $n! \sim \sqrt{2\pi n}(n/e)^n$, and some simple estimates, we find the left side of equation (5.4) is

$$(5.5) \quad O(1) \sum_{k=1}^{n/2} \frac{\sqrt{n}}{\sqrt{k}\sqrt{n-k}} \left[\frac{k}{e(n-k)} \right]^k.$$

This sum can be estimated by splitting it into two parts according as $k \leq \log n$ or $k > \log n$. Then we find that for $1 \leq k \leq \log n$, the value of the sum is $O(n^{-2})$ and for $\log n < k \leq n/2$ it is $O(n^{-1}(\log n)^{-1/2})$. Therefore we have the following theorem.

THEOREM 4. *Almost all general cubic graphs are connected.*

For convenience, let

$$(5.6) \quad F(n) = \frac{(6n)!}{2^{3n}(3n)!(3!)^{2n}}.$$

Then the equation (4.4) can be written

$$(5.7) \quad g(2n) \sim F(n) \cdot e^2.$$

Let $gl(2n)$ be the number of general cubics with at least 1 loop. It follows from the equation (4.2) with $l = 0$ that the number of general cubics with no loops is asymptotic to $F(n)$. (But from the equation (4.3), we have $O(1)F(n)$ instead of $F(n)$. Hence the results we get from now on restricted to the assumptions that are l and $d = o(\sqrt{n})$.) Therefore

$$(5.8) \quad g(2n) \sim gl(2n) + F(n).$$

Since $gl(2n) \sim (g(2n) - F(n)) = F(n) \cdot e^2 - F(n)$, the number of general cubics with at least 1 loop is expressed as follows.

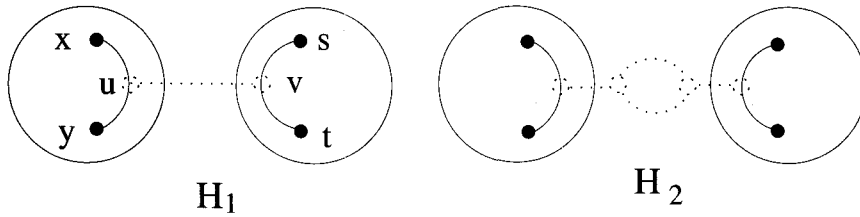


FIGURE 1. Graphs H_1 and H_2

PROPOSITION 3.

$$(5.9) \quad gl(2n) \sim F(n)(e^2 - 1).$$

Note that almost all general cubic graphs are connected. Hence we can say that $F(n)(e^2 - 1)$ is the asymptotic number of connected general cubic graphs with at least one loop and $F(n)$ is the asymptotic number of connected general cubic graphs with no loops. Note that if a connected general cubic graph has a loop then it has $\kappa(G) = 1$, i.e., vertex connectivity one. Now if we show that almost all loopless general cubic graphs are 2-connected, then it can be said that all general cubic graphs with $\kappa(G) = 1$ has at least one loop so the asymptotic number of general cubic graphs with $\kappa(G) = 1$ is $F(n)(e^2 - 1)$. In order to show that almost all loopless general cubic graphs are 2-connected, we need to show that the loopless general cubic graphs with $\kappa(G) = 1$ are negligible. There are two kinds of loopless general cubic graphs with $\kappa(G) = 1$ as shown in Figure 1. Let us consider the first case - the connected loopless general cubic graph constructed from type H_1 which is the graph with solid line and dark vertices x, y, s and t not including u and v in Figure 1. Let G be a connected general cubic graph with $2n$ vertices and no loops, rooted at a bridge, say uv . Therefore $\kappa(G) = 1$. Then let x and y be the vertices other than v which are adjacent to u , and s and t be the vertices other than u that are adjacent to v . Since uv is a bridge, x, y, s and t are all distinct. Delete the vertices u and v with incident edges from G and add new edges xy and st to the graph. Then the graph obtained, say H_1 , is a disconnected general cubic graph with $2n - 2$ vertices, no loops, and two root edges xy and st . In the Figure 1, H_1 is shown and G is shown as H_1 plus the dotted vertices and edges. This operations can be reversed, i.e., G can be constructed from H_1 by adding the two vertices u and v on the two root edges xy and st .

It can be seen from the reversal of this construction that the number of connected general cubic graphs with no loops, rooted at a bridge is bounded above by

$$(5.10) \quad \binom{3n-3}{2} (g(2n-2) - g_1(2n-2)) 2n(2n-1).$$

The factor $g(2n-2) - g_1(2n-2)$ counts the number of disconnected general cubics of order $2n-2$. The binomial coefficient is the number of ways to choose two edges xy and st in different components. The last factors account for the number of ways to label the vertices u and v of a bridge.

Hence (5.10) is also an upper bound for the number of connected general cubic graphs with no loops and at least one bridge. Therefore to show that with high probability this do not exist, it is sufficient to show that

$$(5.11) \quad \frac{\binom{3n-3}{2} (g(2n-2) - g_1(2n-2)) 2n(2n-1)}{F(n)} = o(1).$$

Thus we need, since $g(2n) \sim F(n)e^2$,

$$(5.12) \quad \frac{O(n^4)(g(2n-2) - g_1(2n-2))}{g(2n)} = o(1).$$

Now $g(2n-2)$ and $g_1(2n-2)$ are related by the following equation

$$(5.13) \quad g(2n-2) = \sum_{k=2}^{n-1} \binom{2n-2}{2k} \frac{k}{n-1} g_1(2k) g(2n-2k-2).$$

On dividing both sides by $g(2n)$ after extract the $(n-1)$ th term from the right sum, we have

$$(5.14) \quad \begin{aligned} \frac{g(2n-2)}{g(2n)} - \frac{g_1(2n-2)}{g(2n)} &= \sum_{k=2}^{n-2} \binom{2n-2}{2k} \frac{k}{n-1} \frac{g_1(2k)g(2n-2k-2)}{g(2n)} \\ &\leq \sum_{k=2}^{n-2} \binom{2n-2}{2k} \frac{k}{n-1} \frac{g(2k)g(2n-2k-2)}{g(2n)}. \end{aligned}$$

Since

$$(5.15) \quad \begin{aligned} O(n^4) \left(\frac{g_1(2n-2)}{g(2n)} - \frac{g(2n-2)}{g(2n)} \right) \\ \leq O(n^4) \sum_{k=2}^{n/2} \binom{2n-2}{2k} \frac{g(2k)g(2n-2k-2)}{g(2n)}, \end{aligned}$$

it is enough to show that right side of this equation is $o(1)$. Using the formula (4.3), Stirling's formula and simple estimates, we find that right side of equation (5.15) is

$$(5.16) \quad O(n^4) \sum_{k=2}^{n/2} \frac{1}{\sqrt{k}} \frac{1}{(n-1)^{3/2}(n-k-1)^{3/2}} \left[\frac{k}{e(n-k-1)} \right]^k.$$

This sum can be estimated by splitting it into two parts according as $k \leq \log n$ or $k > \log n$. Then we find that for $2 \leq k \leq \log n$, the value of the sum is $O(n^{-1})$ and for $\log n < k \leq n/2$ it is $O((\log n)^{-1/2})$. For the second case - the connected general loopless graphs constructed from type H_2 , let a graph G which is a connected general cubic graph with $2n$ vertices and no loops, rooted at a double edge and two single edges as shown in the right of the Figure 1. Therefore $\kappa(G) = 1$ also. And it also can be constructed from the graph H_2 rooted at two edges with $2n - 4$ vertices by adding 4 vertices, one double and two single edges, and vice versa as above. We want to show that the number of these graphs are negligible also. And it is enough to show that

$$\frac{\binom{3n-6}{2}(g(2n-4) - g_1(2n-4))2n(2n-1)(2n-2)(2n-3)}{F(n)} = o(1).$$

This can be done similarly as above. We considered all connected general cubic graphs with no loops, and $\kappa(G) = 1$, and showed that they are negligible. And we have the following result:

PROPOSITION 4. *Almost all loopless general cubic graphs are 2-connected.*

COROLLARY 6. *The asymptotic number of general cubic graphs G with $\kappa(G) = 1$ is*

$$(5.17) \quad F(n)(e^2 - 1).$$

COROLLARY 7. *The asymptotic number of 2-connected general cubic graphs is $F(n)$.*

In a 3-connected general cubic graph, there are no loops and no double edges. Note that if a general cubic graph has no loops and no double edges, then it is just a cubic graph. We know that almost all cubic graphs are 3-connected ([10]). Therefore by letting $l = 0$ and $d = 0$ in the equation (4.2), we obtain

$$(5.18) \quad g(2n, 0, 0) \sim F(n) \cdot e^{-2},$$

General Cubic graphs $\frac{g(2n) \sim F(n) e^2}{(almost\ surely\ 1\text{-connected})}$

<p style="text-align: center;">At least one loop (l > 0)</p> <p style="text-align: center;">$gl(2n) \sim F(n)(e^2 - 1)$</p> <p style="text-align: center;">$\frac{gl(2n)}{g(2n)} \sim 86.47\%$</p> <p style="text-align: center;">a.s. $\kappa(G)=1$.</p>	<p style="text-align: center;">No loops (l = 0) No doubles (d = 0)</p> <p style="text-align: center;">$g(2n,0,0) \sim F(n) e^2$</p> <p style="text-align: center;">$\frac{g(2n,0,0)}{g(2n)} \sim 1.83\%$</p> <p style="text-align: center;">a.s. 3-connected (Wormald) a.s. $\kappa(G)=3$.</p>
	<p style="text-align: center;">No loops (l = 0) At least one double (d > 0)</p> <p style="text-align: center;">$F(n)(1 - e^{-2})$</p> <p style="text-align: center;">11.7%</p> <p style="text-align: center;">a.s. 2-connected hence a.s. $\kappa(G)=2$.</p>

$$F(n) = \frac{(6n)!}{2^{3n} (3n)!(3!)^{2n}}$$

NO LOOPS $\sim F(n)$
13.53%
a.s. 2-connected.

FIGURE 2. Summary

so the asymptotic number of 3-connected general cubic graphs is

(5.19)
$$F(n) \cdot e^{-2}.$$

COROLLARY 8. *The asymptotic number of general cubic graphs with $\kappa(G) = 2$ is $F(n)(1 - e^{-2})$.*

These results are summarized in the Figure 2.

References

- [1] E. A. Bender and E. R. Canfield, *The asymptotic number of labeled graphs with given degree sequences*, J. Combin. Theory Ser. A **24** (1978), 296–307.
- [2] Bollobás, *A probabilistic proof of an asymptotic formula for the number of labeled regular graphs*, European J. Combin. **1** (1980), 311–316.
- [3] G. Chae, E. M. Palmer, and R. W. Robinson, *Counting labeled general cubic graphs*, preprint.
- [4] G. Chae, *Inclusion and Exclusion for finitely many types of properties*, submitted.

- [5] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, Chapman & Hall, 2-6 Boundary Row, London, 1996.
- [6] B. D. McKay, E. M. Palmer, R. C. Read, and R. W. Robinson, *The asymptotic number of claw-free cubic graphs*, *Discrete Math.* **272** (2003), 107–118.
- [7] E. M. Palmer, *Graphical Evolution*, John Wiley & Sons, 1985.
- [8] E. M. Palmer, R. C. Read, and R. W. Robinson, *Counting claw-free cubic graphs*, *SIAM J. Discrete Math.* **16** (2002), 65–73.
- [9] N. C. Wormald, *Some problems in the enumeration of labelled graphs*, Doctoral Thesis, University of Newcastle, NSW, Australia, 1978.
- [10] ———, *Enumeration of labelled graphs II : Cubic graphs with a given connectivity*, *J. London Math. Soc. (2)* **20** (1979), 1–7.

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