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ERROR ESTIMATES OF NONSTANDARD FINITE DIFFERENCE SCHEMES FOR GENERALIZED CAHN-HILLIARD AND KURAMOTO-SIVASHINSKY EQUATIONS

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Abstract. Nonstandard finite difference schemes are considered for a generalization of the Cahn-Hilliard equation with Neumann boundary conditions and the Kuramoto-Sivashinsky equation with periodic boundary conditions, which are of the type

$$u_t + \frac{\partial^2}{\partial x^2} g(u, u_x, u_{xx}) = \frac{\partial^\alpha}{\partial x^\alpha} f(u, u_x),\ \alpha = 0, 1, 2.$$

Stability and error estimate of approximate solutions for the corresponding schemes are obtained using the extended Lax-Richtmyer equivalence theorem. Three examples are provided to apply the nonstandard finite difference schemes.

1. Introduction

Consider the partial differential equation

$$u_t + \frac{\partial^2}{\partial x^2} g(u, u_x, u_{xx}) = \frac{\partial^\alpha}{\partial x^\alpha} f(u, u_x),\ x \in \Omega,\ 0 < t \leq T, \tag{1.1}$$

with an initial condition

$$u(x, 0) = u_0(x),\ x \in \Omega, \tag{1.2}$$

and either Neumann boundary conditions

$$\frac{\partial u}{\partial x} = 0,\ \frac{\partial^3 u}{\partial x^3} = 0,\ (x, t) \in \partial\Omega \times (0, T], \tag{1.3a}$$

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or a periodic boundary condition

$$u(x,t) = u(x+1,t), \ (x,t) \in \Omega \times (0,T]. \tag{1.3b}$$

Here, $\alpha = 0, 1, 2$, $\Omega = (0,1)$ and f and g are given functions. We assume that $g(u, u_x, u_{xx}) = g_1(u, u_x)u_{xx} + g_2(u, u_x)$ and there exist constants $\mathcal{G}_0$ and $\mathcal{G}_1$ such that $0 < \mathcal{G}_0 \leq g_1(x,y) \leq \mathcal{G}_1$ for all real x and y. The regularity of g and f is given in Theorems 3.3–3.4. In the case of using the Neumann boundary conditions (1.3a), we also assume $g_1(x,-y) = g_1(x,y)$ and $g_2(x,-y) = g_2(x,y)$ for all x, y, in order to use an integration by parts in the proof of Theorem 3.3.

In case of $\alpha = 2$, the partial differential equation (1.1)–(1.2) can be found as a generalization of the Cahn-Hilliard equation with $g_1(u, u_x) = q, g_2(u, u_x) = 0, f(u, u_x) = -pu + su^3$ and the Neumann boundary conditions (1.3a). Here, p, q and s are positive constants. Global existence and uniqueness of the solution for the Cahn-Hilliard equation have been shown by Elliott and Zheng[9]. Finite element Galerkin approximate solutions have been obtained by Elliott and French[7]–[8]. Furihata[10]–[11] has proposed a finite difference scheme which has inherited the decrease of the total energy. Choo and Chung[4] and Choo, Chung and Kim[5] have considered a conservative nonlinear difference scheme and obtained the corresponding error estimates.

In case of $\alpha = 0$, the equations (1.1)–(1.2) with the periodic boundary condition (1.3b) can be considered as the Kuramoto-Sivashinsky equation with $g_1(u, u_x) = 1, g_2(u, u_x) = -u, f(u, u_x) = \frac{1}{2}u_x^2$. Existence and uniqueness of the solution for the Kuramoto-Sivashinsky equation have been shown by Tadmor[16]. Finite difference schemes and finite element Galerkin methods have been applied by Akrivis([2],[3]). For the Kuramoto-Sivashinsky equation with Dirichlet boundary conditions, Manickam, Moudgalya and Pani[13] have applied a second order splitting method with orthogonal cubic spline collocation methods.

In this paper, we consider error estimates of approximate solutions for nonstandard finite difference methods. It is necessary to introduce nonstandard finite difference schemes in order to preserve positivity and energy decay of analytical solutions(see Mickens[14]). In Section 2, we introduce the general nonstandard finite difference scheme for (1.1)–(1.2) with either the boundary conditions (1.3a) or (1.3b). Some preliminary lemmas and discrete norms are given. In Sections 3, we briefly recall the Lax-Richtmyer equivalence theorem and obtain stability and error estimates for the difference equation by following the idea in Lopez-Marcos and Sanz-Serna[12]. In Section 4, we give specific examples.

2. Nonstandard finite difference scheme

Let $h = \frac{1}{M}$ be the uniform step size in the spatial direction for a positive integer M and $\Omega_h = \{x_i = ih | i = -2, -1, 0, \cdots, M, M+1, M+2\}$. Let $k = \frac{T}{N}$ denote the uniform step size in the temporal direction for a positive integer N. Denote $V_i^n = V(x_i, t_n)$ for $t_n = nk, n = 0, 1, \cdots, N$. For a function $V^n = (V_{-2}^n, V_{-1}^n, V_0^n, \cdots, V_M^n, V_{M+1}^n, V_{M+2}^n)$ defined on Ω_h, define the difference operators as for $0 \le i \le M$,

$$\nabla_+ V_i^n = \frac{V_{i+1}^n - V_i^n}{h}, \ \nabla_- V_i^n = \frac{V_i^n - V_{i-1}^n}{h}, \ \bar{\nabla} V_i^n = \frac{1}{2}(\nabla_+ + \nabla_-)V_i^n,$$
$$\nabla^2 V_i^n = \nabla_+(\nabla_- V_i^n) \text{ and } \nabla^4 V_i^n = \nabla^2(\nabla^2 V_i^n).$$

Further, define operators $V^{n+1/2}$ and $\partial_t V^n$, respectively, as

$$V_i^{n+1/2} = \frac{V_i^{n+1} + V_i^n}{2} \text{ and } \partial_t V_i^n = \frac{V_i^{n+1} - V_i^n}{k}.$$

Then the approximate solution U^n for (1.1)–(1.2) is defined as a solution of

$$\begin{aligned} \partial_t U_i^n + \nabla^2 &\left\{ g_1\left(U_i^{n+1/2}, \bar{\nabla} U_i^{n+1/2}\right) \nabla^2 U_i^{n+1/2} \right\} \\ &= -\nabla^2 g_2\left(U_i^{n+1/2}, \bar{\nabla} U_i^{n+1/2}\right) \\ &\quad + \nabla^\alpha P\left(U_i^n, U_i^{n+1}, \nabla_- U_i^{n+1/2}, \nabla_+ U_i^{n+1/2}\right) \end{aligned} \tag{2.1}$$

with the initial condition

$$U_i^0 = u_0(x_i), \ 0 \le i \le M \tag{2.2}$$

and the boundary conditions either

$$\bar{\nabla} U_i^n = 0, \ \bar{\nabla}\nabla^2 U_i^n = 0, \ i = 0, M, \ 1 \le n \le N \tag{2.3a}$$

or

$$U_i^n = U_{i+M}^n, \ -\infty < i < \infty, \ 0 \le n \le N. \tag{2.3b}$$

Here, $\nabla^1 = \bar{\nabla}$ and P may be any $C^{\alpha+2}(\mathbb{R}^4)$-function satisfying

$$P(u_i^n, u_i^{n+1}, \nabla_- u_i^{n+1/2}, \nabla_+ u_i^{n+1/2}) = f(u_i^{n+1/2}, u_{i,x}^{n+1/2}) + O(k^\gamma + h^\zeta)$$

with $1 < \gamma$ and $1 \le \zeta$. In the case of using the Neumann boundary conditions (1.3a) and $\alpha = 2$, P also must satisfy

$$P(x, y, -z, -w) = P(x, y, w, z), \ \forall x, y, z, w \tag{2.4}$$

in order to use an integration by parts in the proof of Theorem 3.3.

For simplicity, we denote $P\left(U_i^n, U_i^{n+1}, \nabla_- U_i^{n+1/2}, \nabla_+ U_i^{n+1/2}\right)$ by $[PU]_i^{n+1/2}$. Note that the discretized Neumann boundary conditions (2.3a) is equal to $U_{-1}^n = U_1^n$, $U_{M+1}^n = U_{M-1}^n$, $U_{-2}^n = U_2^n$ and $U_{M+2}^n = U_{M-2}^n$.

In order to consider the error estimates corresponding to the finite difference equation (2.1) with the initial condition (2.2) and the boundary condition (2.3a), we now introduce the discrete L^2-inner product

$$(V,W)_h = h\sum_{i=0}^{M}{}'' V_i W_i = h\Big\{\frac{1}{2}(V_0W_0 + V_M W_M) + \sum_{i=1}^{M-1} V_i W_i\Big\}$$

and the corresponding discrete L^2-norm

$$\|V\|_h = (V,V)_h^{\frac{1}{2}}$$

for functions $V = (V_{-2}, \cdots, V_{M+2})$ and $W = (W_{-2}, \cdots, W_{M+2})$ satisfying (2.3a). For the maximum norm, we define

$$\|V\|_\infty = \max_{0\le i\le M} |V_i|.$$

Hereafter, whenever there is no confusion, $(\cdot,\cdot)$ and $\|\cdot\|$ will denote $(\cdot,\cdot)_h$ and $\|\cdot\|_h$, respectively.

It follows from summation by parts that the following Lemma 2.1 holds.

LEMMA 2.1. *For functions V and W defined on Ω_h and satisfying the first equality in* (2.3a), *the following identity and inequalities hold.*

(1) $(\nabla^2 V, W) = -h\sum_{i=1}^{M}(\nabla_- V_i)(\nabla_- W_i) = (V, \nabla^2 W)$.
(2) $\|\nabla_+ V\|^2 \le 2\|V\|\|\nabla^2 V\|$.
(3) $\|\nabla_- V\|^2 \le 2\|V\|\|\nabla^2 V\|$.

The following lemma can be verified by summation by parts and using minimum eigenvalue of a symmetric matrix. For a proof, see Agarwal[1].

LEMMA 2.2. *Let M be any positive integer. If $V_0 = 0$, then*

$$\left\{2\sin\frac{\pi}{2(2M+1)}\right\}^2 \sum_{i=1}^{M} V_i^2 \le \sum_{i=1}^{M}(V_i - V_{i-1})^2.$$

LEMMA 2.3. *For a function V defined on Ω_h, the inverse inequality*

$$\|\bar{\nabla} V\| \le \frac{1}{h}\|V\|$$

holds.

Proof. From the definition of the discrete norm, it holds that

$$\|\bar{\nabla}V\|^2 = h\sum_{i=0}^{M}{}''\Big(\frac{V_{i+1}-V_{i-1}}{2h}\Big)^2 \leq \frac{2}{4h}\sum_{i=0}^{M}{}''(V_{i+1}^2+V_{i-1}^2)$$
$$= \frac{1}{2h}\Big[V_0^2+V_M^2+2\sum_{i=1}^{M-1}V_i^2\Big] = \frac{1}{h^2}\|V\|^2.$$

Thus, we obtain the inverse inequality. □

Using Lemma 2.2 and the definition of $\bar{\nabla}(\bar{\nabla})$, we obtain the following inequalities.

LEMMA 2.4. *Let V be a function defined on Ω_h and satisfying* (2.3a). *Then*

(1) $\|\bar{\nabla}V\| \leq 2\sqrt{2}\|\nabla^2 V\|$.
(2) $\|\bar{\nabla}(\bar{\nabla}V)\| \leq \frac{3\sqrt{2}}{4}\|\nabla^2 V\|$.

It follows from Lemma 2.3 and Lemma 2.4 that the following lemma holds.

LEMMA 2.5. *Let V be a function defined on Ω_h and satisfying* (2.3a). *Then*

(1) $\|V\|_\infty^2 \leq 3\|V\|^2 + 8\|V\|\|\bar{\nabla}V\|$.
(2) $\|\bar{\nabla}V\|_\infty^2 \leq 5\|\bar{\nabla}V\|^2 + \frac{9}{4}\|\nabla^2 V\|^2$.
(3) $\|\nabla^2 V\|_\infty^2 \leq (3+\frac{8}{h})\|\nabla^2 V\|^2$.

Proof. Let μ be an odd index such that

$$V_\mu^2 = \min\{V_{2\ell+1}^2 \mid 0 \leq 2\ell+1 \leq M, \ell \text{ is an integer}\}.$$

Then

$$V_\mu^2 = h\sum_{i=0}^{M}{}''V_\mu^2 \leq 3h\sum_{i=0}^{M}{}''V_i^2 = 3\|V\|^2.$$

For all positive integers j such that $\mu+2j \leq M$, we obtain

$$V_{\mu+2j}^2 - V_\mu^2 = \sum_{i=0}^{j-1}\{V_{2(i+1)+\mu}+V_{2i+\mu}\}\{V_{2(i+1)+\mu}-V_{2i+\mu}\}$$
$$= 2h\sum_{i=0}^{j-1}\{V_{2(i+1)+\mu}+V_{2i+\mu}\}\bar{\nabla}V_{2i+1+\mu}$$
$$< 8\|V\|\|\bar{\nabla}V\|.$$

This implies

$$V_{\mu+2j}^2 \le V_\mu^2 + 8\|V\|\|\bar{\nabla}V\| \le 3\|V\|^2 + 8\|V\|\|\bar{\nabla}V\|.$$

Similarly, we obtain for all positive integers j such that $\mu - 2j \ge 0$.

$$V_{\mu-2j}^2 \le V_\mu^2 + 8\|V\|\|\bar{\nabla}V\| \le 3\|V\|^2 + 8\|V\|\|\bar{\nabla}V\|.$$

Letting ν be an even index such that

$$V_\nu^2 = \min\{V_{2\ell}^2 \mid 0 \le 2\ell \le M,\ \ell \text{ is an integer}\}$$

and following the above idea, we obtain

$$V_{\nu+2j}^2 \le 3\|V\|^2 + 8\|V\|\|\bar{\nabla}V\| \text{ and } V_{\nu-2j}^2 \le 3\|V\|^2 + 8\|V\|\|\bar{\nabla}V\|.$$

Thus, the first inequality (1) is proved.

The second inequality (2) follows from the inequality (1) and Lemma 2.4.

Since we obtain

$$\|V\|_\infty^2 \le (3 + \frac{8}{h})\|V\|^2$$

from the inequality (1) and Lemma 2.3, the inequality (3) is proved. □

REMARK 2.1. For V and W defined on Ω_h and satisfying $V_i = V_{M+i}$ and $W_i = W_{M+i}$ with $i = 0, 1$, we obtain

$$(\nabla^2 V, W) = (V, \nabla^2 W).$$

And for the periodic boundary condition (2.3b), we also obtain similar results in Lemmas 2.1–2.5 which are sufficient for the proof of Theorem 3.3. For proofs, we refer to Ortega and Sanz-Serna[15].

3. Convergence of approximate solution

We recall the extension of Lax-Richtmyer equivalence theorem in Lopez-Marcos and Sanz-Serna[12] which makes us avoid the difficulty of direct proof for convergence arising specially in nonlinear problems. Let u be a solution of a problem $\Phi(u) = 0$ and u_h be a discrete evaluation of u on Ω_h. Let U_h be an approximate solution of u, which is obtained by solving the discrete equation

$$\Phi_h(U_h) = 0, \tag{3.1}$$

where $\Phi_h : \mathbf{X}_h \to \mathbf{Y}_h$ is a continuous mapping and $\mathbf{X}_h, \mathbf{Y}_h$ are normed spaces having the same dimension. The scheme (3.1) is said to be convergent if (3.1) has a solution U_h such that $\lim_{h\to 0} \|U_h - u_h\|_{\mathbf{X}_h} = 0$. The discretization (3.1) is said to be consistent if $\lim_{h\to 0} \|\Phi_h(u_h)\|_{\mathbf{Y}_h} = 0$.

The scheme (3.1) is said to be stable in the threshold R_h if there exists a positive constant C such that for an open ball $B(u_h, R_h) \subset \mathbf{X}_h$,

$$\|V_h - W_h\|_{\mathbf{X}_h} \leq C\|\Phi_h(V_h) - \Phi_h(W_h)\|_{\mathbf{Y}_h}, \ \forall\ V_h, W_h \in B(u_h, R_h).$$

The following theorem is the extended Lax-Richtmyer equivalence theorem which gives existence and convergence of approximate solutions. For the proof, see [12].

THEOREM 3.1. *Assume that the discrete equation* (3.1) *is consistent and stable in the threshold* R_h. *If* Φ_h *is continuous in* $B(u_h, R_h)$ *and* $\|\Phi_h(u_h)\|_{\mathbf{Y}_h} = o(R_h)$ *as* $h \to 0$, *then* (3.1) *has a unique solution* U_h *in* $B(u_h, R_h)$ *and there exists a constant* C *such that*

$$\|U_h - u_h\|_{\mathbf{X}_h} \leq C\|\Phi_h(u_h)\|_{\mathbf{Y}_h}.$$

According to Theorem 3.1, we have only to show that (2.1) is consistent and stable in the threshold in order to show the unique existence and convergence of approximate solutions.

Let Z_h^n be the set of all functions defined on Ω_h satisfying the discretized Neumann boundary condition (2.3a) at time level n $(0 \leq n \leq N)$. We take $\mathbf{X}_h = \mathbf{Y}_h = \prod_{n=0}^{N} Z_h^n$ and define a mapping $\mathbf{\Phi}_h : \mathbf{X}_h \to \mathbf{Y}_h$ by $\mathbf{\Phi}_h(\mathbf{U}) = \tilde{\mathbf{U}}$, where for $n = 0, \cdots, N-1$,

$$\tag{3.2} \begin{aligned} \tilde{U}_i^{n+1} &= \partial_t U_i^n + \nabla^2\Big\{g_1\left(U_i^{n+1/2}, \bar{\nabla} U_i^{n+1/2}\right)\nabla^2 U_i^{n+1/2}\Big\} \\ &\quad + \nabla^2 g_2\left(U_i^{n+1/2}, \bar{\nabla} U_i^{n+1/2}\right) - \nabla^\alpha [PU]_i^{n+1/2} \end{aligned}$$

and

$$\tag{3.3} \tilde{U}^0 = U^0 - u_0.$$

We take norms $\|\cdot\|_{\mathbf{X}_h}$ and $\|\cdot\|_{\mathbf{Y}_h}$ on $\mathbf{X}_h$ and $\mathbf{Y}_h$, respectively, such that

$$\|\mathbf{U}\|_{\mathbf{X}_h}^2 = \max_{0 \leq n \leq N} \|U^n\|^2 + k\sum_{n=0}^{N-1} \|\nabla^2 U^{n+1/2}\|^2$$

and

$$\|\tilde{\mathbf{U}}\|_{\mathbf{Y}_h}^2 = \|\tilde{U}^0\|^2 + k\sum_{n=1}^{N} \|\tilde{U}^n\|^2.$$

The consistency of the scheme (2.1)–(2.2) is obtained using Taylor's Theorem and the Mean Value Theorem.

THEOREM 3.2. *Let u be the solution of (1.1)–(1.2) with bounded derivatives $\frac{\partial^3 u}{\partial t^3}, \frac{\partial^6 u}{\partial x^6}$ and u_h be the discretized evaluation of u. Assume $g_1, g_2 \in C^4(\mathbb{R}^2), f \in C^{\alpha+2}(\mathbb{R}^2)$ and $P \in C^{\alpha+2}(\mathbb{R}^4)$. Then there exists a constant C such that for $1 < \gamma$ and $1 \le \zeta$,*

$$\|\mathbf{\Phi}_h(u_h)\|_{\mathbf{Y}_h} \le C(k^\gamma + h^\zeta).$$

We now consider the stability of the approximate solution in the threshold R_h.

THEOREM 3.3. *Let $\mathbf{\Phi}_h(\mathbf{U}) = \tilde{\mathbf{U}}$ and $\mathbf{\Phi}_h(\mathbf{V}) = \tilde{\mathbf{V}}$. Assume that $u_{xx} \in L^\infty(\Omega \times (0,T)), g_1, g_2, f \in C^1(\mathbb{R}^2), P \in C^1(\mathbb{R}^4)$ and $R_h = O(h^r k^s)$ with $r \ge \frac{1}{2}$ and $s \ge \frac{1}{2}$. Then there exists a constant C such that for any $\mathbf{U}$ and $\mathbf{V}$ in the ball $B(u_h, R_h)$,*

$$\|\mathbf{U} - \mathbf{V}\|_{\mathbf{X}_h} \le C\|\mathbf{\Phi}_h(\mathbf{U}) - \mathbf{\Phi}_h(\mathbf{V})\|_{\mathbf{Y}_h}.$$

Proof. Let $e^n = U^n - V^n$ and $\tilde{K}^n = \tilde{U}^n - \tilde{V}^n$. Replacing U^n and $\tilde{U}^n$ in (3.2) by V^n and $\tilde{V}^n$, respectively, and subtracting this result from (3.2), we obtain

$$\begin{aligned}
(3.4)\quad & \\
& \partial_t e_i^n + \nabla^2 \left\{ g_1 \left(U_i^{n+1/2}, \bar{\nabla} U_i^{n+1/2} \right) \nabla^2 e_i^{n+1/2} \right\} \\
&= \tilde{K}_i^{n+1} \\
&\quad + \nabla^2 \left[\left\{ g_1 \left(V_i^{n+1/2}, \bar{\nabla} V_i^{n+1/2} \right) - g_1 \left(U_i^{n+1/2}, \bar{\nabla} U_i^{n+1/2} \right) \right\} \nabla^2 V_i^{n+1/2} \right] \\
&\quad - \nabla^2 \left\{ g_2 \left(U_i^{n+1/2}, \bar{\nabla} U_i^{n+1/2} \right) - g_2 \left(V_i^{n+1/2}, \bar{\nabla} V_i^{n+1/2} \right) \right\} \\
&\quad + \nabla^\alpha \left([PU]_i^{n+1/2} - [PV]_i^{n+1/2} \right).
\end{aligned}$$

It follows from Lemma 2.5, the definition of $\|\cdot\|_{\mathbf{X}_h}$ and $R_h = O(h^r k^s)$ with $r \ge \frac{1}{2}$ and $s \ge \frac{1}{2}$ that for $\mathbf{V}$ in the ball $B(u_h, R_h)$,

$$\begin{aligned}
\left\| \nabla^2 V^{n+1/2} \right\|_\infty &\le \left\| \nabla^2 \left(V^{n+1/2} - u_h^{n+1/2} \right) \right\|_\infty + \left\| \nabla^2 u_h^{n+1/2} \right\|_\infty \\
&\le C \left(1 + \frac{1}{h^{1/2}} \right) \frac{h^r k^s}{k^{1/2}} + C \\
&\le C \left\{ 1 + k^{s-1/2} \left(\sum_{\ell=0}^{1} h^{r-\ell/2} \right) \right\}.
\end{aligned} \tag{3.5}$$

Using the Mean Value Theorem, Lemma 2.1 and (3.5), we obtain for some $0 < \xi_1 < 1$ and $0 < \xi_2 < 1$,

(3.6)
$$
\begin{aligned}
&\Big\|\Big\{g_1\Big(U^{n+1/2},\bar{\nabla}U^{n+1/2}\Big)-g_1\Big(V^{n+1/2},\bar{\nabla}V^{n+1/2}\Big)\Big\}\\
&\quad\times\nabla^2V^{n+1/2}\Big\|\Big\|\nabla^2e^{n+1/2}\Big\|\\
&\le\Big\|\nabla^2V^{n+1/2}\Big\|_\infty\Big\{\Big\|g_{1,x}\Big(\xi_1U^{n+1/2}+(1-\xi_1)V^{n+1/2},\bar{\nabla}U^{n+1/2}\Big)\Big\|_\infty\\
&\quad\times\Big\|e^{n+1/2}\Big\|+\Big\|g_{1,y}\Big(V^{n+1/2},\bar{\nabla}\xi_2U^{n+1/2}+\bar{\nabla}(1-\xi_2)V^{n+1/2}\Big)\Big\|_\infty\\
&\quad\times\Big\|\bar{\nabla}e^{n+1/2}\Big\|\Big\}\Big\|\nabla^2e^{n+1/2}\Big\|\\
&\le C\Big\|\nabla^2V^{n+1/2}\Big\|_\infty\Big(\Big\|e^{n+1/2}\Big\|+\Big\|\bar{\nabla}e^{n+1/2}\Big\|\Big)\Big\|\nabla^2e^{n+1/2}\Big\|\\
&\le C\Big\|\nabla^2V^{n+1/2}\Big\|_\infty\Big(\Big\|e^{n+1/2}\Big\|+\Big\|\nabla_-e^{n+1/2}\Big\|+\Big\|\nabla_+e^{n+1/2}\Big\|\Big)\Big\|\nabla^2e^{n+1/2}\Big\|\\
&\le C\Big\|\nabla^2V^{n+1/2}\Big\|_\infty\Big(\Big\|e^{n+1/2}\Big\|+\Big\|e^{n+1/2}\Big\|^{\frac12}\Big\|\nabla^2e^{n+1/2}\Big\|^{\frac12}\Big)\Big\|\nabla^2e^{n+1/2}\Big\|\\
&\le C\Big(\Big\|\nabla^2V^{n+1/2}\Big\|_\infty^2+\Big\|\nabla^2V^{n+1/2}\Big\|_\infty^4\Big)\Big\|e^{n+1/2}\Big\|^2+\frac{\mathcal{G}_0}{4}\Big\|\nabla^2e^{n+1/2}\Big\|^2\\
&\le C\Big\{1+\sum_{i=0}^{2}k^{(s-1/2)2^i}\Big(\sum_{\ell=0}^{1}h^{r-\ell/2}\Big)^{2^i}\Big\}\Big\|e^{n+1/2}\Big\|^2+\frac{\mathcal{G}_0}{4}\Big\|\nabla^2e^{n+1/2}\Big\|^2.
\end{aligned}
$$

Further, since $[PV]^{n+1/2} = P(V^n, V^{n+1}, \nabla_-V^{n+1/2}, \nabla_+V^{n+1/2})$, it follows from the Mean Value Theorem and Lemma 2.1 that for $\alpha = 0, 1$,

$$
\begin{aligned}
&\Big(\nabla^\alpha\Big\{[PU]^{n+1/2}-[PV]^{n+1/2}\Big\},e^{n+1/2}\Big)\\
&\le C\left\{\Big\|e^n\Big\|+\Big\|e^{n+1}\Big\|+\Big\|e^{n+1/2}\Big\|^{1/2}\Big\|\nabla^2e^{n+1/2}\Big\|^{1/2}\right\}\Big\|e^{n+1/2}\Big\|
\end{aligned}
\tag{3.7}
$$

and for $\alpha = 2$,

(3.8)
$$
\begin{aligned}
&\Big(\nabla^\alpha\Big\{[PU]^{n+1/2}-[PV]^{n+1/2}\Big\},e^{n+1/2}\Big)\\
&\le C\left\{\Big\|e^n\Big\|+\Big\|e^{n+1}\Big\|+\Big\|e^{n+1/2}\Big\|^{1/2}\Big\|\nabla^2e^{n+1/2}\Big\|^{1/2}\right\}\Big\|\nabla^2e^{n+1/2}\Big\|.
\end{aligned}
$$

Taking an inner product between (3.4) and $2e^{n+1/2}$, and applying Lemma 2.1, (3.6) and (3.7)–(3.8), we obtain

$$
\begin{aligned}
&\partial_t\|e^n\|^2 + 2\Big\|\sqrt{g_1(U^{n+1/2},\bar{\nabla}U^{n+1/2})}\nabla^2 e^{n+1/2}\Big\|^2 \\
&\quad\le \|\tilde{K}^{n+1}\|^2 + \|e^{n+1/2}\|^2 \\
&\quad + C\left\{1+\sum_{i=0}^{2} k^{(s-1/2)2^i}\left(\sum_{\ell=0}^{1} h^{r-\ell/2}\right)^{2^i}\right\}(\|e^n\|^2+\|e^{n+1}\|^2) \\
&\quad + \frac{\mathcal{G}_0}{2}\|\nabla^2 e^{n+1/2}\|^2 + C(\|e^n\|^2+\|e^{n+1}\|^2) + \frac{\mathcal{G}_0}{2}\|\nabla^2 e^{n+1/2}\|^2.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\partial_t\|e^n\|^2 + \mathcal{G}_0\|\nabla^2 e^{n+1/2}\|^2 \\
&\le \|\tilde{K}^{n+1}\|^2 + C\Big\{1+\sum_{i=0}^{2} k^{(s-1/2)2^i}\big(\sum_{\ell=0}^{1} h^{r-\ell/2}\big)^{2^i}\Big\}(\|e^n\|^2+\|e^{n+1}\|^2). \qquad (3.9)
\end{aligned}
$$

Applying the discrete Gronwall's inequality to (3.9), we obtain for $m \ge 0$,

$$
\|e^{m+1}\|^2 + k\sum_{n=0}^{m}\|\nabla^2 e^{n+1/2}\|^2 \le \mathcal{S}\Big\{\|e^0\|^2 + k\sum_{n=1}^{m+1}\|\tilde{K}^n\|^2\Big\},
$$

where $\mathcal{S} = \exp\left(\frac{C\{1+\sum_{i=0}^{2} k^{(s-1/2)2^i}(\sum_{\ell=0}^{1} h^{r-\ell/2})^{2^i}\}}{1-kC\{1+\sum_{i=0}^{2} k^{(s-1/2)2^i}(\sum_{\ell=0}^{1} h^{r-\ell/2})^{2^i}\}}T\right)$.

Since

$$
e^0 = U^0 - V^0 = \tilde{U}^0 - \tilde{V}^0 = \tilde{K}^0,
$$

the desired result is obtained. □

It follows from Theorem 3.1 that for $k = O(h^\ell)$ with $0 < \frac{r}{\gamma-s} < \ell < \frac{\zeta-r}{s}$, $r \ge \frac{1}{2}, s \ge \frac{1}{2}, \gamma > 1$ and $\zeta \ge 1$,

$$
\frac{\|\Phi_h(u_h)\|_{\mathbf{Y}_h}}{R_h} = O\Big(\frac{k^\gamma + h^\zeta}{h^r k^s}\Big) = O(h^{\ell(\gamma-s)-r} + h^{\zeta-r-\ell s}) \to 0 \text{ as } h \to 0. \qquad (3.10)
$$

Hence, applying Theorems 3.2–3.3 and (3.10) to Theorem 3.1, we obtain the following error estimate for (2.1)–(2.2).

THEOREM 3.4. *Suppose that hypotheses of Theorem 3.3 hold. Let* $\mathbf{U}$ *be a solution of* (2.1)–(2.3a). *Then for* $k = O(h^\ell)$ *with* $0 < \frac{r}{\gamma - s} < \ell < \frac{\zeta - r}{s}$, $r \geq \frac{1}{2}$ *and* $s \geq \frac{1}{2}$, *there exists a constant* C *such that for* $1 < \gamma$ *and* $1 \leq \zeta$,

$$\|\mathbf{U} - u_h\|_{\mathbf{X}_h} \leq C(k^\gamma + h^\zeta).$$

REMARK 3.1. Under the periodic boundary condition (2.3b), we may show that Theorem 3.4 also holds.

REMARK 3.2. Consider the partial differential equation, for $\eta > 0$,

$$u_t - \eta u_{xxt} + \frac{\partial^2}{\partial x^2} g(u, u_x, u_{xx}) = \frac{\partial^\alpha}{\partial x^\alpha} f(u, u_x)$$

with an initial condition (1.2) and the boundary conditions either (1.3a) or (1.3b). Let U^n be approximate solutions for

$$\begin{aligned} &\partial_t U_i^n - \eta \nabla^2 \partial_t U_i^n + \nabla^2 \Big\{ g_1(U_i^{n+1/2}, \bar{\nabla} U_i^{n+1/2}) \nabla^2 U_i^{n+1/2} \Big\} \\ &\quad = -\nabla^2 g_2(U_i^{n+1/2}, \bar{\nabla} U_i^{n+1/2}) \\ &\quad + \nabla^\alpha P(U_i^n, U_i^{n+1}, \nabla_- U_i^n, \nabla_- U_i^{n+1}, \nabla_+ U_i^n, \nabla_+ U_i^{n+1}), \end{aligned}$$

where P may be any function satisfying $P(u_i^n, u_i^{n+1}, \nabla_- u_i^n, \nabla_- u_i^{n+1}, \nabla_+ u_i^n, \nabla_+ u_i^{n+1}) = f(u_i^{n+1/2}, u_{i,x}^{n+1/2}) + O(k^\gamma + h^\zeta)$ with $1 < \gamma$ and $1 \leq \zeta$ and when $\alpha = 2$, P also must satisfy

(3.11)
$$P(x, y, -z_1, -z_2, -w_1, -w_2) = P(x, y, w_1, w_2, z_1, z_2), \ \forall x, y, z_1, z_2, w_1, w_2$$

in order to use an integration by parts in the proof of Theorem 3.3. Then we may show that Theorem 3.4 holds.

REMARK 3.3. We may show that Theorem 3.4 also holds for some nonlinear finite difference schemes such as P in (2.1) having additional independent variables $U_{i-1}^n, U_{i+1}^n, U_{i-1}^{n+1}$ and U_{i+1}^{n+1}.

4. Applications

In this section, we apply the nonstandard finite difference scheme to two partial differential equations. And we define $[Pu]_i^{n+1/2}$ satisfying

$$\begin{aligned} &P(u_i^n, u_i^{n+1}, \nabla_- u_i^n, \nabla_- u_i^{n+1}, \nabla_+ u_i^n, \nabla_+ u_i^{n+1}) \\ &\quad = f(u_i^{n+1/2}, u_{i,x}^{n+1/2}) + O(k^\gamma + h^\zeta) \end{aligned}$$

with $1 < \gamma$ and $1 \leq \zeta$.

EXAMPLE 4.1. Consider the Cahn-Hilliard equation

$$u_t = \frac{\partial^2}{\partial x^2}\left(\frac{\delta G}{\delta u}\right), \quad x \in \Omega, \ 0 < t \le T, \tag{4.1}$$

with the initial condition (1.2) and the boundary condition (1.3a). Here $\frac{\delta G}{\delta u} = -pu + su^3 - qu_{xx}$ with positive constants p, q, and s.

Furihata[10] proposed a stable finite difference scheme for (4.1) as follows.

$$\partial_t U_i^n = \nabla^2 \left(\frac{\delta G_d}{\delta U}\right)_i^{n+1/2}, \tag{4.2}$$

$$\left(\frac{\delta G_d}{\delta U}\right)_i^{n+1/2} = -pU_i^{n+1/2} + \frac{s}{4}\sum_{j=0}^{3}\left(U_i^{n+1}\right)^{3-j}\left(U_i^n\right)^j - q\nabla^2 U_i^{n+1/2}. \tag{4.3}$$

He showed that the scheme (4.2)–(4.3) inherits the conservation of mass and the decrease of the total energy from the equation (4.1).

In order to apply the nonstandard finite difference scheme (2.1) to (4.1)–(4.3), we rewrite (4.1)–(4.3) as

$$u_t + \frac{\partial^2}{\partial x^2}\left(q\frac{\partial^2 u}{\partial x^2}\right) = \frac{\partial^2}{\partial x^2}(-pu + su^3)$$

and

$$\partial_t U_i^n + \nabla^2(q\nabla^2 U_i^{n+1/2}) = \nabla^2 [PU]_i^{n+1/2},$$

where

$$[PU]_i^{n+1/2} = -pU_i^{n+1/2} + \frac{s}{4}\sum_{j=0}^{3}(U_i^{n+1})^{3-j}(U_i^n)^j.$$

These are the forms of (1.1) and (2.1) with

$$[Pu]_i^{n+1/2} = -pu_i^{n+1/2} + s(u_i^{n+1/2})^3 + O(k^2)$$

and P satisfies (2.4).

EXAMPLE 4.2. Consider the viscous Cahn-Hilliard equation

$$u_t - \eta\frac{\partial^2 u_t}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left(\frac{\delta G}{\delta u}\right), \quad x \in \Omega, \ 0 < t \le T, \tag{4.4}$$

with the initial condition (1.2) and the boundary condition (1.3a). Here η is a positive constant and $\frac{\delta G}{\delta u} = -pu + su^3 - \frac{1}{2}q_u(u)(u_x)^2 - q(u)u_{xx}$

with $0 < \mathcal{G}_0 \leq q(x) \leq \mathcal{G}_1$ for constants $\mathcal{G}_0$ and $\mathcal{G}_1$. Choo,Chung and Lee[6] proposed a finite difference scheme for (4.4) as follows.

$$\partial_t U_i^n - \eta \nabla^2 \partial_t U_i^n = \nabla^2 \left(\frac{\delta G_d}{\delta U} \right)_i^{n+1/2}, \tag{4.5}$$

(4.6)

$$\begin{aligned}\left(\frac{\delta G_d}{\delta U} \right)_i^{n+1/2} =& -pU_i^{n+1/2} + \frac{s}{4} \sum_{\ell=0}^{3} (U_i^{n+1})^{3-\ell} (U_i^n)^\ell \\ &+ \frac{1}{8} \frac{dq}{d\left(U_i^{n+1}, U_i^n\right)} \left\{ \sum_{\ell=0}^{1} \left(\nabla_+ U_i^{n+\ell} \right)^2 + \left(\nabla_- U_i^{n+\ell} \right)^2 \right\} \\ &- q(U)_i^{n+1/2} \nabla^2 U_i^{n+1/2} - \frac{1}{2} \left\{ \nabla_- q(U_i)^{n+1/2} \right\} \nabla_- U_i^{n+1/2} \\ &- \frac{1}{2} \left\{ \nabla_+ q(U_i)^{n+1/2} \right\} \nabla_+ U_i^{n+1/2},\end{aligned}$$

where

$$\frac{dq}{d(U_i, V_i)} = \begin{cases} \frac{q(U)_i - q(V)_i}{U_i - V_i}, & \text{for } U_i \neq V_i, \\ \frac{dq}{dU}(U_i), & \text{for } U_i = V_i. \end{cases}$$

They showed that the scheme (4.5)–(4.6) inherits the conservation of mass, the decrease of the total energy and the convergence of approximate solutions of (4.1).

In order to apply the nonstandard finite difference scheme (2.1) to (4.4)–(4.6), we rewrite (4.4)–(4.6) as

$$u_t - \eta u_{xxt} + \frac{\partial^2}{\partial x^2} \left\{ q(u) \frac{\partial^2 u}{\partial x^2} \right\} = \frac{\partial^2}{\partial x^2} \left\{ -pu + su^3 - \frac{1}{2} q_u(u)(u_x)^2 \right\}$$

and

$$\partial_t U_i^n - \eta \nabla^2 \partial_t U_i^n + \nabla^2 \{ q(U)_i^{n+1/2} \nabla^2 U_i^{n+1/2} \} = \nabla^2 [PU]_i^{n+1/2},$$

where

$$\begin{aligned}&[PU]_i^{n+1/2} \\ &= -pU_i^{n+1/2} + \frac{s}{4} \sum_{\ell=0}^{3} \left(U_i^{n+1}\right)^{3-\ell} (U_i^n)^\ell \\ &+ \frac{1}{8} \frac{dq}{d(U_i^{n+1}, U_i^n)} \left\{ \sum_{\ell=0}^{1} \left(\nabla_+ U_i^{n+\ell} \right)^2 + \left(\nabla_- U_i^{n+\ell} \right)^2 \right\} \\ &- \frac{1}{2} \left\{ \nabla_- q(U)_i^{n+1/2} \right\} \nabla_- U_i^{n+1/2} - \frac{1}{2} \left\{ \nabla_+ q(U)_i^{n+1/2} \right\} \nabla_+ U_i^{n+1/2}.\end{aligned}$$

These are the forms in Remark 3.2 with

$$[Pu]_i^{n+1/2} = -\, pu_i^{n+1/2} + s\left(u_i^{n+1/2}\right)^3 - \frac{1}{2}q_u\left(u_i^{n+1/2}\right)\left(u_{i,x}^{n+1/2}\right)^2 + O(k^2+h^2)$$

and P satisfies (3.11).

EXAMPLE 4.3. Consider the Kuramoto-Sivashinsky equation

$$u_t + \frac{\partial^2}{\partial x^2}(\nu u_{xx} + u) + uu_x = 0,\ x \in \mathbb{R},\ 0 < t \le T, \tag{4.7}$$

with the initial condition (1.2), the boundary condition (1.3b) and $\nu > 0$.

Akrivis[2] proposed a Crank-Nicolson-type finite difference scheme for (4.7) as follows.

$$\begin{aligned}&\partial_t U_i^n + \nabla^2\left(\nu\nabla^2 U_i^{n+1/2} + U_i^{n+1/2}\right)\\ &\quad + \frac{1}{3}\left(U_{i-1}^{n+1/2} + U_i^{n+1/2} + U_{i+1}^{n+1/2}\right)\bar{\nabla} U_i^{n+1/2} = 0.\end{aligned} \tag{4.8}$$

He derived the optimal-order error estimation of the approximation.

In order to apply the nonlinear finite difference scheme (2.1) to (4.7)–(4.8), we rewrite (4.7)–(4.8) as

$$u_t + \frac{\partial^2}{\partial x^2}\left(\nu u_{xx} + u\right) = -uu_x$$

and

$$\partial_t U_i^n + \nabla^2(\nu\nabla^2 U_i^{n+1/2} + U_i^{n+1/2}) = [PU]_i^{n+1/2},$$

where

$$[PU]_i^{n+1/2} = -\left(U_{i-1}^{n+1/2} + U_i^{n+1/2} + U_{i+1}^{n+1/2}\right)\frac{1}{2}\left(\nabla_- U_i^{n+1/2} + \nabla_+ U_i^{n+1/2}\right)$$

These are the forms in Remark 3.3 with

$$[Pu]_i^{n+1/2} = u_i^{n+1/2} u_{i,x}^{n+1/2} + O(k^2+h^2).$$

References

[1] R. P. Agarwal, *Difference equations and inequalities*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 155, Theory, Methods, and Applications, Marcel Dekker Inc., New York, 1992.

[2] G. D. Akrivis, *Finite difference discretization of the Kuramoto-Sivashinsky equation*, Numer. Math. **63** (1992), no. 1, 1–11.

[3] ———, *Finite element discretization of the Kuramoto-Sivashinsky equation*, Numerical Analysis and Mathematical Modelling, Banach Center Publ., vol. 29, Polish Acad. Sci., Warsaw, 1994, pp. 155–163.

[4] S. M. Choo and S. K. Chung, *Conservative nonlinear difference scheme for the Cahn-Hilliard equation*, Comput. Math. Appl. **36** (1998), no. 7, 31–39.

[5] S. M. Choo, S. K. Chung, and K. I. Kim, *Conservative nonlinear difference scheme for the Cahn-Hilliard equation. II*, Comput. Math. Appl. **39** (2000), no. 1-2, 229–243.

[6] S. M. Choo, S. K. Chung, and Y. J. Lee, *A Conservative difference scheme for the viscous Cahn-Hilliard equation with a nonconstant gradient energy coefficient*, Appl. Numer. Math. **51** (2004), no. 2-3, 207–219

[7] C. M. Elliott and D. A. French, *Numerical studies of the Cahn-Hilliard equation for phase separation*, IMA J. Appl. Math. **38** (1987), no. 2, 97–128.

[8] ———, *A nonconforming finite-element method for the two-dimensional Cahn-Hilliard equation*, SIAM J. Numer. Anal. **26** (1989), no. 4, 884–903.

[9] C. M. Elliott and S. Zheng, *On the Cahn-Hilliard equation*, Arch. Ration. Mech. Anal. **96** (1986), no. 4, 339–357.

[10] D. Furihata, *A stable and conservative finite difference scheme for the Cahn-Hilliard equation*, Numer. Math. **87** (2001), no. 4, 675–699.

[11] ———, *Finite difference schemes for* $\frac{\partial u}{\partial t} = (\frac{\partial}{\partial x})^{\alpha} \frac{\delta G}{\delta u}$, J. Comput. Phys. 181–205.

[12] J. C. López Marcos and J. M. Sanz-Serna, *Stability and convergence in numerical analysis. III. Linear investigation of nonlinear stability*, IMA J. Numer. Anal. **8** (1988), no. 1, 71–84.

[13] A. V. Manickam, K. M. Moudgalya, and A. K. Pani, *Second-order splitting combined with orthogonal cubic spline collocation method for the Kuramoto-Sivashinsky equation*, Comput. Math. Appl. **35** (1998), no. 6, 5–25.

[14] R. E. Mickens, *Applications of nonstandard finite difference schemes*, World Scientific, New Jersey, 2000.

[15] T. Ortega and J. M. Sanz-Serna, *Nonlinear stability and convergence of finite-difference methods for the "good" Boussinesq equation*, Numer. Math. **58** (1990), no. 2, 215–229.

[16] E. Tadmor, *The well-posedness of the Kuramoto-Sivashinsky equation*, SIAM J. Math. Anal. **17** (1986), no. 4, 884–893.

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