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SPECTRAL DUALITIES OF MV-ALGEBRAS

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Abstract. Hong and Nel in [8] obtained a number of spectral dualities between a cartesian closed topological category $\mathbf{X}$ and a category of algebras of suitable type in $\mathbf{X}$ in accordance with the original formalism of Porst and Wischnewsky[12]. In this paper, there arises a dual adjointness $S \vdash C$ between the category $\mathbf{X} = \mathcal{L}im$ of limit spaces and that $\mathbf{A}$ of MV-algebras in $\mathbf{X}$. We firstly show that the spectral duality: $S(\mathbf{A})^{op} \simeq C(\mathbf{X}^{op})$ holds for the dualizing object $K = I = [0,1]$ or $K = 2 = \{0,1\}$. Secondly, we study a duality between the category of Tychonoff spaces and the category of semi-simple MV-algebras. Furthermore, it is shown that for any $X \in \mathcal{L}im\,(X \neq \emptyset)$ $C(X, I)$ is densely embedded into a cube $I^{|H|}$, where H is a set.

1. Introduction

MV-algebras were originaly defined by C. C. Chang[4] as an algebraic counterpart to the Lukasiewicz infinite valued propositional calculus. In [11], Mundici established a categorical equivalence between the category of MV-algebras and that of abelian l-groups with order unit.

The aim of this paper is to investigate the dual adjunction between topological spaces (more generally limit spaces) and (topological) MV-algebras. In [8], the authors studied spectral dualities between the category $\mathbf{X} = \mathcal{L}im$ of limit spaces and that $\mathbf{A}$ of rings with unit in $\mathcal{L}im$ and pointed vector spaces in $\mathcal{L}im$.

In general, one considers a cartesian closed topological category $\mathbf{X}$, a category $\mathbf{A}$ of universal algebras of suitable type in $\mathbf{X}$ and a basic object $K \in \mathbf{A}$.

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Namely, $A \in \mathbf{A}$ is an $\mathbf{X}$-object and all $\mathbf{A}$-structures of a set of operators are $\mathbf{X}$-morphisms. In [12], they regarded a very general situations in which $\mathbf{A}$ is defined by $\mathbf{X}$-monad. But in this paper, we deal with $\mathbf{A}$ as an alternative method of admissible categories based on methods of universal algebras.

Let $C : \mathbf{X}^{op} \to \mathbf{A}$ be the function algebra functor and $S : \mathbf{A} \to \mathbf{X}^{op}$ be the spectral space functor: namely $C(X)$ is the $\mathbf{A}$-object of all $X \to K$ and $S(A)$ is the $\mathbf{X}$-object of all $A \to K$.

For a suitable choice of $\mathbf{X}, \mathbf{A}, K$, it may happen that $S(\mathbf{A})^{op}$ and $C(\mathbf{X}^{op})$ are categorically equivalent. This dual equivalence is called a spectral duality, which is then the largest duality in the $(\eta, \varepsilon) : S \dashv C$.

In this paper, we first establish a spectral duality between $\mathbf{X} = \mathcal{L}im$ and the category $\mathfrak{M}_v$ of MV-algebras in $\mathbf{X}$ while K is the unit interval topological MV-algebra I, and subsequently the discrete two point Boolean algebra $2 = \{0, 1\}$. This result reduced the classical Stone duality.

In the second part of this paper, the dualities between the category of Tychonoff spaces and that of semisimple MV-algebras were investigated. We obtained useful characterization theorems that the counit ε is surjective i.e., an isomorphism. This criterion is a general result from the case of $\mathbf{Zdim}$ of zero-dimensional spaces with $K = 2$ to the case of Tychonoff spaces with $K = I$.

2. Spectral duality of MV-algebras

Let $\mathbf{X} = \mathcal{L}im, A = \mathfrak{M}_v$ and $K = I = [0, 1]$ with the usual topology. In this section, we first prove that for these $\mathbf{X}, \mathbf{A}$ and K, the spectral duality holds.

We note that I is a topological MV-algebra so that $I \in A$. Indeed, all operations of I are continuous. The function algebra functor $C : \mathbf{X}^{op} \to \mathbf{A}$ is defined by $C(X) = \hom_{\mathbf{X}}(X, I)$ for $X \in \mathbf{X}$, thus $C(X) \in A$. And the spectral space functor $S(A) = \hom_{\mathbf{A}}(A, I)$ for $A \in \mathbf{A}$, thus $S(A) \in \mathbf{X}$. The unit η is defined by $\eta_A : A \to CS(A)$ with $\eta_A(a)(u) = u(a)$ for each $a \in A, u \in S(A)$ and each $A \in \mathbf{A}$. And the counit ε is that for $X \in \mathbf{X}^{op}$, $\varepsilon_X : X \to SC(X)$ with $\varepsilon_X(x)(f) = f(x)$ for each $x \in X$ and each $f \in C(X)$. Then it is a routine calculation that S is a left adjoint to C via η and ε.

Let Fixη be the isomorphism closed subcategory of $\mathbf{A}$ determined by objects $A \in \mathbf{A}$ such that η_A is an isomorphism. Dually Fixε the isomorphism closed subcategory of $\mathbf{X}$ determined by objects $X \in \mathbf{X}$

such that ε_X is an isomorphism in $\mathbf{X}$. Then generally we have the largest duality $(\mathrm{Fix}\varepsilon)^{op} \simeq \mathrm{Fix}\eta$ in any $(\eta, \varepsilon) : S \dashv C$.

Recall that a category $\mathbf{X}$ is called an $(\mathbf{E}, \mathbf{M})$-category if for the class $\mathbf{E}$ of epimorphisms, the class $\mathbf{M}$ of monomorphism, and for any $\mathbf{X}$-morphism f, $f = m \cdot e$ for some $m \in \mathbf{M}$ and some $e \in \mathbf{E}$. Moreover, for $e \in \mathbf{E}$, $m \in \mathbf{M}$, and $f, g \in \mathbf{X}$, if $mf = ge$, then there exists $h \in \mathbf{X}$ uniquely such that $he = f$ and $mh = g$.

It is well known for an example that $\mathcal{L}im$ is an (**Onto Embedding**) category.

For $(\eta, \varepsilon) : S \dashv C$, if $\varepsilon_X \in \mathbf{M}$ for $X \in \mathbf{X}$, then X is said to be $\mathbf{M}$-embeddable. The class of all M-embeddable objects in $\mathbf{X}$ is denoted by **Emb**. Then **Emb** is an $\mathbf{E}$-reflective in $\mathbf{X}$. Furthermore, $S(A) \in$**Emb** for any $A \in \mathbf{A}$ [12].

By a completely regular filter on a Tychonoff space X, we mean a filter $\mathcal{F}$ on X which have an open base $\mathcal{B}$ such that for each $B \in \mathcal{B}$, there exists $C \in \mathcal{B}$ and $f \in C(X, I)$ such that $C \subset B, f(C) = \{0\}$ and $f(X - B) = \{1\}$. Recall that a completely regular filter $\mathcal{F}$ is maximal iff for two open sets $D \subset C$ and for $f \in C(X, I)$ with $f(D) = \{0\}$ and $f(X - C) = \{1\}$, we have either $C \in \mathcal{F}$ or $C \notin \mathcal{F}$, and $\mathcal{F}$ has a member B such that $B \cap D = \emptyset$.

Recall that the full subcategory $\mathbf{T}$ of Tychonoff spaces is a reflective hull of I in $\mathbf{X} = \mathcal{L}im$. Once again, consider $S \dashv C : \mathbf{X}^{op} \to \mathbf{A}$, where $\mathbf{X} = \mathcal{L}im, \mathbf{A} = \mathfrak{M}_v$, and $K = I$. Then we have the following lemma :

LEMMA 1. *Let $X \in \mathbf{X}$ be a Tychonoff space. For any $h \in SC(X, I)$, there exists a maximal completely regular filter $\mathcal{F}$ on X such that $h(f) = \lim f(\mathcal{F})$ for any $f \in C(X, I)$.*

Proof. For $f \in h^{-1}(0)$ and $\varepsilon > 0$, let $w(f, \varepsilon) = \{x \in X | f(x) < \varepsilon\}$. Let $\mathcal{B} = \{w(f, \varepsilon) | f \in h^{-1}(0), \varepsilon > 0\}$. Then $\mathcal{B}$ is a filter base on X. Indeed, $w(f, \varepsilon) \neq \emptyset$ for any $f \in h^{-1}(0)$ and any $\varepsilon > 0$. For, if it is empty, $f(x) \geq \varepsilon$ for all $x \in X$. Then $(nf)(x) \geq n \cdot \varepsilon = 1$ for a large $n \in Z^+$ in the MV-algebra $C(X)$. Thus $nf = \mathbf{1}$ and hence $n \cdot h(f) = 1$. This is absurd. Since h preserves the join-operation : $f \vee g = f + f^* g$, it is easy to see that $w(f, \varepsilon) \cap w(g, \delta)$ contains $w(f \vee g, \varepsilon \wedge \delta)$. Let $\mathcal{F}$ be the filter generated by $\mathcal{B}$ on X. Show that $\mathcal{F}$ is a complete regular filter. Indeed for $B = w(f, \varepsilon) \in \mathcal{B}$, choose $C = w(f, \varepsilon/2) \in \mathcal{B}$.

Now define a continuous map g from I into I as follows :

$$g(x) = \begin{cases} 0 & \text{if} \quad 0 \leq x \leq \varepsilon/2, \\ (2/\varepsilon)x - 1 & \text{if} \quad \varepsilon/2 \leq x \leq \varepsilon, \\ 1 & \text{if} \quad \varepsilon \leq x \leq 1. \end{cases}$$

Let $e = g \circ f$. Then $e \in C(X)$. Further $e(C) = \{0\}$, $e(X - B) = \{1\}$. Thus $\mathcal{F}$ is completely regular.

Now we claim that $\mathcal{F}$ is maximal. Consider two open subsets D and C of X with $D \subset C$ and $f \in C(X)$ such that $f(D) = \{0\}$ and $f(X - C) = \{1\}$. Suppose that $C \notin \mathcal{F}$. Then $h(f) \neq 0$. For, if $h(f) = 0$, then $w(f, \frac{1}{2}) \subset C$ because of $w(f, \frac{1}{2}) \cap (X - C) = \emptyset$. Thus $C \in \mathcal{F}$, a contradiction. If $h(f) = r \neq 0$, then for the constant function $\mathbf{r}$ of r, $w(f^*\mathbf{r}, r/2) \in \mathcal{F}$ and it is disjoint from D because if $x \in w(f^*\mathbf{r}, r/2) \cap D$ then, since $h(\mathbf{r}) = r$ we have $(1 - f(x)) \cdot r < r/2$ and $f(x) = 0$, thus $r < r/2$. This is impossible. Hence by the criterion of maximality, $\mathcal{F}$ is a maximal completely regular filter. Finally, we show that $h(f) = \lim f(\mathcal{F})$ for each $f \in C(X)$. Indeed, if $h(f) = r \in I$, then $h(f\mathbf{r}^*) = 0$ and $h(\mathbf{r} \cdot f^*) = 0$. Thus $w(f\mathbf{r}^*, \varepsilon)$ and $w(\mathbf{r}f^*, \varepsilon) \in \mathcal{B} \subset \mathcal{F}$. For any basic nbd$(r - \varepsilon, r + \varepsilon)$ of r in I, we have $f(w(\mathbf{r}f^*, \varepsilon) \cap w(f\mathbf{r}^*, \varepsilon)) \subset (r - \varepsilon, r + \varepsilon)$. Hence $\lim f(\mathcal{F}) = r = h(f)$ for each $f \in C(X)$. The proof is complete. □

Let $S(\mathbf{A})$ be the isomorphism closed full subcategory of $\mathbf{X}$ determined by $S(A)$ for all $A \in \mathbf{A}$, and let $C(\mathbf{X}^{op})$ be the isomorphism closed full subcategory of $\mathbf{A}$ determined by $C(X)$ for all $X \in \mathbf{X}$.

Then we have the spectral duality theorem :

THEOREM 2. *For* $\mathbf{X} = \mathcal{L}im, \mathbf{A} = \mathfrak{M}_v$ *and* $K = I$, *we have that* $S(\mathbf{A})^{op} \simeq C(\mathbf{X}^{op})$. *Furthermore,* $S(\mathbf{A}) \simeq \mathrm{Fix}\varepsilon$ *and* $C(\mathbf{X}^{op}) \simeq \mathrm{Fix}\eta$.

Proof. Since $\mathbf{X} = \mathcal{L}im$ is an (**Onto**, **Embedding**)-category, it is enough to show that ε_X is onto for every $X \in \mathbf{X}$.

Since $\mathbf{T}$ is epi-reflective in **Top**(=category of topological spaces) and **Top** is bireflective in $\mathcal{L}im$, $\mathbf{T}$ is epi-reflective in $\mathcal{L}im$.

Let $c : X \to cX$ be the $\mathbf{T}$-reflective of $X \in \mathbf{X} = \mathcal{L}im$ and $cX \in \mathbf{T}$. For any $h \in SC(X)$ for any $X \in \mathbf{X}$, let $h_c = SC(c)h$. By lemma 1, for h_c there exists a maximal completely regular filter $\mathcal{F}$ on cX such that $h_c(f) = \lim f(\mathcal{F})$ for each f in $C(cX)$.

We claim $\mathcal{F}$ is convergent on cX, which can be shown as in the proof of (theorem 1(a) in [8]). Note that since $K = I$, $\mathcal{F}$ is not necessary to have the finite intersection property. Say $\mathcal{F}$ converges $c(x) \in cX$, where $x \in X$. For each $f \in C(X)$ there exists f_c uniquely such that $f_c \cdot c = f$. Thus $h(f) = h(f_c \circ c) = h_c(f_c) = \lim f_c(\mathcal{F}) = f_c(c(x)) = f(x) = \varepsilon_X(x)(f)$. Thus $h = \varepsilon_X(x)\,(x \in X)$. The proof is complete. □

Next consider the same $\mathbf{X} = \mathcal{L}im$, $\mathbf{A} = \mathfrak{M}_v$ but $K = 2 = \{0, 1\}$ with $0 < 1$ and the discrete topology. Clearly $2 \in \mathbf{A}$. Namely $C(X) = \hom_{\mathbf{X}}(X, 2)$, $S(A) = \hom_{\mathbf{A}}(A, 2)$.

THEOREM 3. *For* $\mathbf{X} = \mathcal{L}im, \mathbf{A} = \mathfrak{M}_v$ *and* $K = 2$, *we have that* $S(\mathbf{A})^{op} \simeq C(\mathbf{X}^{op})$. *Furthermore,* $S(\mathbf{A}) \simeq$Fixε *and* $C(\mathbf{X}^{op}) \simeq$Fix$\eta$.

Proof. It is enough to show that for $X \in \mathbf{X}, \varepsilon_X$ is surjective. For each $h \in SC(X), h^{-1}(0)$ is a maximal ideal in $C(X)$. Consider $\mathcal{F} = \{f^{-1}(0) | f \in h^{-1}(0)\}$. One can easily show that $\mathcal{F}$ is a maximal clopen filter (base) on X. The proof is virtually the same as the ring case.

Now we claim that $h(f) = \lim f(\mathcal{F})$ for all $f \in C(X)$. Indeed, suppose that $h(f) = 0$. Then since $f^{-1}(0) \in \mathcal{F}$ we have $\lim f(\mathcal{F}) = 0$. Now suppose that $h(f) = 1$. Let g be the characteristic function of the set $f^{-1}(0)$. Then $g \in C(X)$ and $f \cdot g = \mathbf{0}$ in $C(X)$. For, $(fg)(x) = f(x) \cdot g(x) = \max\{0, f(x) + g(x) - 1\} = 0$ for all $x \in X$. Thus $fg = \mathbf{0}$.
Hence $h(g) = 0$, thus $g^{-1}(0) \in \mathcal{F}$. We have $f(g^{-1}(0)) = \{1\}$. Hence $\lim f(\mathcal{F}) = h(f)$.

Now let **Zdim** be the full subcategory of zero-dimensional spaces. It is well known that **Zdim** is an epi-reflective in $\mathbf{X}$. Let $d : X \to dX$ be the **Zdim**-reflection of $X \in \mathbf{X}$, and let $SC(d)(h) = h_d$ for each $h \in SC(X)$. Then for any $f \in C(X)$, there exists $\overline{f} \in C(dX)$ with $\overline{f}d = f$ uniquely. Moreover $h(f) = h(\overline{f}d) = h(C(d)\overline{f}) = SC(d)(h)(\overline{f}) = h_d(\overline{f})$.

For any $h \in SC(X)$, i.e., $h_d \in SC(dX)$, there exists a clopen filter $\mathcal{F}$ on dX such that $h_d(\overline{f}) = \lim \overline{f}(\mathcal{F})$ for all $\overline{f} \in C(dX)$. The proof that $\mathcal{F}$ is convergent, and the remaining proofs are virtually the same as in the proof of Theorem 2. □

Let **TBoo** be the full subcategory of $\mathfrak{M}_v$ of topological Boolean algebras. Then we obtain the classical Stone duality in the following corollary :

COROLLARY. $(\mathbf{Zdim})^{op} \simeq \mathbf{TBoo}$.

Proof. One can easily see that $S(\mathbf{TBoo}) \simeq \mathbf{Zdim}$ and $C((\mathbf{Zdim})^{op}) \simeq \mathbf{TBoo}$. □

3. Dualities between Tychonoff spaces and semi-simple MV-algebras

REMARK 1. In [3], Belluce showed that for an MV-algebra A the following statements are equivalent.

(i) A is isomorphic to the set of all fuzzy subsets of some nonempty set X, i.e., $A \cong I^X$ (cube).

(ii) A is complete and sub-atomic.

It is easy to that these statements are equivalent to

(iii) A is complete, $B(A)$ is atomic and A is atomless.

REMARK 2. In [6], it is shown that for each semi-simple MV-algebra A, A is embedded into I^H, say the embedding e, where $H = |S(A, I)|$. But I^H is a topological MV-algebra which is compact Hausdorff.

We have $A \hookrightarrow e(A) \subset \delta(A) = \Gamma(e(A))$ in I^H, where Γ is the closure operation of the product topology of I^H. Then $\delta(A)$ is a compact topological MV-algebra which is complete and $B(\delta(A))$ is complete atomic. Let X be the set of all atoms of $B(\delta(A))$. Then $A \cong e(A) \subset \delta(A) \cong I^{|H_0|} \times \Pi\{A(m)|m \in \Lambda \subset Z^+\}$, where $|H_0| \cup |\Lambda| = |X|$ (disjoint). This $\delta(A)$ is called *the δ-completion of A.*

For $a \in X$, we have $a \in H_0$ when $\downarrow a = Aa$ is atomless and we have $a \in \Lambda$ when $\downarrow a = Aa \cong A(m)$ for some $m \in Z^+$, where $\downarrow a = \{x | x \leq a\}$. Hence if A is atomless then $\delta(A) \cong I^{|X|}$. We would say that any semi-simple MV-algebra that is atomless is densely embedded into a cube $I^{|X|}$. Of course, A itself is not necessary to be a topological MV-algebra but $e(A)$ has the relative topology under which it is regarded as a topological MV-algebra.

REMARK 3. In [6], it is also shown that for $e : A \hookrightarrow \delta(A)$ (embedding) e is I-extensive. This reduces that if B is complete and its idempotents are atomic then for a homomorphism $f : A \to B$, there exists a unique homomorphism $F : \delta(A) \to B$ such that $F \circ e = f$.

LEMMA 4. *Let $X \in \mathbf{X} = \mathcal{L}im$, and let $|X|$ be the underlying set of X with the discrete structure. Then $|X|$ and $|SC(|X|)|$ are equipotent.*

Proof. Consider $C(|X|)$. Clearly, it is isomorphic to $I^{|X|}$. Now we find the all maximal ideals of $I^{|X|}$. Clearly $B(I^{|X|})$ is a complete Boolean algebra whose atoms are $\{\overline{e}_x | x \in X\}$, where $\overline{e}_x$ has all zero components but the x^{th}-component is 1 for each $x \in X$. Since $I^{|X|}$ is a compact MV-algebra, $B(I^{|X|})$ is a power set Boolean algebra with atoms $\{\overline{e}_x | x \in X\}$. Every maximal ideal M of $I^{|X|}$ is compact because it is closed. There exists an $\overline{e}^*_x$ such that $M = \downarrow \overline{e}^*_x$, where $\overline{e}^*_x$ are the coatoms of $B(I^{|X|})$. Hence $|X|$ and $|\mathfrak{M}(I^{|X|})|$ are equivalent under the map $x \longmapsto \overline{e}^*_x$. On the other hand $\mathfrak{M}(I^{|X|})$ and $S(I^{|X|})$ are also equipotent and $|X|$ and $|SI(|X|)|$ are equipotent. □

THEOREM 5. *Let $X \in \mathcal{L}im, (X \neq \emptyset)$. Then the δ-completion of the semi-simple MV-algebra $C(X)$ is isomorphic to a cube $I^{|SC(X)|}$.*

Proof. Firstly we have to show that for any non void $X \in \mathcal{L}im, C(X)$ is a semi-simple MV-algebra that is atomless. Clearly $C(X)$ is embedded into $I^{|X|}$. Thus it is semi-simple. Now show that $C(X)$ is atomless. For any $f \in C(X)$ with $f \neq 0$, we claim that $\frac{1}{2}f < f$ in $C(X)$. Indeed, setting $g = \frac{1}{2}f$, we have $g \leq g \vee f = f + f^*g$ in $C(X)$, $g(x) \leq g(x) \vee f(x) = f(x) + f^*(x)g(x) = f(x)$ in I for each $x \in X$. Show that $gf^* = 0$, i.e., $g \leq f$. Since $g(x) \leq f(x), (gf^*)(x) = 0$ for any $x \in X$. Thus $gf^* = 0$. Obviously, $g \neq f$, and hence $g < f$. Hence $C(X)$ has no atom. By Remark 2, $\delta(C(X)) \cong I^{S(C(X))}$. □

LEMMA 6. *For $X \in \mathbf{T}$, the maximal space $(\mathfrak{M}(I^{|X|}), \tau_z)$ of $I^{|X|}$ with the Zarski topology is the discrete space.*

Proof. For each $M \in \mathfrak{M}(I^{|X|})$ we have $M = \downarrow \overline{e}_x^*$ for some $x \in |X|$. For any non void subfamily $\{M_x | x \in \Gamma\}$ of $\mathfrak{M}(I^{|X|})$, let $a = \inf\{\overline{e}_x^* | x \in \Gamma\}$ and let $b = \inf\{\overline{e}_y^* | y \notin \Gamma\}$. Since $I^{|X|}$ is semi-simple, we have $a \wedge b = 0$.

Furthermore, $a^* = \sup\{\overline{e}_x | x \in \Gamma\}$ and $b^* = \sup\{\overline{e}_y | y \notin \Gamma\}$. Claim that $\hat{b} = \{M_x | x \in \Gamma\}$. Indeed, if $b \in M_x = \downarrow \overline{e}_x^*$, then $\overline{e}_x \leq b^*$. This is impossible. Thus $M_x \in \hat{b}$ for all $x \in \Gamma$. Hence τ_z is discrete topology. Hence $|X|$ and $\mathfrak{M}(I^{|X|}, \tau_z)$ are homeomorphic as the discrete spaces. □

LEMMA 7. *For $X \in \mathbf{T}, \varepsilon_{|X|} : |X| \to S(I^{|X|}, \tau_p)$ is surjective.*

Proof. By [6], $\Phi : S(I^{|X|}, \tau_p) \to \mathfrak{M}(I^{|X|}, \tau_z)$ defined by $\Phi(u) = u^{-1}(0)$ for $u \in S(I^{|X|})$ is a continuous bijection. Let $\Psi : |X| \to \mathfrak{M}(I^{|X|}, \tau_z)$ be the homeomorphism by $\Phi(x) = \downarrow \overline{e}_x^*$ for each $x \in |X|$. Thus $\Phi \cdot \varepsilon_{|X|} = \Psi$. It follows that $\varepsilon_{|X|}$ is surjective. □

The following criterion whether ε_X for $X \in \mathbf{T}$ is surjective is a generalized result from the case of **Zdim** with $k = 2$ to the case $\mathbf{T}$ with $K = I$.

LEMMA 8. *For $(\eta, \varepsilon) : S \dashv C$, ε_X is onto for $X \in \mathbf{T}$ iff $\bigcap_{r \in I} \mathcal{U}_h^{(r)} \neq \emptyset$, where $\mathcal{U}_h^{(r)} = \bigcap \{u^{-1}(r) | u \in h^{-1}(r)\}$ for each $r \in I$.*

Proof. Assume that ε_X is onto. Then for each $h \in SC(X)$ there exists $x \in X$ such that $h = \varepsilon_X(x)$. For each $r \in I$ and for each $u \in h^{-1}(r)$, $h(u) = \varepsilon_X(x)u = u(x) = r$.

Thus $x \in u^{-1}(r)$, i.e., $x \in \mathcal{U}_h^{(r)}$ and $x \in \bigcap_{r \in I} \mathcal{U}_h^{(r)}$.

Conversely, $\bigcap_{r \in I} \mathcal{U}_h^{(r)} \neq \emptyset$. Then $\bigcap \mathcal{U}_h^{(r)} = \{x\}$, for some $x \in X$. For, if $x, y \in \bigcap \mathcal{U}_h^{(r)}$ and $x \neq y$, then there exists $u \in C(X)$ such that

$u(x) \neq u(y)$, say $u(x) = r$ and $u(y) = s$. Let $h(u) = t$. Then $x \in \mathcal{U}_h^{(t)} \cap \mathcal{U}_h^{(r)}$, $x \in u^{-1}(t)$ and $x \in u^{-1}(r)$, $u \in h^{-1}(t)$ and $u \in h^{-1}(r)$. Thus $t = r = h(u)$. Similarly $t = s$, a contradiction. Now show that $\varepsilon_X(x) = h$. For each $u \in C(X)$, let $h(u) = r$. Then $u \in h^{-1}(r)$. Since $x \in \bigcap \mathcal{U}_h^{(r)}$, $x \in u^{-1}(r)$, i.e., $u(x) = r$. i.e., $\varepsilon_X(x)(u) = h(u)$. Thus $h = \varepsilon_X(x)$. □

For $A \in \mathbf{A}$, a family $\{A_r | r \in I\}$ are said to be *completely separated* if there exists $h \in S(A)$ such that $A_r \subset h^{-1}(r)$ for each $r \in I$. For any object $X \in \mathbf{X}$, the underlying set $|X|$ of X with discrete topology is also an object of $\mathbf{X}$. For the identity $\mathrm{id}_X : |X| \to X$ in $\mathbf{X}$, $C(\mathrm{id}_X) : C(X) \to C(|X|) = I^X$ is an embedding.

THEOREM 9. *For an object $X \in \mathbf{T}$, the following statements are equivalent :*

(1) *ε_X is an epimorphism, i.e., an isomorphism.*
(2) *For each $h \in SC(X)$ $\bigcap_{r \in I} \mathcal{U}_h^{(r)} = \{x\}$ for some $x \in X$, where $\mathcal{U}_h^{(r)}$ is the same as in* lemma 6.
(3) *Any completely separated family $\{A_r | r \in I\}$ in $C(X)$ is also completely separated in $I^{|X|}$.*
(4) *For each $h \in SC(X), \{\Gamma(h^{-1}(r)) | r \in I\}$ are mutually disjoint in $I^{|X|}$, where Γ is the closure operation.*
(5) *$SC(i_x)$ is surjective.*

Proof. (1) $\iff$ (2) We have proved in lemma 6.

(2) $\implies$ (3) Let $\{A_r | r \in I\}$ be a completely separated family in $C(X)$, i.e., there exists a $h \in SC(X)$ such that $A_r \subset h^{-1}(r)$ for each $r \in I$. If $u \in A_r \subset h^{-1}(r)$, then $\mathcal{U}_h^{(r)} \subset u^{-1}(r)$ for each $r \in I$. Since $x \in \bigcap \mathcal{U}_h^{(r)}, u(x) = r$. $u \in pr_x^{-1}(r)$ for each r, and hence $A_r \subset pr_x^{-1}(r)$, where pr_x is the x^{th}-projection $C(|X|)$ onto I. But on the other hand, $pr_x \in SC(|X|) = S(I^{|X|})$. Thus $\{A_r | r \in I\}$ is completely separated in $I^{|X|}$.

(3) $\implies$ (4) Since $\{h^{-1}(r) | r \in I\}$ is complete separated in $C(X)$, by (3) it is completely separated in $I^{|X|}$. Thus there exists a $\overline{h} \in S(I^{|X|})$ such that $h^{-1}(r) \subset \overline{h}^{-1}(r)$. But $Cl(h^{-1}(r)) \subset \overline{h}^{-1}(r)$. Hence (4) holds.

(4) $\implies$ (5) To show this end, we prove firstly that (4) implies (1) as follows: for each $h \in SC(X), \{h^{-1}(r) | r \in I\}$ is complete separated. By (4), they are complete separated in $I^{|X|}$. It follows that there exists

$\overline{h} \in S(I^{|X|})$ such that $h^{-1}(r) \subset \overline{h}^{-1}(r)$ for each $r \in I$. But by lemma 8, $\varepsilon_{|X|}$ is surjective. Hence there exists $x \in |X|$ such that $\overline{h} = \varepsilon_{|X|}(x)$.

Claim that for the same $x \in X, h = \varepsilon_X(x)$. Indeed for any $u \in C(X)$, $h(u) = r$ iff $u \in h^{-1}(r)$ implies $u \in \overline{h}^{-1}(r)$ iff $\overline{h}(u) = r$ iff $\varepsilon_{|X|}(x)(u) = r$ iff $u(x) = r$. Thus $h(u) = u(x) = \varepsilon_X(x)(u)$, which says that $h = \varepsilon_X(x)$ for $x \in X$. Hence $X \in \mathrm{Fix}\varepsilon$.

On the other hand, the δ-completion $\delta(C(X))$ of $C(X)$ is isomorphic to a cube $I^{|SC(X)|}$ by theorem 5 which is isomorphic to $I^{|X|}$ because $SC(X) \cong X$ by ε_X. By the universal property of δ, for a homomorphism $h : C(X) \to I$, there exists an extension $\overline{h} : \delta(C(X)) \to I$ such that $\overline{h}e = h$, where e is the embedding of $C(X)$ into $\delta(C(X))$. Let $i_X : |X| \to X$ be the identity function. Then i_X is a $\mathbf{X}$-morphism. Thus $C(i_X) : C(X) \to C(|X|)(\cong I^{|X|})$. Further, $\overline{h} \cdot C(i_X) = h$ because $C(i_X)$ and e are identified since $\delta C(X) \cong C(|X|)$. Namely $SC(i_X)(\overline{h}) = h$. Hence $SC(i_X)$ is surjective.

(5) $\Longrightarrow$ (1) By $(\eta, \varepsilon)S \dashv C$ we have that for $X \in \mathbf{X}$, and let $i_X : |X| \to X$, $\varepsilon_X i_X = SC(i_X)\varepsilon_{|X|}$. Since $\varepsilon_{|X|}$ and $SC(i_X)$ are surjective, ε_X is also surjective. Thus ε_X is an isomorphism. The proof is complete. □

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