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ERROR ANALYSIS FOR APPROXIMATION OF HELIX BY BI-CONIC AND BI-QUADRATIC BEZIER CURVES

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ABSTRACT. In this paper we approximate a cylindrical helix by bi-conic and bi-quadratic Bezier curves. Each approximation method is G^1 end-points interpolation of the helix. We present a sharp upper bound of the Hausdorff distance between the helix and each approximation curve. We also show that the error bound has the approximation order three and monotone increases as the length of the helix increases. As an illustration we give some numerical examples.

1. Introduction

In recently twenty years, many methods for approximation of circular arc or helix by Bezier curves have been developed. Since circular arcs cannot be represented by polynomials in explicit form, circular arc approximations with Bezier curves have been developed in many papers [3, 5, 6, 8, 11, 12, 13, 17]. Since helices cannot be represented by polynomials or rational polynomials in explicit form, the helix approximations with rational Bezier curves have been also developed in many papers. They are focused on the rational Bezier curves of degree three [14], of degree three and four [16], of degree from four to six [23], or of degree five [25]. Circular arc and helix have a property whose segments subdivided by equi-length are congruent. Thus, if a error analysis for the approximation of a segment by Bezier curve is obtained, then that for the approximation of whole curve is also done. So, the error analysis and the method for the approximation of circular arc or helix are interesting problems in Computer Aided Geometric Design.

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In the paper [2] the helix was approximated by G^1 quadratic rational/polynomial Bezier curves, and a good error analysis was presented. But the method in [2] have an important demerit which does not yield G^1 end-point interpolation of helix segment. That is to say, the helix segment and the approximated quadratic curve do not have the same tangent direction at both end-points. In this paper we present a end-point interpolation of helix segment by biconic and biquadratic Bezier curves. A sharp upper bound of the Hausdorff distance between the helix and each approximation curve is presented All upper bounds of the Hausdorff distances we present are monotone increasing as α increases so that the minimum subdivision schemes can be achieved within given tolerance. All upper bounds are of approximation order three $\mathcal{O}(\alpha^3)$ which is optimal order [7] with spatial quadratic rational/polynomial Bezier curves. All error bound analysis for the helix approximation with the biquadratic polynomial curve are well done by the help of Floater's error analysis [11], which is restated in Proposition 2.1 in this paper.

The paper is organized as follows. In section 2, G^1 end-point interpolation of the helix by biconic and biquadratic Bezier curves are presented, and the error analysis is given. In section 3, our approximation method is applied to some examples. In section 4, we summarize our work.

2. Helix approximations with biquadratic rational and polynomial Bezier curves

In this section we present G^1 end-points interpolations of the helix segment. Using affine transform all circular helix could be represented by

$$\mathbf{h}(\theta) = (r\cos\theta, r\sin\theta, p\theta), \qquad \theta \in [-\alpha, \alpha] \tag{2.1}$$

for some positive real numbers α, p and r. For $0 < \alpha < \pi$, we define the biconic and the biquadratic Bezier curves as

$$\bar{\mathbf{r}}(t) = \begin{cases} (\sum_{i=0}^{2} \bar{w}_i \bar{\mathbf{b}}_i B_i(t))/(\sum_{i=0}^{2} \bar{w}_i B_i(t)) & 0 \le t \le 1, \\ (\sum_{i=0}^{2} \bar{w}_i \bar{\mathbf{b}}_{i+2} B_i(t-1))/(\sum_{i=0}^{2} \bar{w}_i B_i(t-1)) & 1 < t \le 2, \end{cases}$$

$$\bar{\mathbf{q}}(u) = \begin{cases} \sum_{i=0}^{2} \bar{\mathbf{b}}_i B_i(u) & 0 \le u \le 1, \\ \sum_{i=0}^{2} \bar{\mathbf{b}}_{i+2} B_i(u-1) & 1 < u < 2, \end{cases}$$

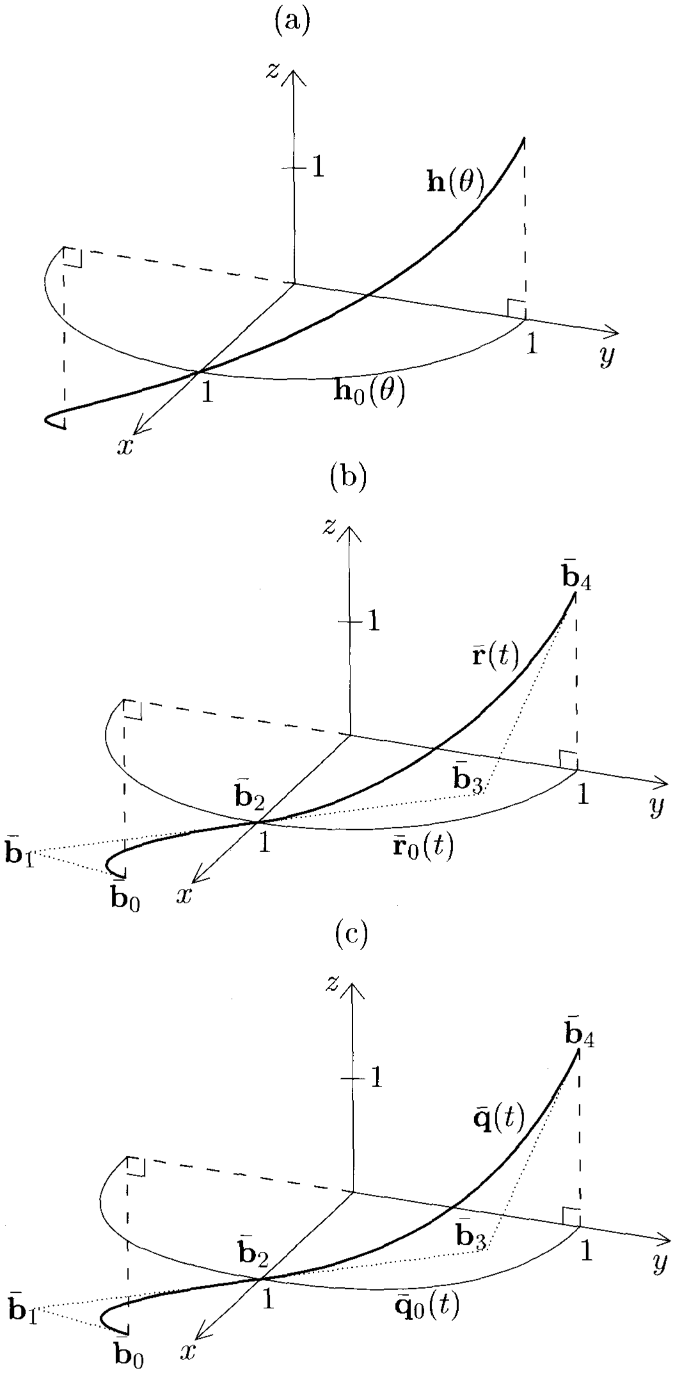

FIGURE 1. (a) The helix curve $\mathbf{h}(\theta)$, $\theta \in [-\pi/2, \pi/2]$, and its projection $\mathbf{h}_0(\theta)$ on xy-plane, when $p = r = 1$ and $\alpha = \pi/2$. (b) The biconic approximation $\bar{\mathbf{r}}(t)$, $t \in [0, 2]$, and its projection $\bar{\mathbf{r}}_0(t)$. (c) The biquadratic Bezier approximation $\bar{\mathbf{q}}(u)$, $u \in [0, 2]$, and its projection $\bar{\mathbf{q}}_0(u)$. The dotted lines are control polygon $\bar{\mathbf{b}}_0\bar{\mathbf{b}}_1\bar{\mathbf{b}}_2\bar{\mathbf{b}}_3\bar{\mathbf{b}}_4$.

having the control points

$$\begin{aligned}
\bar{\mathbf{b}}_0 &= (\bar{x}_0, \bar{y}_0, \bar{z}_0) = (r\cos\alpha, -r\sin\alpha, -p\alpha),\\
\bar{\mathbf{b}}_1 &= (\bar{x}_1, \bar{y}_1, \bar{z}_1) = (r, -r\tan\frac{\alpha}{2}, -p(\alpha - \tan\frac{\alpha}{2}),\\
\bar{\mathbf{b}}_2 &= (\bar{x}_2, \bar{y}_2, \bar{z}_2) = (r, 0, 0),\\
\bar{\mathbf{b}}_3 &= (\bar{x}_3, \bar{y}_3, \bar{z}_3) = (r, r\tan\frac{\alpha}{2}, p(\alpha - \tan\frac{\alpha}{2}),\\
\bar{\mathbf{b}}_4 &= (\bar{x}_4, \bar{y}_4, \bar{z}_4) = (r\cos\alpha, r\sin\alpha, p\alpha),
\end{aligned}$$

and the weights $\bar{w}_0 = 1, \bar{w}_1 = \cos(\alpha/2), \bar{w}_2 = 1$, as shown in Figures 1(a)-(c), where $B_i(t) = \binom{2}{i} t^i (1-t)^{2-i}$, $i = 0, 1, 2$, is the quadratic Bernstein polynomial. The six points of $\bar{\mathbf{r}}(t)$, $t = 0, 1, 2$ and $\bar{\mathbf{q}}(u)$, $u = 0, 1, 2$ lie on the helix and all points of $\bar{\mathbf{r}}(t)$, $0 \le t \le 2$, lie on the cylinder $x^2 + y^2 = r^2$ containing the helix. The control points $\bar{\mathbf{b}}_1$ and $\bar{\mathbf{b}}_3$ are chosen so that $\mathbf{h}'(-\alpha)$, $\bar{\mathbf{q}}'(0)$ and $\bar{\mathbf{r}}'(0)$ are mutually parallel, $\mathbf{h}'(\alpha)$, $\bar{\mathbf{q}}'(2)$ and $\bar{\mathbf{r}}'(2)$ are also mutually parallel, and $\bar{\mathbf{q}}(u)$ and $\bar{\mathbf{r}}(t)$ are C^1-continuous on $[0, 2]$. Note that

$$\begin{aligned}
\mathbf{h}'(-\alpha) &= (r\sin\alpha, r\cos\alpha, p),\\
\bar{\mathbf{q}}'(0) &= 2(\bar{\mathbf{b}}_1 - \bar{\mathbf{b}}_0) = 2\tan\frac{\alpha}{2}(r\sin\alpha, r\cos\alpha, p),\\
\bar{\mathbf{r}}'(0) &= 2\cos\frac{\alpha}{2}(\bar{\mathbf{b}}_1 - \bar{\mathbf{b}}_0)
\end{aligned}$$

are mutually parallel. Since all curves $\mathbf{h}(\theta)$, $\bar{\mathbf{r}}(t)$ and $\bar{\mathbf{q}}(u)$, on the domains $\theta \in [-\alpha, \alpha]$, $t \in [0, 2]$ and $u \in [0, 2]$, respectively, are symmetric with respect to x-axis, $\mathbf{h}'(\alpha)$, $\bar{\mathbf{r}}'(2)$ and $\bar{\mathbf{q}}'(2)$ are parallel and so the biquadratic rational/polynomial Bezier curves $\bar{\mathbf{r}}(t)$ and $\bar{\mathbf{q}}(u)$ are G^1 endpoints interpolation of the helix $\mathbf{h}(\theta)$, $\theta \in [-\alpha, \alpha]$. Since the weight $\cos(\alpha/2)$ is less than one, the biquadratic rational Bezier curve $\bar{\mathbf{r}}(t)$ is an biellipse. For $0 \le t \le 1$, putting

$$\begin{aligned}
\bar{w}(t) &= \sum_{i=0}^{2} \bar{w}_i B_i(t) = (1-t)^2 + 2\cos\frac{\alpha}{2}\, t(1-t) + t^2,\\
\bar{x}(t) &= \sum_{i=0}^{2} \bar{w}_i \bar{x}_i B_i(t) = r(\cos\alpha(1-t)^2 + 2\cos(\frac{\alpha}{2})t(1-t) + t^2),\\
\bar{y}(t) &= \sum_{i=0}^{2} \bar{w}_i \bar{y}_i B_i(t) = -r(1-t)(\sin\alpha(1-t) + 2\sin\frac{\alpha}{2}t),
\end{aligned}$$

$$\bar{z}(t) = \sum_{i=0}^{2} \bar{w}_i \bar{z}_i B_i(t) = -p(1-t)(\alpha(1-t) + 2(\alpha\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2})t),$$

we have $\bar{\mathbf{r}}(t) = (\bar{x}(t), \bar{y}(t), \bar{z}(t))/\bar{w}(t)$ for $t \in [0,1]$.

We find a sharp upper bound of the Hausdorff distance between the helix and each approximation curve $\mathbf{p}(t)$, $t \in [a,b]$, is presented, where the Hausdorff distance is defined [1, 11] by

$$d_H(\mathbf{h},\mathbf{p}) = \max\left\{ \max_{-\alpha\le\theta\le\alpha} \min_{a\le t\le b} \left|\mathbf{h}(\theta) - \mathbf{p}(t)\right|, \max_{a\le t\le b} \min_{-\alpha\le\theta\le\alpha} \left|\mathbf{h}(\theta) - \mathbf{p}(t)\right| \right\}$$

The following proposition was presented by Floater[11], which is needed to analyze the error bounds proposed in this paper.

PROPOSITION 2.1. *Let a conic $\mathbf{r}(t)$ and a quadratic Bezier curve $\mathbf{q}(u)$ have the same control points $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$, and $\mathbf{r}(t)$ have the weights 1, w, 1, in order. Then there is a reparametrisation* (or one-to-one and onto mapping) *$t(u)$ such that $\mathbf{r}(t(u)) - \mathbf{q}(u)$ is parallel with $\mathbf{p}_0 - 2\mathbf{p}_1 + \mathbf{p}_2$ and*

$$|\mathbf{r}(t(u)) - \mathbf{q}(u)| \le \frac{|1-w|}{4(1+w)} |\mathbf{p}_0 - 2\mathbf{p}_1 + \mathbf{p}_2|.$$

PROOF. See Proposition 2.1 and Corollary 2.2 in Floater[11]. □

PROPOSITION 2.2. *For each α, p and r, the helix approximations with biquadratic rational/polynomial curves $\bar{\mathbf{r}}(t)$, $t \in [0,2]$, and $\bar{\mathbf{q}}(u)$, $u \in [0,2]$, have the error bounds*

$$d_H(\mathbf{h}, \bar{\mathbf{r}}) \le p\bar{E}(\alpha), \tag{2.2}$$

$$d_H(\mathbf{h}, \bar{\mathbf{q}}) \le \sqrt{(p(\bar{E}(\alpha) + \bar{F}(\alpha)))^2 + (rF(\alpha/2))^2}, \tag{2.3}$$

where

$$\bar{t}_A = \frac{\alpha - \sin\alpha}{\alpha - \sin\alpha + 2\sin(\alpha/2) - \alpha\cos(\alpha/2)},$$

$$\bar{E}(\alpha) = -\arctan\frac{\bar{y}(\bar{t}_A)}{\bar{x}(\bar{t}_A)} + \frac{\bar{z}(\bar{t}_A)}{p\bar{w}(\bar{t}_A)},$$

$$\bar{F}(\alpha) = \frac{1}{4}\tan^2\frac{\alpha}{4}(2\tan\frac{\alpha}{2} - \alpha),$$

$$F(\alpha) = 2\sin^4\frac{\alpha}{2}\sec\alpha.$$

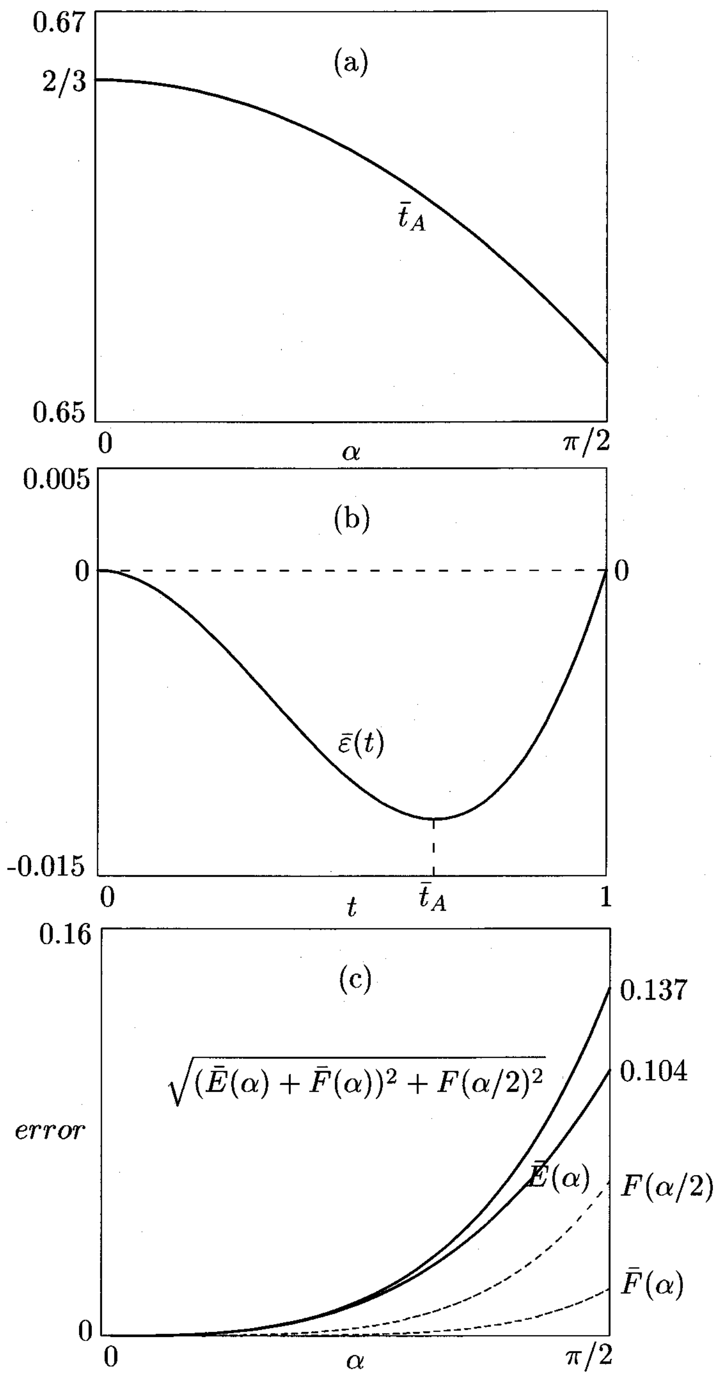

FIGURE 2. (a) $\bar{t}_A$ for $\alpha \in [0,\pi/2]$. (b) $\bar{\varepsilon}(t)$ for $t \in [0,1]$ when $\alpha = \pi/2$. (c) The upper bounds $p\bar{E}(\alpha)$ and $\sqrt{p^2(\bar{E}(\alpha)+\bar{F}(\alpha))^2+r^2F(\alpha/2)^2}$ (solid lines) with $F(\alpha/2)$ and $\bar{F}(\alpha)$ for $\alpha \in [0,\pi/2]$ (dash lines) when $p=r=1$.

PROOF. Since $\mathbf{h}(\theta)$, $\bar{\mathbf{r}}(t)$ and $\bar{\mathbf{q}}(u)$ are symmetric with respect to x-axis, it suffices to find the upper bound of the Hausdorff distances $d_H(\mathbf{h},\bar{\mathbf{r}})$ and $d_H(\mathbf{h},\bar{\mathbf{q}})$ on the half domains $\theta \in [-\alpha, 0]$, $t \in [0,1]$, and $u \in [0,1]$. It is well known [1, 11, 12] that

$$d_H(\mathbf{h},\bar{\mathbf{r}}) \le \max_{0\le t\le 1} |\mathbf{h}(\theta(t)) - \bar{\mathbf{r}}(t)| \tag{2.4}$$

for a reparametrisation (or one-to-one and onto mapping) $\theta = \theta(t)$. With the reparametrisation $\theta = \theta(t) = \arctan(\bar{y}(t)/\bar{x}(t))$, we have

$$\mathbf{h}(\theta(t)) - \bar{\mathbf{r}}(t) = \left(0, 0, p \arctan\frac{\bar{y}(t)}{\bar{x}(t)} - \frac{\bar{z}(t)}{\bar{w}(t)}\right)$$

for $0 \le t \le 1$. Let $\bar{\varepsilon}(t)$ be the third component in the last equation. It suffices to find the maximum of $|\bar{\varepsilon}(t)|$ for $0 \le t \le 1$. Since $\bar{x}(t)^2 + \bar{y}(t)^2 = r^2\bar{w}(t)^2$, we have

$$\bar{\varepsilon}'(t) = \frac{p(\bar{y}'\bar{x} - \bar{y}\bar{x}') - r^2(\bar{z}'\bar{w} - \bar{z}\bar{w}')}{r^2\bar{w}^2}.$$

The numerator of the derivative $\bar{\varepsilon}'(t)$ in the last equation is the quadratic polynomial

$$2pr^2t\{(\alpha - \sin\alpha + 2\sin(\alpha/2) - \alpha\cos(\alpha/2))t - (\alpha - \sin\alpha)\}$$

and has zeros at

$$\bar{t}_A = \frac{\alpha - \sin\alpha}{\alpha - \sin\alpha + 2\sin(\alpha/2) - \alpha\cos(\alpha/2)} \quad \text{and} \quad \bar{t}_B = 0.$$

Since $0 < \bar{t}_A < 1$, $\bar{\varepsilon}(t)$ decreases on $(0, \bar{t}_A)$ and increases on $(\bar{t}_A, 1)$, as shown in Figures 2(a)–(b). Since $\bar{\varepsilon}(0) = \bar{\varepsilon}(1) = 0$, $\bar{\varepsilon}(t)$ has the absolute maximum $-\bar{\varepsilon}(\bar{t}_A)$ at $\bar{t}_A$ in the closed interval $[0,1]$. Thus with the reparametrisation $\theta = \arctan(\bar{y}(t)/\bar{x}(t))$, $\mathbf{h}(\theta(t)) - \bar{\mathbf{r}}(t)$ is parallel with z-axis, and

$$|\mathbf{h}(\theta(t)) - \bar{\mathbf{r}}(t)| \le -\bar{\varepsilon}(\bar{t}_A) = -p\arctan\frac{\bar{y}(\bar{t}_A)}{\bar{x}(\bar{t}_A)} + \frac{\bar{z}(\bar{t}_A)}{\bar{w}(\bar{t}_A)} = p\bar{E}(\alpha)$$

for all $t \in [0,2]$, and the error bound (2.2) of $d_H(\mathbf{h},\bar{\mathbf{r}})$ is obtained.

We find the upper bound (2.3) of the Hausdorff distance between the helix $\mathbf{h}(\theta)$ and the biquadratic Bezier curve $\bar{\mathbf{q}}(u)$. By Proposition 2.1, there exists a reparametrisation $t = t(u)$ such that $\bar{\mathbf{r}}(t(u)) - \bar{\mathbf{q}}(u)$ is parallel with $\bar{\mathbf{b}}_0 - 2\bar{\mathbf{b}}_1 + \bar{\mathbf{b}}_2$ and

$$|\bar{\mathbf{r}}(t(u)) - \bar{\mathbf{q}}(u)| \le \frac{1-\cos(\alpha/2)}{4(1+\cos(\alpha/2))}\left|\bar{\mathbf{b}}_0 - 2\bar{\mathbf{b}}_1 + \bar{\mathbf{b}}_2\right| = \frac{1}{4}\tan^2\frac{\alpha}{4}\left|\bar{\mathbf{b}}_0 - 2\bar{\mathbf{b}}_1 + \bar{\mathbf{b}}_2\right|.$$

By simple calculations, we have

$$(\bar{\mathbf{b}}_0 - 2\bar{\mathbf{b}}_1 + \bar{\mathbf{b}}_2) = -\left(2r\sin^2\frac{\alpha}{2}, 2r\sin^2\frac{\alpha}{2}\tan\frac{\alpha}{2}, p\left(2\tan\frac{\alpha}{2} - \alpha\right)\right)$$

and so

$$\frac{1}{4}\tan^2\frac{\alpha}{4}\left(\bar{\mathbf{b}}_0 - 2\bar{\mathbf{b}}_1 + \bar{\mathbf{b}}_2\right) = -2r\sin^4\frac{\alpha}{4}\left(1, \tan\frac{\alpha}{2}, 0\right) - p\bar{F}(\alpha)(0,0,1).$$

With the reparametrisations $\theta = \theta(t)$ and $t = t(u)$,

$$\begin{aligned}|\mathbf{h}(\theta) - \bar{\mathbf{q}}(u)| &= |(\mathbf{h}(\theta(t(u))) - \bar{\mathbf{r}}(t(u))) + (\bar{\mathbf{r}}(t(u)) - \bar{\mathbf{q}}(u))| \\ &\le \left| \pm p(0,0,\bar{E}(\alpha) + \bar{F}(\alpha)) - 2r\sin^4\frac{\alpha}{4}\left(1, \tan\frac{\alpha}{2}, 0\right)\right| \\ &\le \sqrt{(p(\bar{E}(\alpha) + \bar{F}(\alpha)))^2 + (rF(\alpha/2))^2}.\end{aligned}$$

Thus we obtain the upper bound (2.3) of the Hausdorff distance $d_H(\mathbf{h}, \bar{\mathbf{q}})$. □

As an illustration, for the given helix $\mathbf{h}(\theta) = (\cos\theta, \sin\theta, \theta)$, $\theta \in [-\pi/2, \pi/2]$, we obtain the conic and the quadratic Bezier approximations as shown in Figures 1(a)-(c). By the proposition above, the upper bounds of $d_H(\mathbf{h}, \bar{\mathbf{r}})$ and $d_H(\mathbf{h}, \bar{\mathbf{q}})$ are 1.04×10^{-1} and 1.37×10^{-1}, respectively. By some numerical method, we also find the real Hausdorff distances $d_H(\mathbf{h}, \bar{\mathbf{r}}) = 7.37 \times 10^{-2}$ and $d_H(\mathbf{h}, \bar{\mathbf{q}}) = 1.06 \times 10^{-1}$.

We show that the error bounds (2.2) and (2.3) are monotone increasing as shown in Figure 2(c), and have the approximation order three $\mathcal{O}(\alpha^3)$.

PROPOSITION 2.3. *The error bounds of $d_H(\mathbf{h}, \bar{\mathbf{q}})$ and $d_H(\mathbf{h}, \bar{\mathbf{r}})$ are monotone increasing as α increases and are of approximation order three $\mathcal{O}(\alpha^3)$.*

PROOF. It is clear that $F(\alpha)$ and $\bar{F}(\alpha)$ are increasing as α increases. We show that $\bar{E}(\alpha)$ is also monotone increasing. Using the chain rule for the multi-variables function we have

$$pE'(\alpha) = -\frac{p(\bar{x}\bar{y}_\alpha - \bar{x}_\alpha\bar{y}) - r^2(\bar{z}_\alpha\bar{w} - \bar{z}\bar{w}_\alpha)}{r^2\bar{w}^2}\Bigg|_{t=\bar{t}_A}. \tag{2.5}$$

Simple calculations yield

$$p\bar{E}'(\alpha) = \frac{p\sin\alpha\ \bar{t}_A^2(1-\bar{t}_A)\ (\sin\alpha(1-\bar{t}_A)+\alpha\cos(\alpha/2)\bar{t}_A)}{(1+\cos\alpha)\ \bar{w}(t_A)^2} > 0$$

since $0 < \bar{t}_A < 1$. Thus the upper bounds of $d_H(\mathbf{h}, \bar{\mathbf{r}})$ and $d_H(\mathbf{h}, \bar{\mathbf{q}})$ are strictly increasing as α increases.

It is also clear that $F(\alpha)$ and $\bar{F}(\alpha)$ are of approximation order four $\mathcal{O}(\alpha^4)$. We show that $\bar{E}(\alpha)$ is of approximation order three $\mathcal{O}(\alpha^3)$. By the Taylor expansion of the followings at $\alpha = 0$,

$$\begin{aligned}
\bar{t}_A &= \frac{2}{3} - \frac{1}{180}\alpha^2 + \mathcal{O}(\alpha^4),\\
\bar{x}(\bar{t}_A) &= r - \frac{r}{9}\alpha^2 + \mathcal{O}(\alpha^4),\\
\bar{y}(\bar{t}_A) &= -\frac{r}{3}\alpha + \frac{r}{45}\alpha^3 + \mathcal{O}(\alpha^5),\\
\bar{z}(\bar{t}_A) &= -\frac{p}{3}\alpha + \frac{11p}{270}\alpha^3\mathcal{O}(\alpha^5),\\
\bar{w}(\bar{t}_A) &= 1 - \frac{1}{18}\alpha^2 + \mathcal{O}(\alpha^4),\\
\frac{\bar{y}(\bar{t}_A)}{\bar{x}(\bar{t}_A)} &= \frac{-\frac{r}{3}\alpha + \frac{r}{45}\alpha^3 + \mathcal{O}(\alpha^5)}{r - \frac{r}{9}\alpha^2 + \mathcal{O}(\alpha^4)} = -\frac{1}{3}\alpha - \frac{2}{135}\alpha^3 + \mathcal{O}(\alpha^5),\\
\frac{\bar{z}(\bar{t}_A)}{w(\bar{t}_A)} &= \frac{-p\alpha/3 + \mathcal{O}(\alpha^3)}{1 + \mathcal{O}(\alpha^2)} = -\frac{p}{3}\alpha + \frac{p}{45}\alpha^3 + \mathcal{O}(\alpha^5),
\end{aligned}$$

we have

$$\begin{aligned}
\bar{E}(\alpha) &= -\left\{\left(\frac{\bar{y}(\bar{t}_A)}{\bar{x}(\bar{t}_A)}\right) - \frac{1}{3}\left(\frac{\bar{y}(\bar{t}_A)}{\bar{x}(\bar{t}_A)}\right)^3 + \cdots\right\} + \frac{\bar{z}(\bar{t}_A)}{p\bar{w}(\bar{t}_A)}\\
&= \frac{2}{81}\alpha^3 + \mathcal{O}(\alpha^5).
\end{aligned}$$

Thus the upper bounds of $d_H(\mathbf{h}, \bar{\mathbf{r}})$ and $d_H(\mathbf{h}, \bar{\mathbf{q}})$ are of approximation order three $\mathcal{O}(\alpha^3)$. □

REMARK 1. Since the error bounds are monotone increasing, the subdivision scheme for the helix approximation by the minimum number of segments of the biquadratic rational/polynomial Bezier curves within given tolerance can be obtained.

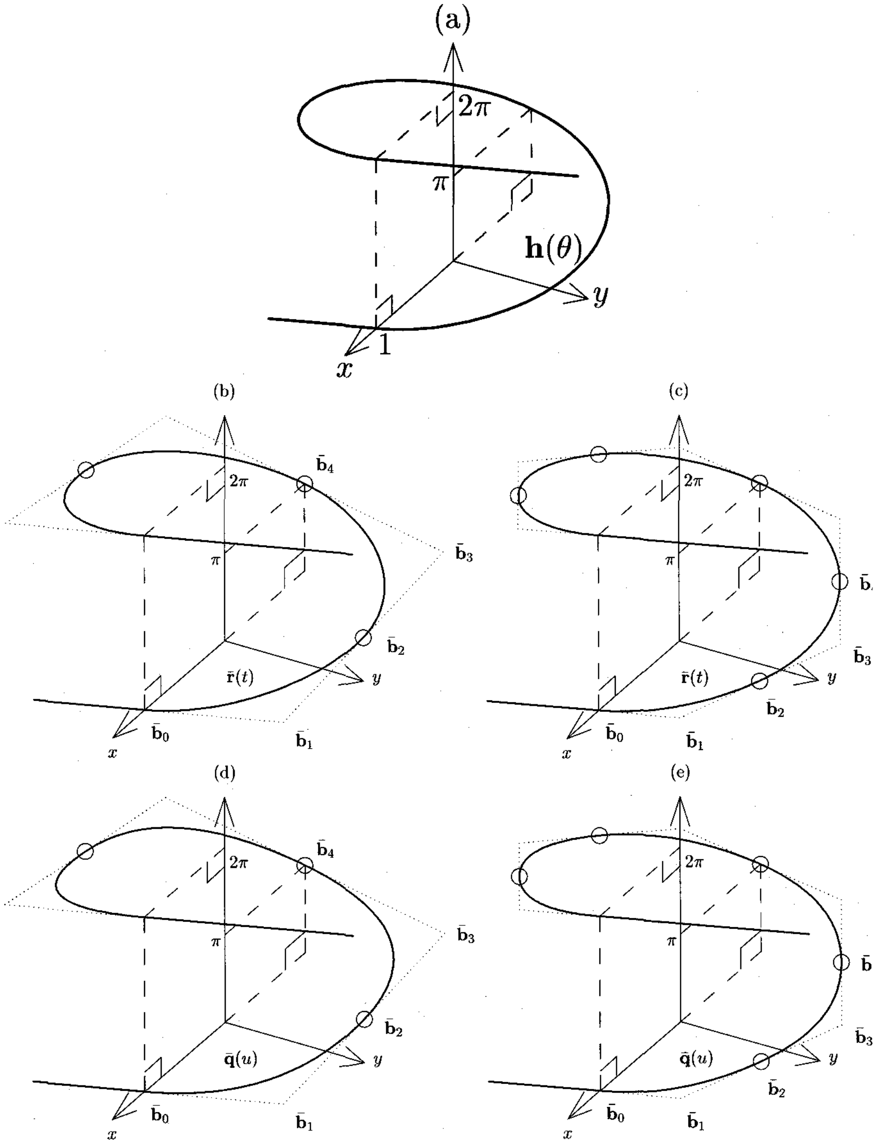

FIGURE 3. (a) helix curve $\mathbf{h}(\theta) = (\cos\theta, \sin\theta, \theta)$, $\theta \in [0, 2\pi]$. (b)-(c) biconic curve $\bar{\mathbf{r}}(t)$ using four and six segments. (d)-(e) biquadratic Bezier curve $\bar{\mathbf{q}}(u)$ using four and six segments. (b)-(e) The dotted lines are control polygons. The circles are the junction points of two consecutive segments.

no. segments	$d_H(\mathbf{h}, \bar{\mathbf{r}})$	app. order.	$d_H(\mathbf{h}, \bar{\mathbf{q}})$	app. order.
4	1.04×10^{-1}		1.37×10^{-1}	
8	1.22×10^{-2}	3.08	1.30×10^{-2}	3.39
16	1.50×10^{-3}	3.02	1.52×10^{-3}	3.10
32	1.87×10^{-4}	3.00	1.88×10^{-4}	3.02

TABLE 1. The error bounds of $d_H(\mathbf{h}, \bar{\mathbf{r}})$ and $d_H(\mathbf{h}, \bar{\mathbf{q}})$ for the given helix curve $\mathbf{h}(\theta) = (\cos\theta, \sin\theta, \theta)$, $\theta \in [0, 2\pi]$, with k-segments, $k = 4, 8, 16$, and 32.

3. Examples

In this section the helix is approximated by the biquadratic rational/polynomial curves/surfaces. Two line segments are connected by the helix which is given by

$$\mathbf{h}(\theta) = (r\cos\theta, r\sin\theta, p\theta), \qquad \theta \in [0, 2\pi]$$

for $r = p = 1$, as shown in Figure 3(a). Using the method proposed in Section two we obtain the G^1 biquadratic rational and polynomial spline curves $\bar{\mathbf{r}}(t)$ and $\bar{\mathbf{q}}(u)$ which are consisted of 'four' segments, i.e., two biquadratic rational/polynomial Bezier curves, respectively, as shown in Figures 5(b) and (d). By Proposition 2.2, we have the error bounds

$$d_H(\mathbf{h}, \bar{\mathbf{r}}) \le 0.104 \qquad \text{and} \qquad d_H(\mathbf{h}, \bar{\mathbf{q}}) \le 0.137.$$

Also, the error bounds for the approximations of the helix by k-segments of $\mathbf{r}(t)$, $\mathbf{q}(u)$, $\bar{\mathbf{r}}(t)$ and $\bar{\mathbf{q}}(u)$, with $k = 8, 16, 32$, are obtained as shown in Table 1. We can see that the approximation order of these approximation methods are three $\mathcal{O}(\alpha^3)$.

Let the tolerance be given by 0.1. Then the subdivision is needed for the helix approximation by the biconic $\bar{\mathbf{r}}(t)$ and the biquadratic $\bar{\mathbf{q}}(u)$. Using the subdivision scheme in Remark 1, the helix approximation by the biconic $\bar{\mathbf{r}}(t)$ and the biquadratic $\bar{\mathbf{q}}(u)$ can be achieved within the tolerance by the number of subdivision $k = 6$, respectively, as shown in Figures 5(c) and (e). They have the new upper bounds of the Hausdorff distances $d_H(\mathbf{h}, \bar{\mathbf{r}}) \le 2.94 \times 10^{-2}$ and $d_H(\mathbf{h}, \bar{\mathbf{q}}) \le 3.30 \times 10^{-2}$, which are less than the tolerance.

REMARK 2. Using our method in this paper, two line segments are connected by the biquadraic rational/polynomial spline with G^1-manner, i.e., continuity of tangent direction. But if the method in [2] is applied, then line segments and approximated quadratic rational/polynomial spline do not have the continuity of tangent direction at the junction points, although the approximated quadratic rational/polynomial spline is G^1 continuous in its interior.

4. Comments

In this paper we presented the approximation method of the cylindrical helix by biconic and biquadratic Bezier curve. For each case we presented the error bound analysis of the Hausdorff distance between the helix and each approximation curve. The approximation methods in this paper are different from those in [2], since our methods are G^1 end-points interpolation of helix, but those in [2] are G^0 end-points interpolation of helix. The approximation methods in this paper can be also extended to the approximation of any sweeping surface of conic section along the helix as well as the approximation of "torus-like helicoid" [2].

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