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DIFFERENTIABILITY OF FRACTAL CURVES

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Abstract. As a tool of measuring the irregularity of curve, fractal dimensions can be used. For an irregular function, fractional calculus are more available. However, to know its fractional differentiability which is related to its complexity is complicated one. In this paper, variants of the Hausdorff dimension and the packing dimension as well as the derivative order are defined and the relations between them are investigated so that the differentiability of fractal curve can be explained through its complexity.

1. Introduction

Many natural and man-made phenomena we encounter in our world are much better described through an inherently irregular fractal geometry than by the traditional regular Euclidean geometry. With this fractal, nowadays, fractional calculus attracted much attention since it allows us to consider the integration and differentiation of non-integral order as well as integral order ([8]). Therefore fractional calculus can be applied to many fractal-related problems in physics and engineering ([7, 9]). In fact, typical data sets in signal analysis can be represented by the irregular functions which are nowhere differentiable ([6]).
As a tool of measuring how the set is irregular, the fractal dimensions are defined. Let $|E|$ denote the diameter of set E in Euclidean space R^n and let $\{U_i\} \in \mathcal{C}_\delta(E)$ mean that $\{U_i\}$ is a countable (or finite) collection of subsets in R^n satisfying $E \subset \cup U_i$ and $|U_i| < \delta$. Then the α-dimensional Hausdorff measure H^α and the Hausdorff dimension $\dim_{\mathrm{H}}$ are defined by

$$\mathrm{H}^\alpha(E) = \lim_{\delta\to 0} \inf_{\mathcal{C}_{\delta(E)}} \sum_{\{U_i\}} |U_i|^\alpha,$$

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and

$$\dim_{\mathrm{H}}(\mathrm{E}) = \sup\{\alpha : \mathrm{H}^\alpha(\mathrm{E}) > 0\} = \inf\{\alpha : \mathrm{H}^\alpha(\mathrm{E}) < \infty\}.$$

On the other hand, when $N_\delta(E)$ denote the smallest number of boxes of size δ that cover E, the box dimension $\dim_{\mathrm{B}}$ is defined by

$$\dim_{\mathrm{B}}(\mathrm{E}) = \lim_{\delta \to 0} \log \mathrm{N}_\delta(\mathrm{E}) / -\log \delta.$$

One usually uses this box dimension due to its relative ease for analytical calculation and empirical estimation but it has some defects such as countable instability ([2]). And since it is not defined by a measure, there are difficulties in its theoretical formulation. Thus as a modification, the packing dimension is defined. Let $\{B_i\} \in \mathcal{P}_\delta(E)$ mean that B_i's are pairwise disjoint balls with centers in E. Then α-dimensional packing pre-measure $\mathcal{P}^\alpha$ and packing measure P^α are defined by

$$\mathcal{P}^\alpha(E) = \lim_{\delta \to 0} \sup_{\mathcal{P}_\delta(E)} \sum_{B_i} |B_i|^\alpha,$$

and

$$\mathrm{P}^\alpha(E) = \inf\left\{\sum \mathcal{P}^\alpha(E_i) : E \subset \cup E_i\right\}.$$

For this measure, packing dimension $\dim_{\mathrm{P}}$ is defined similarly to the case of the Hausdorff dimension. Then it is clear that $\dim_{\mathrm{H}}(\mathrm{E}) \leq \dim_{\mathrm{P}}(\mathrm{E}) \leq \dim_{\mathrm{B}}(\mathrm{E})$. And for the case of the Jordan curve, Γ_f, of continuous function f on [0, 1], there also exist other varieties of dimensions. That is, $\dim_{\mathrm{H}}$ of Γ_f is defined from $\lim_{\delta \to 0} \inf_\pi \sum |f([t_{i-1}, t_i])|^\alpha$, for the infimum taken over all partitions $\pi : 0 = t_0 < t_1 < \cdots < t_n = 1$ satisfying $|f([t_{i-1}, t_i])| < \delta$, and $\dim_{\mathrm{B}}$ is also defined from $\lim_{\delta \to 0} \log M_\delta(\Gamma_f) / -\log \delta$ for the maximum number $M_\delta(\Gamma_f)$, of points $\gamma_0 < \gamma_1 < \cdots < \gamma_n$ on Γ_f with $|\gamma_i - \gamma_{i-1}| = \delta$.

While the complexity of a set can be measured by its fractal dimensions, the irregularity of a function can be measured through its fractional differentiability. Though there exist several definitions of the fractional order derivative, the one defined by Riemann and Liouville is the most commonly used ([4, 8]). As a generalization of the integration to non-integral order, the Riemann-Liouville integral ${}_a\mathcal{D}_x^\alpha$ is defined by

$${}_a\mathcal{D}_x^\alpha f(x) \equiv \frac{d^\alpha f(x)}{d(x-a)^\alpha} = [1/\Gamma(-\alpha)] \int_a^x f(y)/(x-y)^{\alpha+1} dy, \ \alpha < 0.$$

From this, the fractional derivative of non-integral order α is derived by

$$ {}_a\mathcal{D}_x^\alpha f(x) = \begin{cases} [1/\Gamma(1-\alpha)]\dfrac{d}{dx}\left[\displaystyle\int_a^x f(y)(x-y)^{-\alpha}dy\right], & (0<\alpha<1), \\ [1/\Gamma(n-\alpha)]\dfrac{d^n}{dx^n}\left[\displaystyle\int_a^x f(y)(x-y)^{n-\alpha-1}dy\right], & \\ & (n-1<\alpha<n). \end{cases} $$

Note that these definitions have two parameters, the order α and the lower limit a of integration. These definitions are consistent with the usual ones for x^p with $a = 0$ and e^{px} with $a = -\infty$. It is well-known that

$$ \frac{d^\alpha f(bx)}{dx^\alpha} = b^\alpha \frac{d^\alpha f(bx)}{d(bx)^\alpha}. $$

However, except in the case of the positive integer α, the fractional derivative of order α is non-local due to its dependence on the lower limit a. Moreover from the well-known formula

$$ \frac{d^\alpha x^\beta}{dx^\alpha} = [\Gamma(\beta+1)/\Gamma(\beta-\alpha+1)]x^{\beta-\alpha} \quad \text{for } \alpha > -1, $$

we see that the fractional derivative of non-zero constant function is not zero. Therefore changing the lower limit or adding a constant to a function alters the value of the fractional derivative. In [5], Kolwankar et al. define the local fractional derivative by subtracting the value of the function and taking the lower limit at the point where fractional differentiability is to be checked. This makes the value at the differentiation point be zero, washing out the effect of any constant term, and preserve the local nature of the differentiability property. That is, a function $f : [0,\ 1] \to \mathbb{R}$ is order α differentiable at y if the following limit exists;

$$ D^\alpha f(y) = \lim_{x\to y} {}_y\mathcal{D}_x^\alpha (f(x) - f(y)) = \lim_{x\to y} \frac{d^\alpha (f(x)-f(y))}{d(x-y)^\alpha}. $$

Let $|D^\alpha f(y)| = \lim_{x\to y} |{}_y\mathcal{D}_x^\alpha (f(x) - f(y))|$. For the critical value $q(y) = \sup\{0 \le q < 1 : |D^\beta f(y)| < \infty \text{ for } 0 \le \beta < q\}$, let us define the upper derivative order of function f by

$$ \mathrm{UDER}(f) = \sup\{q(y) : y \in [0,\ 1]\}, $$

and the lower derivative order of function f by

$$ \mathrm{LDER}(f) = \inf\{q(y) : y \in [0,\ 1]\}. $$

When $\mathrm{UDER}(f) = \mathrm{LDER}(f)$, it is simply denoted $\mathrm{DER}(f)$ as the derivative order $\mathrm{DER}(f)$.

In this paper, since the fractal curve are treated, we may assume that $0 < \mathrm{LDER}(f) \leq \mathrm{UDER}(f) < 1$.

2. Main results

In this paper, we only consider a continuous real valued function f defined on a closed interval, say, $[0,\ 1]$.
For the interval $I \subset [0,\ 1]$, let us define the variation of f on I by $V(f : I) = \sup\{|f(x) - f(y)| : x, y \in I\}$. For each partition $\pi : 0 = x_0 < x_1 < \cdots < x_n = 1$, $\pi \in \Pi_\delta$ means that $|I_i| = |x_i - x_{i-1}| < \delta,\ i = 1, 2, \cdots, n$. Then by using Munroe's Method II ([10]), the α-dimensional variational Hausdorff (outer) measure of the curve Γ_f is defined by

$$H_V^\alpha(\Gamma_f) = \lim_{\delta \to 0} \inf_{\Pi_\delta} \sum_\pi V(f : I_i)|I_i|^{\alpha-1}$$

and from this measure, the variational Hausdorff dimension of the curve, $\dim_V(\Gamma_f)$, is defined similarly to the case of the Hausdorff dimension. And the α-dimensional variational packing premeasure $\mathcal{P}_V{}^\alpha$ and variational packing (outer) measure P_V^α are defined by

$$\mathcal{P}_V{}^\alpha(\Gamma_f) = \lim_{\delta \to 0} \sup_{\Pi_\delta} \sum_\pi V(f : I_i)|I_i|^{\alpha-1},$$

and

$$P_V^\alpha(\Gamma_f) = \inf \left\{ \sum_{\{I_i\}} \mathcal{P}_V{}^\alpha(\Gamma_{f|I_i}) : [0,\ 1] \subset \cup I_i \right\}.$$

And the variational packing dimension Dim_V are also defined as above. Then we will see that $\dim_H(\Gamma_f) \leq \dim_V(\Gamma_f) \leq \mathrm{Dim}_V(\Gamma_f) \leq \dim_P(\Gamma_f)$.
Note: Strictly speaking, above two defined measures are Borel-regular outer measures and so they can be measures on each σ-algebra consisting of all their measurable sets, respectively. In this sense we simply call them measures as in the cases of Hausdorff and Packing measures.

THEOREM 2.1. $\dim_V(\Gamma_f) \geq 2 - \mathrm{UDER(f)}$.

PROOF. First, suppose $y = 0$, $f(0) = 0$. Let q be taken such that $\mathrm{UDER}(f) < q$ or $|D^q f(0)| = \lim_{x \to 0} |{}_0\mathcal{D}_x^q f(x)| = \infty$. For given $C > 0$ there exists $\delta_0 > 0$ such that $|{}_0\mathcal{D}_x^q f(x)| \geq C$ for $|x| < \delta_0$. Since f is bounded and $q < 1$, ${}_a\mathcal{D}_x^{-p}({}_a\mathcal{D}_x^p f(x)) = f(x)$ holds ([8]). Thus for

$|x| < \delta_0$,

$$f(x) = {}_0\mathcal{D}_x^{-q}[{}_0\mathcal{D}_x^q f(x)] = [1/\Gamma(q)] \int_0^x {}_0\mathcal{D}_t^q f(t)/(x-t)^{-q+1} dt$$
$$\geq [C/\Gamma(q)] \int_0^x 1/(x-t)^{-q+1} dt$$
$$= [C/\Gamma(q+1)]x^q.$$

Put $M = C/\Gamma(q+1) > 0$. Then for $|x| < \delta_0$, $|f(x)| \geq M|x|^q$ holds. Now put $x - y = u$ and $f(x+y) = g(x)$. Then

$$D^q f(y) = \lim_{x \to y} {}_y\mathcal{D}_x^q (f(x) - f(y))$$
$$= \lim_{x \to y} [1/\Gamma(1-q)] \frac{d}{dx} \int_y^x (f(t) - f(y))/(x-t)^{-q+1} dt$$
$$= \lim_{u \to 0} [1/\Gamma(1-q)] \frac{d}{du} \int_0^u (g(t) - g(0))/(u-t)^{-q+1} dt$$
$$= \lim_{u \to 0} {}_0\mathcal{D}_u^q g(u).$$

From this, we can see that $|x-y| < \delta_0$ implies $|f(x) - f(y)| \geq M|x-y|^q$ for above C and δ_0. Therefore for any partition $\pi \in \Pi_\delta$ with $\delta \leq \delta_0$,

$$\sum_\pi V(f; I)|I|^{1-q} \geq \sum_\pi |f(x_i) - f(x_{i-1})||x_i - x_{i-1}|^{1-q}$$
$$\geq M \sum_\pi |x_i - x_{i-1}|$$
$$= M > 0.$$

Thus $H_V^{2-q}(\Gamma_f) = \lim_{\delta \to 0} \inf_{\Pi_\delta} \sum_\pi V(f; I)|I|^{1-q} > 0$, and $\dim_V(\Gamma_f) \geq 2 - q$. Finally we have $\dim_V(\Gamma_f) \geq 2 - \mathrm{UDER}(f)$. □

LEMMA 2.2. *There exists $C > 0$ such that*

$$\lim_{\delta \to 0} \sup_{\pi \in \Pi_\delta} \sum_{I \in \pi} V(f : I)|I|^{1-q}$$
$$\leq C \lim_{\delta \to 0} \sup_{\pi \in \Pi_\delta} \sum_{[x_i, x_{i-1}]} |f(x_i) - f(x_{i-1})||x_i - x_{i-1}|^{1-q}.$$

PROOF. Let a partition π, denoted by $\pi \in \Pi_\delta$, be given such that $\pi : 0 = x_0 < x_1 < \cdots < x_n = 1$ with $|x_i - x_{i-1}| < \delta$. Then for each $I_i = [x_{i-1}, x_i]$, y_i and z_i can be taken in I_i such that $|f(z_i) - f(y_i)| > V(f : I_i)/3$ and $|z_i - y_i| > |x_i - x_{i-1}|/3$. Then

$$|f(z_i) - f(y_i)||z_i - y_i|^{1-q} > V(f : I_i)|I_i|^{1-q}/3^{2-q}.$$

Let π' be the redefined partition consisting of above π, y_i and z_i, $i = 1, 2, \cdots n$. Then

$$\begin{aligned}\sum_{I\in\pi'} V(f:I)|I|^{1-q} &\geq \sum_{[x_{i-1},x_i]\in\pi'} |f(x_i)-f(x_{i-1})||x_i-x_{i-1}|^{1-q} \\ &\geq \sum_{i=1}^{n} |f(z_i)-f(y_i)||z_i-y_i|^{1-q} \\ &\geq [1/3^{2-q}]\sum_{\pi} V(f:I)|I|^{1-q}.\end{aligned}$$

Put $C = 3^{2-q}$. Then for any partition $\pi \in \Pi_\delta$ we can take a corresponding partition $\pi' \in \Pi_\delta$ such that

$$\sum_{\pi} V(f:I)|I|^{1-q} \leq C\sum_{\pi'} |f(x_i)-f(x_{i-1}||x_i-x_{i-1}|^{1-q}.$$

From this, above result can be derived. □

THEOREM 2.3. $\mathrm{Dim_V}(\Gamma_\mathrm{f}) \leq 2 - \mathrm{LDER(f)}$.

PROOF. Let q be taken such that $0 < q < \mathrm{LDER}(f)$. Suppose $y = 0$ with $f(y) = 0$ for the convenience. Since $|D^q f(0)|$ is bounded, there exist $K > 0$ and $\delta > 0$ such that $0 < x < \delta$ implies $|{}_0\mathcal{D}_x^q f(x)| < K$. Then

$$|f(x)| \leq [1/\Gamma(q)]\int_0^x |{}_0\mathcal{D}_t^q f(t)|/|x-t|^{-q+1}dt \leq [K/\Gamma(q+1)]|x|^q.$$

Put $M = K/\Gamma(q+1)$ and generalized above inequality by the change of variables so that for above δ and x, y with $|x-y| < \delta$, $|f(x)-f(y)| \leq M|x-y|^q$ may holds. From the Lemma 2.2, we have some $C > 0$ such that

$$\begin{aligned}\sum_{\pi\in\Pi_\delta} V(f:I)|I|^{1-q} &\leq C\sum_{\pi} |f(x_i)-f(x_{i-1})||x_i-x_{i-1}|^{1-q} \\ &\leq MC\sum_{\pi} |x_i-x_{i-1}| = MC.\end{aligned}$$

Therefore $\mathcal{P}_V^{2-q}(\Gamma_f) = \lim_{\delta\to 0}\sup_\delta \sum_{\pi\in\Pi_\delta} V(f:I)|I|^{1-q} < MC$ holds and we have $\mathrm{Dim_V}(\Gamma_\mathrm{f}) \leq 2 - \mathrm{LDER(f)}$. □

By combining Theorem 2.1 and Theorem 2.3, we get the following results.

COROLLARY 2.4. $2 - \mathrm{UDER(f)} \leq \mathrm{dim_V}(\Gamma_\mathrm{f}) \leq \mathrm{Dim_V}(\Gamma_\mathrm{f}) \leq 2 - \mathrm{LDER(f)}$.

COROLLARY 2.5. *If f is fractionally differentiable with derivative order* DER(f), *then*

$$\dim_V(\Gamma_f) = \mathrm{Dim}_V(\Gamma_f) = 2 - \mathrm{DER}(f).$$

THEOREM 2.6. *For the interval I_y containing y, let $V(f : I_y) = \sup\{|f(x) - f(x')| : x, x' \in I_y\}$ and let $\tilde{V}(f : I_y) = |f(x) - f(x')|$ for two boundary points x and x' of I_y.*

(1) $\sup_y \limsup_{|I_y|\to 0} V(f : I_y)/|I_y|^\alpha < \infty \Longrightarrow \alpha \leq \mathrm{UDER(f)}$.

(2) $\inf_y \liminf_{|I_y|\to 0} \tilde{V}(f : I_y)/|I_y|^\alpha = \infty \Longrightarrow \mathrm{LDER(f)} \leq \alpha$.

PROOF. (1) Let $q < \alpha$. We claim that $q \leq \mathrm{UDER(f)}$. If there exists a sequence $\{x_n\} \to y$ such that

$$D^q f(y) = \lim_{x_n \to y} \frac{d^q(f(x_n) - f(y))}{d(x_n - y)^q} = \pm\infty.$$

Then as in the proof of Theorem 2.1, for any $M > 0$, there exists an $x \in I_y$ such that $|f(x) - f(y)| > M|x - y|^q$ or $V(f : I_x)/|I|^\alpha > M$, which contradicts the hypothesis. Therefore $q < \mathrm{UDER(f)}$. Since q is arbitrary, we have $\alpha \leq \mathrm{UDER(f)}$.

(2) For any $M > 0$, there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| > M|x - y|^\alpha$. Suppose that $y = 0$ and $f(0) = 0$. Let $F(x) = f(x) - Mx^\alpha$. By the continuity, we can take a decreasing sequence $\{x_n\}$ which converges to 0 such that for each n $F(x_n) = 0$ and $F(x) > 0, x_n < x < \epsilon$, for some ϵ (here it is not ruled out that x_n may be zero for some n). Put

$$F_n(x) = \begin{cases} 0, & x \leq x_n, \\ F(x), & \text{otherwise.} \end{cases}$$

For $0 < \alpha < q < 1$ and $x_n \leq x < x_{n-1}$, we may assume the value of

$${}_{x_n}\mathcal{D}_x^q F_n(x) = [1/\Gamma(1-q)] \frac{d}{dx} \int_{x_n}^{x} F(y)/(x - y)^q dy,$$

for otherwise there is nothing to prove. Put $G(x) = \int_{x_n}^{x} F(y)/(x-y)^q dy$. Since $G(x)$ is increasing on (x_n, ϵ) for some $\epsilon > 0$, $G(x_n) = 0$ and $G(x_n + \epsilon) > 0$,

$$0 \leq [1/\Gamma(1-q)] \frac{d}{dx} G(x)|_{x=x_n} = {}_{x_n}\mathcal{D}_x^q F_n(x)|_{x=x_n},$$

and thus $M\{{}_{x_n}\mathcal{D}_x^q(x^\alpha)|_{x=x_n}\} \leq {}_{x_n}\mathcal{D}_x^q f(x)$.

So $D^q f(x_n) = \lim_{x\to x_n} \frac{d^q(f(x) - f(x_n))}{d(x - x_n)^q} = \infty$ as $x_n \to 0$ because the left hand side of the above inequality diverges as $x_n \to 0$. This can be generalized so that $D^q f(y) = \lim_{x\to y} \frac{d^q(f(x)-f(y))}{d(x-y)^q} = \infty$. Therefore LDER(f) $\leq$ q and finally we have LDER(f) $\leq \alpha$. □

As an application of above relations, we may calculate the fractal dimension of the graph of singular function which is almost every where non-differentiable in usual sense. For this, consider the iterated function system (IFS) $(\{s_i\}_{i=1}^k, \{p_i\}_{i=1}^k)$ defined by the similarity maps $s_i : [0,1] \to [0,1]$ with $s_i(x) = r_i x + b_i$, $0 < r_i < 1$, and a probability vector $\{p_i\}$ on $[0,1]$ such that $p_i \neq 1/k$ for i. Then this system takes the attractor domain E in $[0,1]$([2]). Let $\{X_n\}$ be an independent identically distributed random process on $[0,1]$ through the measure μ which has support E with μ, such that $\mu(X_n = i) = p_i$ for every $i = 1, \cdots, k$. Let λ be the Lebesgue measure on $[0,1]$. Then for the random variable $X = \sum_1^\infty X_n/k^n$, its distribution F_X defined by $F_X(x) = \mu(X \leq x)$ has derivative 0 a.e$[\lambda]$. Examples of such distribution are shown in Figure 1.

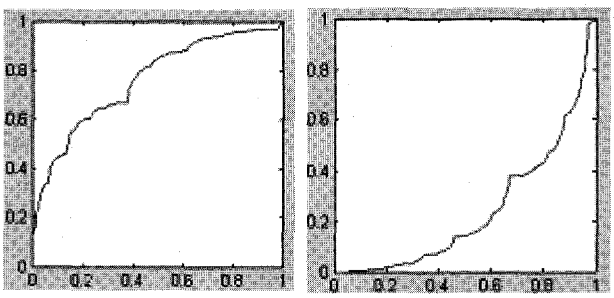

FIGURE 1. Distribution function of IFS. (a) $r_1 = 0.4, r_2 = 0.6, p_1 = 0.7, p_2 = 0.3, b_1 = 0, b_2 = 0.4$ (b) $r_1 = 0.7, r_2 = 0.3, p_1 = 0.4, p_2 = 0.6, b_1 = 0, b_2 = 0.7$

Let C_n be the cylinder defined by $C_n = s_{i_1} \circ s_{i_2} \circ \cdots s_{i_n}(0,1] = \{x \in (0,1] : X_j(x) = i_j, j = 1, \cdots, n\}$. Then for each cylinder C_n, there exists some ℓ, $0 \leq \ell < k^n - 1$, such that

$$\mu(C_n) = F((\ell+1)/k^n) - F(\ell/k^n) = p_{i_1} p_{i_2} \cdots p_{i_n}.$$

Let $n_0(A)$ denote the cardinality of the set A. Let A and B be given by $A = \{x \in (0,1] : \lim_{m\to\infty} n_0\{i : X_i(x) = j\}/m = p_j \text{for } 0 \leq j \leq k-1 \text{ and } i < m\}$ and $B = \{x \in (0,1] : \lim_{m\to\infty} n_0\{i : X_i(x) = j\}/m = 1/n \text{ for } 0 \leq j \leq k-1 \text{ and } i < m\}$. Then two sets A and B are disjoint. Furthermore, $\mu(A) = 1$ by the strong law of large numbers and $\lambda(B) = 1$ by the normal number theorem. Therefore we can see that μ is mutually singular to λ. Also F is not differentiable a.e$[\mu]$ ([1]). For each cylinder $C_n(x)$ of rank n containing x, take

$$a = \lim_{n\to\infty} -\log \mu(C_n(x))/n = \begin{cases} -\sum_{i=1}^{k} p_i \log p_i, & x \in A, \\ -\sum_{i=1}^{k} [1/k] \log p_i, & x \in B. \end{cases}$$

Since $-\sum_{i=1}^{k} p_i \log p_i < \log k$ and $-\sum_{i=1}^{k} [1/k] \log p_i > \log k$, we have

$$\lim_{n\to 0} \mu(C_n(x))/(k^{-n})^{\alpha} = \begin{cases} \infty, & \alpha > a/\log k, \\ 0, & \alpha < a/\log k. \end{cases}$$

Thus DER(F) = a/ log k from the Theorem 2.6. Thus $\dim_{\mathrm{V}}(\Gamma_{\mathrm{F}}) = \mathrm{Dim}_{\mathrm{V}}(\Gamma_{\mathrm{F}}) = 2 - a/\log k$ from the Corollary 2.5.

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