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ON CERTAIN CLASSES OF LINKS AND 3-MANIFOLDS

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Abstract. We construct an infinite family of closed 3-manifolds $\mathcal{M}(2m+1,n,k)$ which are identification spaces of certain polyhedra $\mathcal{P}(2m+1,n,k)$, for integers $m \geq 1, n \geq 3$, and $k \geq 2$. We prove that they are (n/d)-fold cyclic coverings of the 3-sphere branched over certain links $\mathcal{L}_{(m,d)}$, where $d = \gcd(n,k)$, by handle decomposition of orbifolds. This generalizes the results in [3] and [2] as a particular case $m = 2$.

1. Introduction

There are various methods to construct closed 3-manifolds, namely, surgeries on the components of an oriented link in the 3-sphere, branched covering of some link in the 3-sphere, and polyhedral description, that is, the description of closed 3-manifolds as polyhedral 3-balls, whose finitely many boundary faces are identified in pairs. We note that if the polyhedron admits a geometric structure and its face identification is performed by means of geometric isometries, the same geometric structure is inherited by the identification space. In the present paper we use polyhedral description method to construct an infinite family of 3-manifolds $\mathcal{M}(2m+1,n,k)$ which are the identification spaces of certain polyhedra $\mathcal{P}(2m+1,n,k)$ and admit symmetric Heegaard splittings induced by axes of polyhedra $\mathcal{P}(2m+1,n,k)$ as in [1], for integers $m \geq 1, n \geq 3$, and $k \geq 0$. Further we construct Heegaard diagrams of $\mathcal{M}(2m+1,n,k)$ and show that $\mathcal{M}(2m+1,n,k)$ with $\gcd(n,k) = d \geq 1$ are (n/d)-fold cyclic coverings of the 3-sphere branched over links $\mathcal{L}_{(m,d)}$ as shown in Figure 1.a where L_i denotes the $(\frac{1}{m})$-rational tangle for all i's. Moreover $\mathcal{M}(2m+1,n,k)$ for $\gcd(n,k) = d \geq 1$ are n-fold cyclic branched covering

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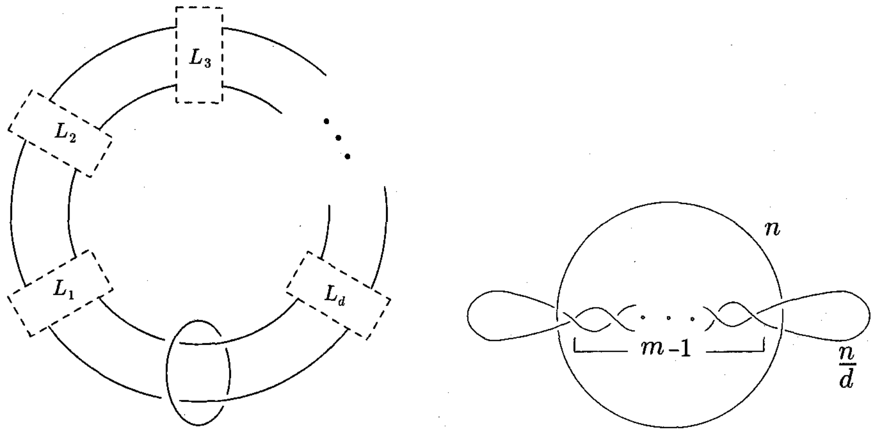
spaces of the 3-space branched over 2-components links $\mathcal{L}_{(m,1)}$ which are hyperbolic for $m > 1$ where the branched indices of its components are n and $\frac{n}{d}$ as shown in Figure 1.b. The technique for our results is the same as one in [2] which basically depends on the cancellation of handles on Heegaard diagrams representing the orbifolds. Our consideration is the generalization of constructions given in [3] and [2] where the particular case $m = 2$ was studied. In [3], Helling, Kim, and Mennicke proved that $\mathcal{M}(5, n, k)$ with $\gcd(n, k) = 1$ are n-fold cyclic coverings of the 3-sphere branched over the Whitehead link. Furthermore Cavicchioli-Paoluzzi[2] proved that $\mathcal{M}(5, n, k)$ with $\gcd(n, k) = d > 1$ are (n/d)-fold cyclic coverings of the 3-sphere branched links $\mathcal{L}_{(2,d)}$.

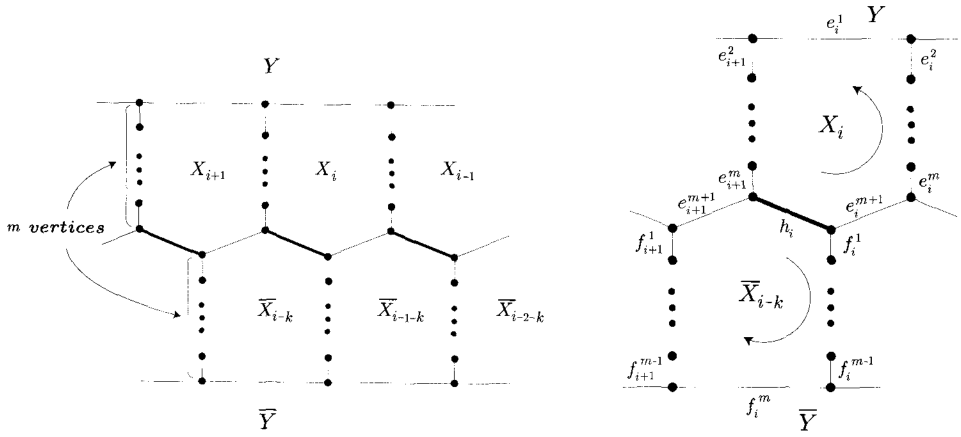


a. $\mathcal{L}_{(m,d)}$ b. $\mathcal{L}_{(m,1)}$

FIGURE 1

2. Constructions of 3-manifolds

We construct an infinite family of 3-manifolds by considering a combinatorial polyhedron together with identifications of pairs of faces on the boundary. For integers $m \geq 1, n \geq 3, k \geq 0$, let $\mathcal{P}(2m + 1, n, k)$ be a polyhedron whose boundary, which can be regarded as the 2-sphere, consists of two n-gons in the northern and southern hemispheres, and two $(2m + 1)$-gon bands in the equatorial zone(see Figure 2.a). Each $(2m + 1)$-gon band has n $(2m + 1)$-gons and so $\mathcal{P}(2m + 1, n, k)$ has $2n + 2$ faces, $2nm + 2n$ edges and $2nm$ vertices.



a. $P(2m+1,n,k)$ b.

FIGURE 2

The identification of pairs of faces of $\mathcal{P}(2m+1,n,k)$ will be done cyclically. That is, X_i and $\overline{X}_i$, and Y and $\overline{Y}$ will be identified. The identification for X_i and $\overline{X_i}$ will be accomplished by the reflection with respect to the thick line followed by the rigid rotation of one of the two $(2m+1)$-bands onto itself by $\frac{2\pi k}{n}$ radians. For this task, we now define labels and orientations of the edges of faces. We first define the boundary path of X_i(resp. $\overline{X}_{i-k}$) as follow(see Figure 2.b):

$$e_i^{m+1}e_i^m\cdots e_i^2e_i^1e_{i+1}^2\cdots e_{i+1}^m h_i(\text{resp. } f_i^1\cdots f_i^{m-1}f_i^m f_{i+1}^{m-1}\cdots f_{i+1}^1 e_{i+1}^{m+1}h_i),$$

where the lower indices of e_i^j and f_i^j are taken mod n and the indices of h_i are taken $\mathrm{mod}(\gcd(n,k))$. It is clear that $\mathcal{P}(2m+1,n,k)$ admits an n-cylindrical symmetry whose action increases lower indices by 1. This symmetry is compatible with the identification and thus passes to the quotient space. We note that once we choose orientations for e_1^1 and h_1, there is a unique choice of orientation for the remaining edges which is compatible with the identification and the n-symmetry. From the identification of X_i and $\overline{X}_i$, we have the following equalities;

$$\begin{cases} e_{j+1}^m = e_{j+k+1}^{m+1}, & e_{j+1}^l = f_{j+k+1}^{m-l} \quad \text{for } 2 \le l \le m-1, \\ e_j^1 = f_{j+k+1}^{m-1}, & e_j^l = f_{j+k}^{m-l+2} \quad \text{for } 2 \le l \le m, \\ e_j^{m+1} = f_{j+k}^1, & h_j = h_{j+k}. \end{cases}$$

We then identify Y and $\overline{Y}$ such that the corresponding oriented edges of polygons carrying the same label are identified.

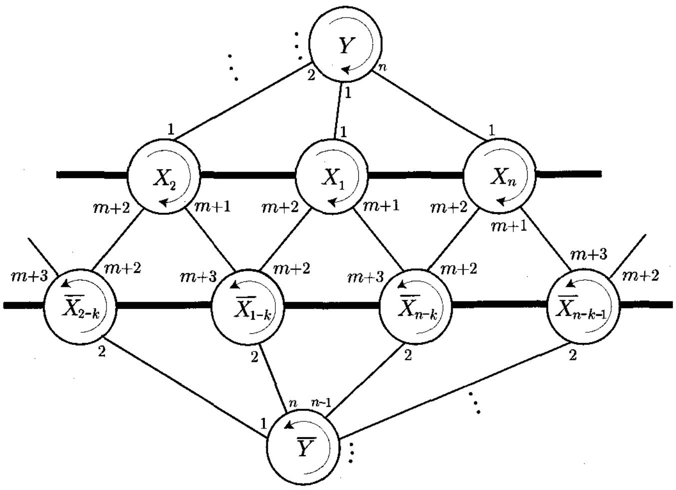


FIGURE 3. $H(2m+1, n, k)$

By the identification of the edges, the quotient space of $\mathcal{P}(2m+1, n, k)$ which we denote by $\mathcal{M}(2m+1, n, k)$, clearly has $n+d$ edges, $n+1$ two-cells and 1 three-cell. Moreover a routine check gives that $\mathcal{M}(2m+1, n, k)$ has d vertices, namely the starting points of the classes of edges represented by the h_i's, where $d = \gcd(n, k)$. Thus its Euler characteristic vanishes and so $\mathcal{M}(2m+1, n, k)$ forms a closed, oriented, compact 3-manifold, according to [5]: *a complex, which is formed by identifying the faces of a polyhedron will be a manifold if and only if its Euler characteristic equals zero.*

3. 3-Manifolds and links $\mathcal{L}_{(m,d)}$

In this section we apply the same argument in [2] to study the fundamental group of $\mathcal{M}(2m+1, n, k)$ by considering a Heegaard diagram of $\mathcal{M}(2m+1, n, k)$, and the quotient spaces of $\mathcal{M}(2m+1, n, k)$ by the action of the cyclic group of rotations induced by the cylindrical symmetries of polyhedra $\mathcal{P}(2m+1, n, k)$. We note that the polyhedron $\mathcal{P}(2m+1, n, k)$ defines a decomposition of $\mathcal{M}(2m+1, n, k)$ into handles. All the information can be shown in a Heegaard diagram $H(2m+1, n, k)$ of $\mathcal{M}(2m+1, n, k)$(see Figure 3, where each thick arc indicates $m-1$ lines), which consists of $(n+1)$ 2-disks, $X_1, X_2, \ldots, X_n$ and Y, joined by certain nonintersecting arcs with suitable labels. From $H(2m+1, n, k)$, we can easily obtain the fundamental group $G(2m+1, n, k)$ of $\mathcal{M}(2m+1, n, k)$ as follows: the presentation has generators $x_1, x_2, \ldots, x_n$ and y associated to the 1-handle $X_1, X_2, \ldots, X_n$ and Y, respectively. The group relations come from the cycle relations determined by edges of $\mathcal{P}(2m+1,n,k)$. We have the following presentation for $G(2m+1,n,k)$.

THEOREM 1. *Let $G(2m+1,n,k)$ be the fundamental group of $\mathcal{M}(2m+1,n,k)$, with $d=\gcd(n,k)$. Then for even m,*

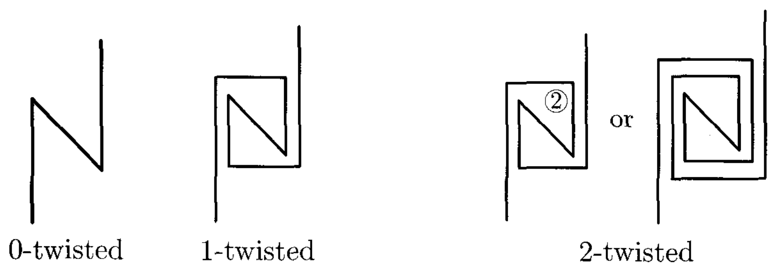
$$G(2m+1,n,k) = \langle x_1, x_2, \ldots, x_n, y | \{(x_i x_{i+1})^{\frac{m}{2}} (x_{i-k+1}^{-1} x_{i-k+2})^{\frac{m}{2}} y^{-1} = 1\}_{i=1,\ldots,n}, \{x_i x_{i+k} \cdots x_{i+(\frac{n}{d}-1)k} = 1\}_{i=1,\ldots,d}, \text{ indices } \bmod n\rangle,$$

and for odd m

$$G(2m+1,n,k) = \langle x_1, x_2, \ldots, x_n, y | \{x_i (x_{i+1}^{-1} x_i)^{\frac{m-1}{2}} (x_{i+k+1} x_{i+k}^{-1})^{\frac{m-1}{2}} x_{i+k+1} y^{-1} = 1\}_{i=1,\ldots,n} \{x_i x_{i+k} \cdots x_{i+(\frac{n}{d}-1)k} = 1\}_{i=1,\ldots,d}, \text{ indices } \bmod n\rangle.$$

Let $\mathcal{L} := \mathcal{L}_0 \cup \cdots \cup \mathcal{L}_r$ be an oriented link with $r+1$ components in the right-hand oriented 3-sphere $\mathbb{S}^3$. By $\mathcal{O}_{n_0,n_1,\ldots,n_r}(\mathcal{L})$ we denote the 3-orbifold which is topologically the 3-sphere, and with singular set the link $\mathcal{L}$, whose i-th component $\mathcal{L}_i$ has branching index $n_i > 1$, for any $i = 0,1,\ldots,r$. In particular, if $n_i = n$ for all i, then we write $\mathcal{O}_n(\mathcal{L})$ instead of $\mathcal{O}_{n_0,n_1,\ldots,n_r}(\mathcal{L})$.

To simplify figures, we define twisted lines as follows:



Naturally we define an n-twisted line for all non-negative integer n.

THEOREM 2. *$\mathcal{M}(2m+1,n,k)$ are (n/d)-fold cyclic coverings of $\mathbb{S}^3$ over $\mathcal{L}_{(m,d)}$, where $\gcd(n,k)=d$.*

PROOF. We note that $\mathcal{M}(2m+1,n,k)$ is an $\frac{n}{d}$-fold covering of the quotient orbifold $\mathcal{O}_{\frac{n}{d}}(L) = \mathcal{M}(2m+1,n,k)/\mathbb{Z}_{n/d}$ which is simply the identification space of the polyhedron $\mathcal{P}(2m+1,d,0)$ of identification

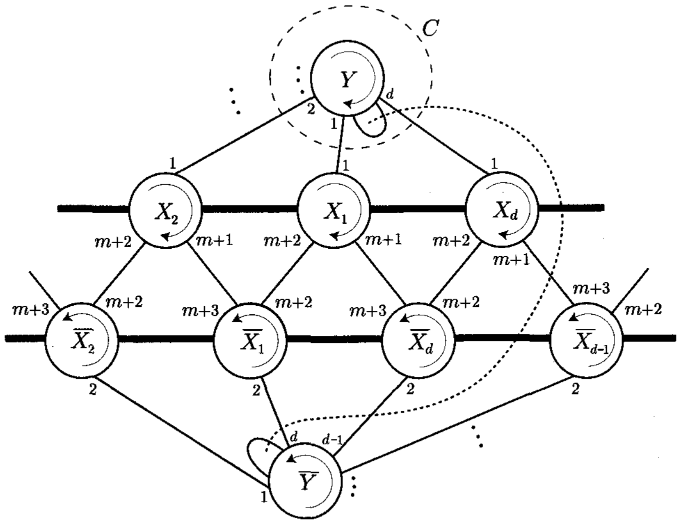


FIGURE 4. $H(2m+1,d,0)$

index 0. Moreover the singular set L is the image in the quotient of the axis of symmetry of the rotation and the image of the edges with label k_i, $i=0,\dots,d-1$. A Heegaard diagram $H(2m+1,d,0)$ contains all information for the handle decompositions of the identification space $\mathcal{O}_{\frac{n}{d}}(L)$, which is depicted in Figure 4, where each thick line indicates $m-1$ lines and the dotted line is the image in the quotient of the axis of symmetry. $H(2m+1,d,0)$ consists of $(d+1)$ meridians $X_1, X_2, \cdots, X_d$, and Y corresponding to d $(2m+1)$-gons and the d-gon Y respectively, and curves corresponding to classes of identified sides of the polygons. We now simplify Heegaard diagrams by using handle-cancellations. We first eliminate 1-handles X_i with 2-handles connecting two points $m+2$ on X_i and $\overline{m+2}$ on $\overline{X}_i$ for $1 \leq i \leq d$. On the other hand we dig a 3-ball along a curve C around Y and glue it back along the pair of disks determined by C to create 1-handle Y'. Then we have the deformed graph as shown in Figure 5, where the broken lines are singular cores of 2-handles and each arc from Y' to $\overline{Y}'$ is an $(m-1)$-twisted line. Finally we cancel 1-handle Y' with a 2-handle and all left 2-handles to get $\mathbb{S}^3$. Further by Reidemeister moves we see that L is equivalent to the link $\mathcal{L}_{(m,d)}$. □

EXAMPLE. We consider $\mathcal{P}(7,4,2)$ and $\mathcal{M}(7,4,2)$. Then $H(7,2,0)$ consists of three 1-handles, X_1, X_2 and Y, five 2-handles, a, b, c, d and e,

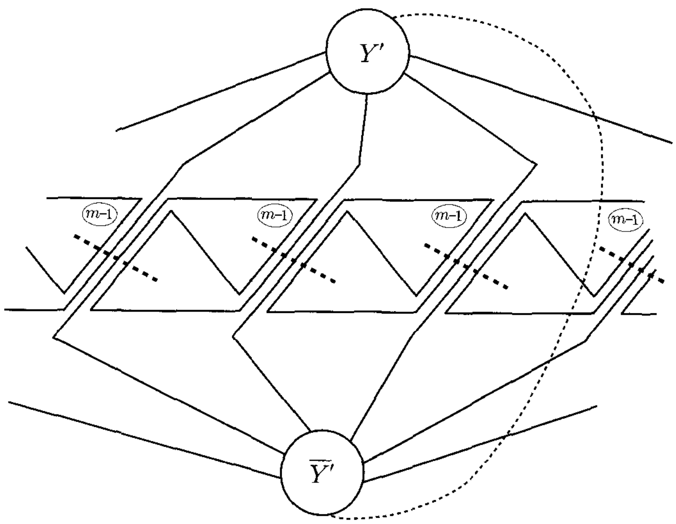


FIGURE 5. After cancelling $X_1, \cdots, X_d$ and Y

and three 3-handles, u, v and w(see Figure 6 where the dotted line is the image in the quotient of the axis of symmetry). We eliminate 1-handles X_1 and X_2 with two 2-handles d and e, respectively. Further we dig a 3-ball along a curve C surrounding 1-handle Y and glue it back along the pair of disks determined by C to create 1-handle Y'. Then we have the deformed graph as shown in Figure 7.a, where the broken lines are singular cores of 2-handles and each arc from Y' to $\overline{Y'}$ is a 2-twisted line. We cancel 1-handle Y' with 2-handle b. Finally we cancel 2-handle c to get $\mathbb{S}^3$ and the branched set L(see Figure 7.b). By Reidemeister moves L is equivalent to $\mathcal{L}_{(3,2)}$ as seen in Figure 8.

THEOREM 3. *$\mathcal{M}(2m+1, n, k)$ for $\gcd(n,k) = d \geq 1$ are n-fold cyclic branched covering spaces of $\mathbb{S}^3$ over $\mathcal{L}_{(m,1)}$ where the branched indices of its components are n and $\frac{n}{d}$.*

PROOF. We note that $\mathcal{M}(2m+1, n, k)$ is an $\frac{n}{d}$-fold cyclic covering of the orbifold $\mathcal{O}_{\frac{n}{d}}(\mathcal{L}_{(m,d)})$ and $\mathcal{O}_{\frac{n}{d}}(\mathcal{L}_{(m,d)})$ has a symmetry of order d which is a rotation around the trivial component. Therefore the quotient space of $\mathcal{O}_{\frac{n}{d}}(\mathcal{L}_{(m,d)})$ under this rotation is the orbifold $\mathcal{O}_{n,\frac{n}{d}}(\mathcal{L}_{(m,1)})$ whose underlying space is $\mathbb{S}^3$, and the singular set is $\mathcal{L}_{(m,1)}$ with branch indices n and $\frac{n}{d}$ as shown in Figure 1.b. □

We recall some notation for a hyperbolic structure of $\mathcal{M}(2m+1, n, k)$. The link $\mathcal{L}$ in the 3-sphere is hyperbolic if its complement $mathcalS^3 - \mathcal{L}$

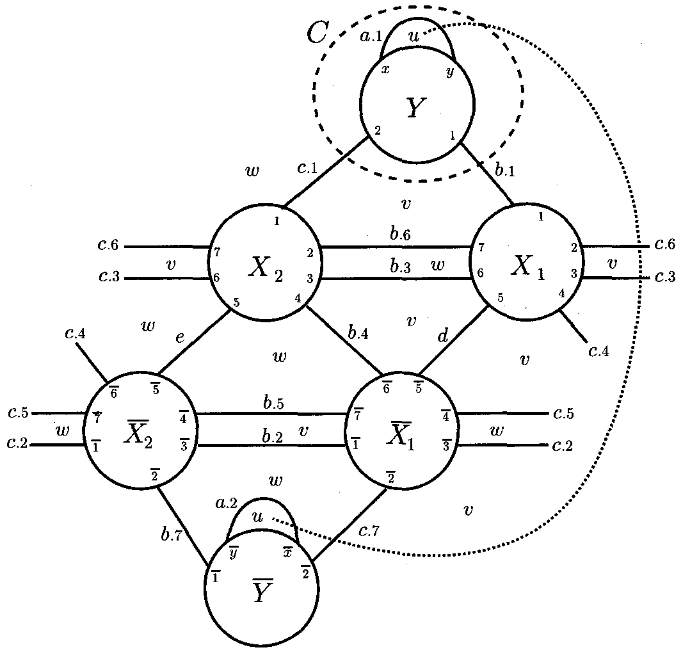
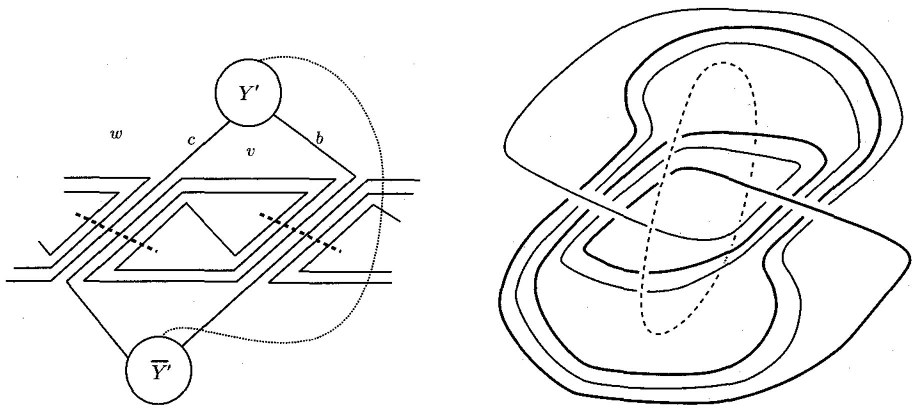


FIGURE 6. $H(7,2,0)$



a. After cancelling X_1, X_2 and Y b. The branched set L

FIGURE 7

admits a complement hyperbolic structure (Riemannian metric of constant negative curvature) of finite volume. From the theory of orbifolds

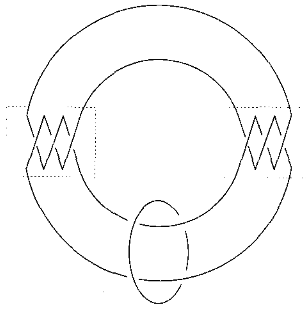


FIGURE 8. $\mathcal{L}_{(3,2)}$

in [6] we have that $\mathcal{O}_{n,\frac{n}{d}}(\mathcal{L})$ is the 3-orbifold whose underlying topological space is the 3-sphere and whose singular sets, of branching indices n, $\frac{n}{d}$, are the link $\mathcal{L}$. Another way of saying this is that $\mathcal{O}_{n,\frac{n}{d}}(\mathcal{L})$ is a hyperbolic 3-orbifold for sufficiently large n and $\frac{n}{d}$; equivalently, the n-fold cyclic branched covering of $\mathcal{L}$ is a hyperbolic 3-manifold and the cyclic covering group acts by isometries. In general, for an explicitly given link $\mathcal{L}$ the hyperbolicity of $\mathcal{L}$ can be checked by direct and more elementary methods, for example by computer using Weeks' SnapPea program (see [4]).

We note that $\mathcal{L}_{(m,1)}$ is hyperbolic for $m > 1$, that is, $\mathcal{O}_{n,\frac{n}{d}}(\mathcal{L}_{(m,1)})$ is a hyperbolic 3-orbifold for sufficiently large n and $\frac{n}{d}$. Actually $\mathcal{L}_{(1,1)}$ is the Double link, and $\mathcal{L}_{(2,1)}$ is the Whitehead link which was considered in [3] and [2] as a special case.

For the covering diagram, we have

$$\mathcal{M}(2m+1,n,k) \xrightarrow{\frac{n}{d}} \mathcal{O}_{\frac{n}{d}}(\mathcal{L}_{(m,d)}) \xrightarrow{d} \mathcal{O}_{n,\frac{n}{d}}(\mathcal{L}_{(m,1)}),$$

where $d = \gcd(n,k)$ and the labels of the maps indicate the degree of the covering. Therefore the manifold $\mathcal{M}(2m+1,n,k)$ is the closed hyperbolic 3-manifold. In particular $\mathcal{M}(2m+1,n,k)$ is the n-fold cyclic branched covering spaces of $\mathbb{S}^3$ over $\mathcal{L}_{(m,1)}$ corresponding to the kernel of the surjection $\psi_{n,k} : \mathcal{O}_{n,\frac{n}{d}}(\mathcal{L}_{(m,1)}) \longrightarrow \mathbb{Z}_n$ which is defined as above. The symmetry group of $\mathcal{L}_{(m,1)}$ is isomorphic to the dihedral group D_4 of order 8 or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Hence we can apply the same argument as in the proof of Theorem 3 [7] to get the following.

THEOREM 4. *Let $n \geq 2$ and $k, k' \neq 0$, with $(n,k) = (n,k') = 1$. The cyclic branched coverings $\mathcal{M}(2m+1,n,k)$ and $\mathcal{M}(2m+1,n,k')$ of $\mathcal{L}_{(m,1)}$ are isometric if and only if $k \equiv \pm k' (\bmod\, n)$ or $kk' \equiv \pm 1 (\bmod\, n)$.*

With $d := d_1 \neq 1$ we have

THEOREM 5. *Let* $n \geq 2$ *and* $k, k' \neq 0$, *with* $(n,k) = (n,k') \neq 1$. *The cyclic branched coverings* $\mathcal{M}(2m+1, n, k)$ *and* $\mathcal{M}(2m+1, n, k')$ *of* $\mathcal{L}_{(m,1)}$ *are isometric if and only if* $k \equiv \pm k' (\mathrm{mod}\, n)$.

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