Commun. Korean Math. Soc. **20** (2005), No. 4, pp. 791–801

SOME RESULTS ON INTUITIONISTIC FUZZY TOPOLOGICAL SPACES DEFINED BY INTUITIONISTIC GRADATION OF OPENNESS

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Abstract. In this paper, we introduce the concepts of closure and interior defined by an intuitionistic gradation of openness. We also introduce the concepts of weakly gp-maps, gp-open maps and several types of compactness, and obtain some characterizations.

1. Introduction

In [8], Chattopadyay et al. introduced the concept of fuzzy topology redefined by a gradation of openness and investigated some fundamental properties. Gayyar, Kerre, Ramadan[7] and Demirci[5, 6] introduced the concepts of fuzzy closure and fuzzy interior in the fuzzy topological space, and obtained some properties of them. Atanassov [1] introduced the concept of intuitionistic fuzzy set which is a generalization of fuzzy set in Zadeh's sense [11]. Çoker introduced the concept of intuitionistic fuzzy topological spaces [4] by using the intuitionistic fuzzy sets, which is an extended concept of fuzzy topological spaces [2] in Chang's sense. In 2002, Mondal and Samanta introduced the concept of intuitionistic gradations of openness [9] which is a generalization of the concept of gradation of openness defined by Chattopadyay.

In this paper, we introduce the concepts of closure and interior defined by intuitionistic gradation of openness. We also introduce the concepts of weakly gp-maps and gp-open maps, intuitionistic fuzzy cover,

Received June 4, 2004.

2000 Mathematics Subject Classification: 54A40.

Key words and phrases: intuitionistic fuzzy sets, intuitionistic gradation of openness, weakly gp-map, gp-open map, intuitionistic fuzzy cover, intuitionistic fuzzy compact, nearly intuitionistic fuzzy compact, almost intuitionistic fuzzy compact.

This work was supported by a grant from Research Institute for Basic Science at Kangwon National University.

intuitionistic fuzzy compactness, nearly intuitionistic fuzzy compactness, almost intuitionistic fuzzy compactness, and investigate their properties.

2. Preliminaries

Let X be a set and $I = [0, 1]$ be the unit interval of the real line. I^X will denote the set of all fuzzy sets of X. 0_X and 1_X will denote the characteristic functions of ϕ and X, respectively.

DEFINITION 2.1. [3, 8, 10] Let X be a non-empty set and $\tau : I^X \to I$ be a mapping satisfying the following conditions:
(1) $\tau(0_X) = \tau(1_X) = 1$;
(2) $\forall A, B \in I^X,\ \tau(A \cap B) \geq \tau(A) \wedge \tau(B)$;
(3) for every subfamily $\{A_i : i \in J\} \subseteq I^X,\ \tau(\cup_{i \in J} A_i) \geq \wedge_{i \in J} \tau(A_i)$.
Then the mapping $\tau : I^X \to I$ is called a fuzzy topology (or gradation of openness [10]) on X. We call the ordered pair (X, τ) a fuzzy topological space. The value $\tau(A)$ is called the degree of openness of A.

DEFINITION 2.2. [1] An intuitionistic fuzzy set A in a set X is an object having the form

$$A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\},$$

where the functions $\mu_A : X \to I$ and $\gamma_A : X \to I$ denote the degree of membership and the degree of nonmembership of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for each $x \in X$.

DEFINITION 2.3. [9] An intuitionistic gradation of openness (briefly IGO) of fuzzy subsets of a set X is an ordered pair (τ, τ^*) of functions $\tau, \tau^* : I^X \to I$ such that
(IGO1) $\tau(A) + \tau^*(A) \leq 1$, for all $A \in I^X$;
(IGO2) $\tau(0_X) = \tau(1_X) = 1, \tau^*(0_X) = \tau^*(1_X) = 0$;
(IGO3) $\forall A, B \in I^X,\ \tau(A \cap B) \geq \tau(A) \wedge \tau(B)$ and $\tau^*(A \cap B) \leq \tau^*(A) \vee \tau^*(B)$;
(IGO4) for every subfamily $\{A_i : i \in J\} \subseteq I^X,\ \tau(\cup_{i \in J} A_i) \geq \wedge_{i \in J} \tau(A_i)$ and $\tau^*(\cup_{i \in J} A_i) \leq \vee_{i \in J} \tau^*(A_i)$.

Then the triplet (X, τ, τ^*) is called an intuitionistic fuzzy topological space (briefly $IFTS$) on X. τ and τ^* may be interpreted as gradation of openness and gradation of nonopenness, respectively.

DEFINITION 2.4. [9] Let X be a nonempty set and two functions $\mathcal{F}, \mathcal{F}^* : I^X \to I$ be satisfying

(IGC1) $\mathcal{F}(A) + \mathcal{F}^*(A) \leq 1$, for all $A \in I^X$;
(IGC2) $\mathcal{F}(0_X) = \mathcal{F}(1_X) = 1, \mathcal{F}^*(0_X) = \mathcal{F}^*(1_X) = 0$;
(IGC3) $\forall A, B \in I^X$, $\mathcal{F}(A \cup B) \geq \mathcal{F}(A) \wedge \mathcal{F}(B)$ and $\mathcal{F}^*(A \cup B) \leq \mathcal{F}^*(A) \vee \mathcal{F}^*(B)$;
(IGC4) for every subfamily $\{A_i : i \in J\} \subseteq I^X$, $\mathcal{F}(\bigcap_{i\in J} A_i) \geq \bigwedge_{i\in J} \mathcal{F}(A_i)$ and $\mathcal{F}^*(\bigcap_{i\in J} A_i) \leq \bigvee_{i\in J} \mathcal{F}^*(A_i)$.

Then the ordered pair $(\mathcal{F}, \mathcal{F}^*)$ is called an intuitionistic gradation of closedness [9] (briefly *IGC*) on X. $\mathcal{F}$ and $\mathcal{F}^*$ may be interpreted as gradation of closedness and gradation of nonclosedness, respectively.

THEOREM 2.5. [9] *Let X be a nonempty set. If (τ, τ^*) is an IGO on X, then the pair $(\mathcal{F}_\tau, \mathcal{F^*}_{\tau^*})$, defined by $\mathcal{F}_\tau(A) = \tau(A^c)$, $\mathcal{F^*}_{\tau^*}(A) = \tau^*(A^c)$ where A^c denotes the complement of A, is an IGC on X. And if $(\mathcal{F}, \mathcal{F}^*)$ is an IGC on X, then the pair $(\tau_\mathcal{F}, \tau^*{}_{\mathcal{F}^*})$, defined by $\tau_\mathcal{F}(A) = \mathcal{F}(A^c)$, $\tau^*{}_{\mathcal{F}^*}(A) = \mathcal{F}^*(A^c)$, is an IGO on X.*

DEFINITION 2.6. [9] Let (X, τ, τ^*) and (Y, σ, σ^*) be two IFTSs. A mapping $f : X \to Y$ is a *gp-map* if $\tau(f^{-1}(A)) \geq \sigma(A)$ and $\tau^*(f^{-1}(A)) \leq \sigma^*(A)$ for every $A \in I^Y$.

All the other definitions and notations which are related to our discussion, can be found in [3, 4, 5, 6, 7, 9].

3. Closure and Interior Operators in IFTS

In this section, we introduce the concepts of closure and interior on an IFTS and investigate some their properties.

DEFINITION 3.1. Let (X, τ, τ^*) be an IFTS and $A \in I^X$. Then the closure (resp., interior) of A, denoted by $\overline{A}$ (resp., A^o), is defined by $\overline{A} = \bigcap\{K \in I^X : \mathcal{F}_\tau(K) > 0$ and $\mathcal{F^*}_{\tau^*}(K) \leq \mathcal{F^*}_{\tau^*}(A), A \subseteq K\}$ (resp., $A^o = \bigcup\{K \in I^X : \tau(K) > 0$ and $\tau^*(K) \leq \tau^*(A), K \subseteq A\}$).

THEOREM 3.2. *Let (X, τ, τ^*) be an IFTS and $A, B \in I^X$. Then*

(1) $\mathcal{F}^*_{\tau^*}(\overline{A}) \leq \mathcal{F}^*_{\tau^*}(A)$,
(2) $\tau^*(A^o) \leq \tau^*(A)$,
(3) $A \subseteq B$ *and* $\mathcal{F}^*_{\tau^*}(B) \leq \mathcal{F}^*_{\tau^*}(A) \Rightarrow \overline{A} \subseteq \overline{B}$,
(4) $A \subseteq B$ *and* $\tau^*(A) \leq \tau^*(B) \Rightarrow A^o \subseteq B^o$.

PROOF. (1) From Definition 2.4 and Definition 3.1, we have

$$
\begin{aligned}
&\mathcal{F}^*_{\tau^*}(\overline{A})\\
&= \mathcal{F}^*_{\tau^*}(\cap\{K \in I^X : \mathcal{F}_\tau(K) > 0 \text{ and } \mathcal{F^*}_{\tau^*}(K) \le \mathcal{F^*}_{\tau^*}(A), A \subseteq K\})\\
&\le \vee\{\mathcal{F}^*_{\tau^*}(K) : \mathcal{F}_\tau(K) > 0 \text{ and } \mathcal{F^*}_{\tau^*}(K) \le \mathcal{F^*}_{\tau^*}(A), A \subseteq K\}\\
&\le F^*_{\tau^*}(A).
\end{aligned}
$$

(2) It is similar to (1).

(3) If L is any element of $\{K \in I^X : \mathcal{F}_\tau(K) > 0 \text{ and } \mathcal{F}^*_{\tau^*}(K) \le \mathcal{F}^*_{\tau^*}(B), B \subseteq K\}$, then it is also in $\{K \in I^X : \mathcal{F}_\tau(K) > 0 \text{ and } \mathcal{F}^*_{\tau^*}(K) \le \mathcal{F}^*_{\tau^*}(A), A \subseteq K\}$. Hence $\overline{A} \subseteq \overline{B}$.

(4) The proof is similar to that of (3). □

THEOREM 3.3. *Let (X, τ, τ^*) be an IFTS and $A \in I^X$. Then*

(1) $(\overline{A})^c = (A^c)^o$,
(2) $\overline{A} = ((A^c)^o)^c$,
(3) $(A^o)^c = \overline{(A^c)}$,
(4) $A^o = (\overline{(A^c)})^c$.

PROOF. (1) From Definition 3.1, we have

$$
\begin{aligned}
(\overline{A})^c &= (\cap\{K \in I^X : \mathcal{F}_\tau(K) > 0 \text{ and } \mathcal{F}^*_{\tau^*}(K) \le \mathcal{F}^*_{\tau^*}(A), A \subseteq K\})^c\\
&= \cup\{K^c : K \in I^X, \tau(K^c) = \mathcal{F}_\tau(K) > 0\\
&\qquad\qquad \text{and } \tau^*(K^c) \le \tau^*(A^c), K^c \subseteq A^c\}\\
&= \cup\{U \in I^X : \tau(U) > 0 \text{ and } \tau^*(U) \le \tau^*(A^c), U \subseteq A^c\}\\
&= (A^c)^o.
\end{aligned}
$$

(2), (3) and (4) are easily obtained from (1). □

THEOREM 3.4. *Let (X, τ, τ^*) be an IFTS and $A, B \in I^X$. Then*

(1) $\overline{(0_X)} = 0_X$,
(2) $A \subseteq \overline{A}$,
(3) $\overline{A} \subseteq \overline{\overline{A}}$,
(4) $\overline{A} \cap \overline{B} \subseteq \overline{(A \cup B)}$.

PROOF. (1) and (2) are easily obtained from Definition 3.1.

(3) It follows directly from (2).

(4) For every $A, B \in I^X$, by Definition 2.4 we have

$$\begin{aligned}
\overline{(A \cup B)} &= \cap\{K \in I^X : \mathcal{F}_\tau(K) > 0 \\
&\qquad \text{and } \mathcal{F}^*_{\tau^*}(K) \leq \mathcal{F}^*_{\tau^*}(A \cup B), A \cup B \subseteq K\} \\
&\supseteq \cap\{K \in I^X : \mathcal{F}_\tau(K) > 0 \\
&\qquad \text{and } \mathcal{F}^*_{\tau^*}(K) \leq \mathcal{F}^*_{\tau^*}(A) \vee \mathcal{F}^*_{\tau^*}(B), A \cup B \subseteq K\} \\
&\supseteq \cap[\{K \in I^X : \mathcal{F}_\tau(K) > 0 \text{ and } \mathcal{F}^*_{\tau^*}(K) \leq \mathcal{F}^*_{\tau^*}(A), A \subseteq K\} \\
&\quad \cup \{K \in I^X : \mathcal{F}_\tau(K) > 0 \text{ and } \mathcal{F}^*_{\tau^*}(K) \leq \mathcal{F}^*_{\tau^*}(B), B \subseteq K\}] \\
&= [\cap\{K \in I^X : \mathcal{F}_\tau(K) > 0 \text{ and } \mathcal{F}^*_{\tau^*}(K) \leq \mathcal{F}^*_{\tau^*}(A), A \subseteq K\}] \\
&\quad \cap [\cap\{K \in I^X : \mathcal{F}_\tau(K) > 0 \text{ and } \mathcal{F}^*_{\tau^*}(K) \leq \mathcal{F}^*_{\tau^*}(B), B \subseteq K\}] \\
&= \overline{A} \cap \overline{B}.
\end{aligned}$$

□

THEOREM 3.5. *Let (X, τ, τ^*) be an IFTS and $A, B \in I^X$. Then*
(1) $(1_X)^o = 1_X$,
(2) $A^o \subseteq A$,
(3) $(A^o)^o \subseteq A^o$,
(4) $(A \cap B)^o \subseteq A^o \cup B^o$.

PROOF. It is similar to the proof of Theorem 3.4. □

THEOREM 3.6. *Let (X, τ, τ^*) be an IFTS and $A \in I^X$. Then*
(1) $\tau(A) > 0 \Rightarrow A^o = A$,
(2) $\mathcal{F}_\tau(A) > 0 \Rightarrow \overline{A} = A$.

PROOF. (1) Let $\tau(A) > 0$; then $A \in \{K \in I^X : \tau(K) > 0$ and $\tau^*(K) \leq \tau^*(A), K \subseteq A\}$, so $A \subseteq A^o$. Thus we get $A^o = A$ by Theorem 3.5(2).
(2) It is similar to (1). □

EXAMPLE 3.7. Let $X = I$ and let N denote the set of all natural numbers. For each $n \in N$, we consider a fuzzy set μ_n in X as the following: $\mu_n(x) = \frac{n-1}{n}x$ for $x \in X$.

Define an intuitionistic gradation of openness $\tau, \tau^* : I^X \to I$ by

$$\begin{aligned}
&\tau(0_X) = \tau(1_X) = 1, \tau^*(0_X) = \tau^*(1_X) = 0, \\
&\tau(\mu_n) = \frac{1}{n},\ \tau^*(\mu_n) = \frac{n-1}{2n}, \\
&\tau(\mu) = 0,\ \tau^*(\mu) = \frac{1}{2} \text{ for all other fuzzy set } \mu \in I^X.
\end{aligned}$$

Let us take a fuzzy set A in X such that $A(x) = x$ for all $x \in X$. Then it follows $A^o = A$ but $\tau(A) = 0,\ \tau^*(A) = \frac{1}{2}$. Thus the converse of the part (1) in Theorem 3.6 is not true in general.

In the same way, we can show that the converse of the part (2) may not be true.

4. Weakly gp-maps and gp-open maps

DEFINITION 4.1. Let (X, τ, τ^*) and (Y, σ, σ^*) be two IFTSs. A mapping $f : X \to Y$ is a *weakly gp-map* if for every $A \in I^Y$, $\sigma(A) > 0 \Rightarrow \tau(f^{-1}(A)) > 0$ and $\tau^*(f^{-1}(A)) \leq \sigma^*(A)$.

It is obvious that every gp-map is a weakly gp-map from the above definition. But we can show that the converse is not always true from the following example:

EXAMPLE 4.2. Let $X = I$ and let N denote the set of all natural numbers. For each $n \in N$, we consider $\mu_n \in I^X$ such that $\mu_n(x) = \frac{1}{n}x$ for $x \in X$.
Define $\tau, \tau^* : I^X \to I$ by

$$\tau(0_X) = \tau(1_X) = 1, \tau^*(0_X) = \tau^*(1_X) = 0;$$
$$\tau(\mu_n) = \frac{1}{n+2}, \tau^*(\mu_n) = \frac{1}{n+2} \text{ for each } n \in N;$$
$$\tau(\mu) = 0, \tau^*(\mu) = 1 \text{ for all other fuzzy set } \mu \text{ in } X.$$

And define $\sigma, \sigma^* : I^X \to I$ by

$$\sigma(0_X) = \sigma(1_X) = 1, \sigma^*(0_X) = \sigma^*(1_X) = 0;$$
$$\sigma(\mu_n) = \frac{1}{n+1},\ \sigma^*(\mu_n) = \frac{1}{n+1} \text{ for each } n \in N;$$
$$\sigma(\mu) = 0, \sigma^*(\mu_n) = 1 \text{ for all other fuzzy set } \mu \text{ in } X.$$

Then the pairs (τ, τ^*) and (σ, σ^*) are two intuitionistc gradations of openness on X. Consider the identity mapping $f : (X, \tau, \tau^*) \to (Y, \sigma, \sigma^*)$. Then f is a weakly gp-map but not a gp-map, since for each fuzzy set μ_n, we get $\tau^*(f^{-1}(\mu_n)) \leq \sigma^*(\mu_n)$ but $\sigma(\mu_n)$ is not less than $\tau(f^{-1}(\mu_n))$.

THEOREM 4.3. *Let (X, τ, τ^*) and (Y, σ, σ^*) be two IFTSs. Then a mapping $f : X \to Y$ is a weakly gp-map iff for every $A \in I^Y$ $\mathcal{F}_\sigma(A) > 0 \Rightarrow \mathcal{F}_\tau(f^{-1}(A)) > 0$ and $\mathcal{F}^*{}_{\tau^*}(f^{-1}(A)) \leq \mathcal{F}^*{}_{\sigma^*}(A)$.*

PROOF. Suppose f is a weakly gp-map and let $\mathcal{F}_\sigma(A) > 0$ for $A \in I^Y$; then $\mathcal{F}_\sigma(A^{cc}) = \sigma(A^c) > 0$, so it follows $\tau(f^{-1}(A^c)) > 0$ and $\tau^*(f^{-1}(A^c)) \leq \sigma^*(A^c)$. Thus we get $\mathcal{F}_\tau(f^{-1}(A)) > 0$ and $\mathcal{F}^*{}_{\tau^*}(f^{-1}(A)) \leq \mathcal{F}^*{}_{\sigma^*}(A)$.
The converse is obvious. □

THEOREM 4.4. *Let (X, τ, τ^*) and (Y, σ, σ^*) be two IFTSs. Then a mapping $f : X \to Y$ is a gp-map iff for every $A \in I^Y$, $\mathcal{F}_\sigma(A) \leq \mathcal{F}_\tau(f^{-1}(A))$ and $\mathcal{F}^*{}_{\tau^*}(f^{-1}(A)) \leq \mathcal{F}^*{}_{\sigma^*}(A)$.*

PROOF. The proof is similar to that of Theorem 4.3. □

THEOREM 4.5. *Let (X, τ, τ^*) and (Y, σ, σ^*) be two IFTSs. If $f : X \to Y$ is a weakly gp-map, then we have*
(1) $f(\overline{A}) \subseteq \overline{f(A)}$ *for every* $A \in I^X$,
(2) $\overline{(f^{-1}(A))} \subseteq f^{-1}(\overline{A})$ *for every* $A \in I^Y$,
(3) $f^{-1}(A^o) \subseteq (f^{-1}(A))^o$ *for every* $A \in I^Y$.

PROOF. (1) Let $A \in I^X$; then by Definition 3.1 and Theorem 4.3, we have

$$\begin{aligned} f^{-1}(\overline{f(A)}) = f^{-1}[\cap\{U \in I^Y : \mathcal{F}_\sigma(U) > 0 \\ \text{and } \mathcal{F}^*{}_{\sigma^*}(U) \leq \mathcal{F}^*{}_{\sigma^*} f(A), f(A) \subseteq U\}] \\ \supseteq \cap\{f^{-1}(U) \in I^X : \mathcal{F}_\tau(f^{-1}(U)) > 0 \\ \text{and } \mathcal{F}^*{}_{\tau^*}(f^{-1}(U)) \leq \mathcal{F}^*{}_{\sigma^*}(U), A \subseteq f^{-1}(U)\}. \end{aligned}$$

Since $\mathcal{F}_\tau(f^{-1}(U)) > 0$, it follows $\overline{A} \subseteq \overline{f^{-1}(U)} = f^{-1}(U)$ from Theorem 3.6, and so $\cap\{f^{-1}(U) \in I^X : \mathcal{F}_\tau(f^{-1}(U)) > 0$ and $\mathcal{F}^*{}_{\tau^*}(f^{-1}(U)) \leq \mathcal{F}^*{}_{\sigma^*}(U), A \subseteq f^{-1}(U)\} \supseteq \overline{A}$.
Consequently, we get $f(\overline{A}) \subseteq \overline{f(A)}$.
(2) It follows from (1).
(3) It is obtained by (2) and Theorem 3.3. □

COROLLARY 4.6. Let (X, τ, τ^*) and (Y, σ, σ^*) be two IFTSs. If $f : X \to Y$ is a gp-map, then we have
(1) $f(\overline{A}) \subseteq \overline{f(A)}$ for every $A \in I^X$,
(2) $\overline{(f^{-1}(A))} \subseteq f^{-1}(\overline{A})$ for every $A \in I^Y$,
(3) $f^{-1}(A^o) \subseteq (f^{-1}(A))^o$ for every $A \in I^Y$.

DEFINITION 4.7. Let (X, τ, τ^*) and (Y, σ, σ^*) be two IFTSs. A mapping $f : X \to Y$ is called a gp-open map (resp., gp-closed map) if $\tau(A) \leq \sigma(f(A))$ (resp., $\mathcal{F}_\tau(A) \leq \mathcal{F}_\sigma(f(A)))$ and $\sigma^*(f(A)) \leq \tau^*(A)$ (resp., $\mathcal{F}^*{}_{\sigma^*}(f(A)) \leq \mathcal{F}^*{}_{\tau^*}(A))$ for every $A \in I^X$.

THEOREM 4.8. *Let (X, τ, τ^*) and (Y, σ, σ^*) be two IFTSs. If a mapping $f : X \to Y$ is a gp-open map, then $f(A^o) \subseteq (f(A))^o$ for every $A \in I^X$.*

PROOF. For every $A \in I^X$, we have

$$\begin{aligned} f(A^o) &= f[\cup\{U \in I^X : \tau(U) > 0 \text{ and } \tau^*(U) \le \tau^*(A), U \subseteq A\}] \\ &\subseteq \cup\{f(U) \in I^Y : \tau(U) > 0 \text{ and } \tau^*(U) \le \tau^*(A), f(U) \subseteq f(A)\} \\ &\subseteq \cup\{f(U) \in I^Y : \sigma(f(U)) > 0 \\ &\qquad\qquad \text{and } \sigma^*(f(U)) \le \tau^*(U), f(U) \subseteq f(A)\}. \end{aligned}$$

Since $\sigma(f(U)) > 0$, it follows $(f(U))^o = f(U) \subseteq (f(A))^o$ from Theorem 3.6. Thus we get $f(A^o) \subseteq (f(A))^o$. □

THEOREM 4.9. *Let (X, τ, τ^*) and (Y, σ, σ^*) be two IFTSs. If $f : X \to Y$ is an injective gp-closed map, then $\overline{(f(A))} \subseteq f(\overline{A})$ for every $A \in I^X$.*

PROOF. Let $A \in I^X$; then since f is an injective gp-closed map, we have

$$\begin{aligned} f(\overline{A}) &= f[\cap\{U \in I^X : \mathcal{F}_\tau(U) > 0 \text{ and } \mathcal{F}^*{}_{\tau^*}(U) \le \mathcal{F}^*{}_{\tau^*}(A), A \subseteq U\}] \\ &= \cap\{f(U) \in I^X : \mathcal{F}_\tau(U) > 0 \\ &\qquad\qquad \text{and } \mathcal{F}^*{}_{\tau^*}(U)) \le \mathcal{F}^*{}_{\tau^*}(A), f(A) \subseteq f(U)\} \\ &\supseteq \cap\{f(U) \in I^X : \mathcal{F}_\sigma(f(U)) > 0 \\ &\qquad\qquad \text{and } \mathcal{F}^*{}_{\sigma^*}(f(U)) \le \mathcal{F}^*{}_{\tau^*}(U), f(A) \subseteq f(U)\}. \end{aligned}$$

Since $\mathcal{F}_\sigma(f(U)) > 0$, we get $f(\overline{A}) \supseteq \overline{f(A)}$. □

5. Several types of compactness on intuitionistic fuzzy topological spaces

In this section, we introduce the concepts of intuitionistic fuzzy compactness, nearly intuitionistic fuzzy compactness and almost intuitionistic fuzzy compactness in an IFTS, and investigate some properties of them.

DEFINITION 5.1. Let (X, τ, τ^*) be an IFTS. A family $\{A_i \in I^X : \tau(A_i) > 0 \text{ and } \tau^*(A_i) \le \tau(A_i), i \in J\}$ is called an intuitionistic fuzzy cover of X if $\cup_{i \in J} A_i = 1_X$.

DEFINITION 5.2. An IFTS (X, τ, τ^*) is said to be intuitionistic fuzzy compact if for every intuitionistic fuzzy cover $\{A_i \in I^X : \tau(A_i) > 0$ and $\tau^*(A_i) \leq \tau(A_i), i \in J\}$ of X, there exists a finite subset J_o of J such that $\cup_{i \in J_o} A_i = 1_X$.

LEMMA 5.3. *Let (X, τ, τ^*) and (Y, σ, σ^*) be two IFTSs and let $f : X \to Y$ be a gp-map (or weakly gp-map). If $\sigma(B) > 0$ and $\sigma^*(B) \leq \sigma(B)$ for $B \in I^Y$, then $\tau(f^{-1}(B)) > 0$ and $\tau^*(f^{-1}(B)) \leq \tau(f^{-1}(B))$.*

PROOF. From the definition of gp-map, we get the following relationships: $0 < \sigma(B) < \tau(f^{-1}(B))$ and $\tau^*(f^{-1}(B)) \leq \sigma^*(B) \leq \sigma(B)$. And these complete the proof. □

THEOREM 5.4. *Let (X, τ, τ^*) and (Y, σ, σ^*) be two IFTSs and let $f : X \to Y$ be a surjective weakly gp-map. If (X, τ, τ^*) is intuitionistic fuzzy compact, then so is (Y, σ, σ^*).*

PROOF. Let $\{B_i \in I^Y : \sigma(B_i) > 0$ and $\sigma^*(B_i) \leq \sigma(B_i), i \in J\}$ be an intuitionistic fuzzy cover of Y. Then by Lemma 5.3, we have an intuitionistic fuzzy cover $\{f^{-1}(B_i) \in I^X : \tau(f^{-1}(B_i)) > 0$ and $\tau^*(f^{-1}(B_i)) \leq \tau(f^{-1}(B_i)), i \in J\}$ of X. Since X is intuitionistic fuzzy compact, there exists a finite subset J_o of J such that $\cup_{i \in J_o} f^{-1}(B_i) = 1_X$. From the surjectivity of f, it follows $\cup_{i \in J_o} B_i = 1_Y$. Thus Y is intuitionistic fuzzy compact. □

DEFINITION 5.5. An IFTS (X, τ, τ^*) is said to be nearly intuitionistic fuzzy compact if for every intuitionistic fuzzy cover $\{A_i \in I^X : \tau(A_i) > 0$ and $\tau^*(A_i) \leq \tau(A_i), i \in J\}$ of X, there exists a finite subset J_o of J such that $\cup_{i \in J_o} (\overline{A_i})^o = 1_X$.

THEOREM 5.6. *An intuitionistic fuzzy compact space (X, τ, τ^*) is nearly intuitionistic fuzzy compact.*

PROOF. Let $\{A_i \in I^X : \tau(A_i) > 0$ and $\tau^*(A_i) \leq \tau(A_i), i \in J\}$ be an intuitionistic fuzzy cover of X; then there exists a finite subset J_o of J such that $\cup_{i \in J_o} A_i = 1_X$. Since $\tau(A_i) > 0$ for all $i \in J$, we have $A_i = (A_i)^o \subseteq (\overline{A_i})^o$. Thus $1_X = \cup_{i \in J_o} A_i \subseteq \cup_{i \in J_o} (\overline{A_i})^o$. Hence (X, τ, τ^*) is nearly intuitionistic fuzzy compact. □

REMARK 5.7. In Theorem 5.6, the converse implication may not be true. For if (X, τ, τ^*) is an IFTS and $\tau^*(\mu) = 0$ for all $\mu \in I^X$, then the (X, τ, τ^*) is a fuzzy topological space in Sostak's sense. In general, a nearly fuzzy compact space is not fuzzy compact, so we can say a nearly intuitionistic fuzzy compact space (X, τ, τ^*) is not always intuitionistic fuzzy compact.

DEFINITION 5.8. An IFTS (X, τ, τ^*) is said to be almost intuitionistic fuzzy compact if for every intuitionistic fuzzy cover $\{A_i \in I^X : \tau(A_i) > 0$ and $\tau^*(A_i) \leq \tau(A_i), i \in J\}$ of X, there exists a finite subset J_o of J such that $\cup_{i\in J_o}\overline{A_i} = 1_X$.

THEOREM 5.9. *A nearly intuitionistic fuzzy compact space (X, τ, τ^*) is almost intuitionistic fuzzy compact.*

PROOF. Since $(\overline{A})^o \subseteq \overline{A}$ for each $A \in I^X$, it is obvious that (X, τ, τ^*) is almost intuitionistic fuzzy compact. □

As Remark 5.7, we can show that the almost intuitionistic fuzzy compactness is not always the nearly intuitionistic fuzzy compactness.

THEOREM 5.10. *Let (X, τ, τ^*) and (Y, σ, σ^*) be two IFTSs and let $f : X \to Y$ be a surjective weakly gp-map. If X is almost intuitionistic fuzzy compact, then so is Y.*

PROOF. Let $\{A_i \in I^Y : \sigma(A_i) > 0$ and $\sigma^*(A_i) \leq \sigma(A_i), i \in J\}$ be an intuitionistic fuzzy cover of Y. By Lemma 5.3, $\{f^{-1}(A_i) \in I^X : \tau(f^{-1}(A_i)) > 0$ and $\tau^*(f^{-1}(A_i)) \leq \tau(f^{-1}(A_i)), i \in J\}$ is an intuitionistic fuzzy cover of X.

Since X is almost intuitionistic fuzzy compact, there exists a finite subset J_o of J such that $\cup_{i\in J_o}\overline{f^{-1}(A_i)} = 1_X$. From the surjectivity of f and Theorem 4.5, we have $1_Y = \cup_{i\in J_o} f(\overline{f^{-1}(A_i)}) \subseteq \cup_{i\in J_o} f(f^{-1}(\overline{A_i})) = \cup_{i\in J_o}\overline{A_i}$. Thus Y is almost intuitionistic fuzzy compact. □

COROLLARY 5.11. Let (X, τ, τ^*) and (Y, σ, σ^*) be two IFTSs and let $f : X \to Y$ be a surjective weakly gp-map. If X is nearly intuitionistic fuzzy compact, then Y is almost intuitionistic fuzzy compact.

THEOREM 5.12. *Let (X, τ, τ^*) and (Y, σ, σ^*) be two IFTSs and let $f : X \to Y$ be a surjective, weakly gp-map and gp-open map. If X is nearly intuitionistic fuzzy compact, then so is Y.*

PROOF. Let $\{A_i \in I^Y : \sigma(A_i) > 0$ and $\sigma^*(A_i) \leq \sigma(A_i), i \in J\}$ be an intuitionistic fuzzy cover of Y. Since f is a weakly gp-map, $\{f^{-1}(A_i) \in I^X : \tau(f^{-1}(A_i)) > 0$ and $\tau^*(f^{-1}(A_i)) \leq \tau(f^{-1}(A_i)), i \in J\}$ is an intuitionistic fuzzy cover of X. Since X is nearly intuitionistic fuzzy compact, there exists a finite subset J_o of J such that $\cup_{i\in J_o}(\overline{f^{-1}(A_i)})^o = 1_X$.

From the surjectivity of f, Theorem 4.5 and Theorem 4.8, we have

$$\begin{aligned} 1_Y &= \cup_{i\in J_o} f((\overline{f^{-1}(A_i)})^o) \\ &\subseteq \cup_{i\in J_o} (f(\overline{f^{-1}(A_i)}))^o \\ &\subseteq \cup_{i\in J_o} (f(f^{-1}(\overline{A_i})))^o \\ &= \cup_{i\in J_o} (\overline{A_i})^o. \end{aligned}$$

Thus Y is nearly intuitionistic fuzzy compact. □

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