

## A FAMILY OF QUANTUM MARKOV SEMIGROUPS

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ABSTRACT. For a given gauge invariant state  $\omega$  on the CAR algebra  $\mathcal{A}$  isomorphic with the  $C^*$ -algebra of  $2 \times 2$  complex matrices, we construct a family of quantum Markov semigroups on  $\mathcal{A}$  which leave  $\omega$  invariant. By analyzing their generators, we decompose the algebra  $\mathcal{A}$  into four eigenspaces of the semigroups and show some properties.

### 1. Introduction

Let  $(\mathcal{M}, \omega)$  be a quantum system, where  $\mathcal{M}$  is a  $C^*$ -algebra (or von Neumann algebra) with identity  $\mathbf{1}$  and  $\omega$  a state on  $\mathcal{M}$ . A semigroup  $\{S_t\}_{t \geq 0}$  on  $\mathcal{M}$  is called a *quantum Markov semigroup* (q.M.s.) which leaves  $\omega$  invariant if it satisfies

- (a)  $S_t(\mathbf{1}) = \mathbf{1}$  (identity preserving),
- (b)  $0 \leq A \Rightarrow 0 \leq S_t(A)$  (positivity preserving),
- (c)  $\omega(S_t(A)) = \omega(A)$  ( $\omega$  - invariance),

for all  $A \in \mathcal{M}$  and all  $t \geq 0$ . The conditions (a) and (b) imply that the semigroup has the following properties: for all  $A \in \mathcal{M}$  and all  $t \geq 0$

- (i)  $S_t(A)^* = S_t(A^*)$  (real),
- (ii)  $0 \leq A \leq \mathbf{1} \Rightarrow 0 \leq S_t(A) \leq \mathbf{1}$  (sub-Markov),
- (iii)  $\|S_t(A)\| \leq \|A\|$  (contraction).

Let  $G$  be the generator of the semigroup  $\{S_t\}_{t \geq 0}$ :  $S_t = e^{-tG}$ . If  $\{S_t\}_{t \geq 0}$  satisfies the strongly positivity  $S_t(A^*A) \geq S_t(A)^*S_t(A) \forall A \in \mathcal{M}$ ,  $t \geq 0$ , then it is real and positivity preserving[7, 12]. Hence, to construct a q.M.s.  $\{S_t\}_{t \geq 0}$  on  $\mathcal{M}$  which leave  $\omega$  invariant, it is sufficient that we find the generator  $G$  satisfying for all  $A \in D(G)$

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- (a)  $\mathbf{1} \in D(G)$  and  $G(\mathbf{1}) = 0$ ,
- (b)  $G(A)^* = G(A^*)$ ,
- (c)  $G(A^*A) \leq G(A^*)A + A^*G(A)$ ,
- (d)  $\omega(G(A)) = 0$ .

The purpose of this paper is to find the explicit expressions of the generators of Markov semigroups on the CAR algebra  $\mathcal{A}$  isomorphic with the  $C^*$ -algebra of  $2 \times 2$  complex matrices, which leave a given (fixed) gauge invariant state  $\omega$  on  $\mathcal{A}$  invariant (Theorem 2.1). By analyzing the generators, we decompose the algebra  $\mathcal{A}$  into four eigenspaces of the semigroups and show some properties (Theorem 2.2), and give a quantum dynamical system of the single two-level atom. In particular case, this is one of the type introduced by G. Alli and E. L. Sewell[1].

The need to construct Markov semigroups on quantum systems with a non-tracial state is clear for various applications to open systems[8], quantum statistical mechanics[6], and quantum probability theory[15]. Trial of construction of symmetric Markov semigroups with respect to arbitrary KMS states of quantum models was by Zegalinski-Majewski[12, 13] under assumptions of asymptotic abelian property. The theory of Dirichlet forms and Markov semigroups on von Neumann algebra with non-tracial state was developed recently by Cipriani[7], and in [14], the author gave a general construction method of Dirichlet forms on standard forms of von Neumann algebras, and the method developed in [14] has been extended to construct symmetric Markov semigroups on the CCR and CAR algebras with respect to quasi-free states[3, 4] and on quantum mechanical systems[2]. Quantum Ornstein-Uhlenbeck semigroups[10] were constructed by means of noncommutative Dirichlet forms. In this paper, we directly construct a family of quantum Markov semigroups on CAR algebras over one dimensional Hilbert space correspondence to those on the CCR algebras[10] without using the theory of Dirichlet forms.

We would like to mention that the family of semigroups obtained in this paper may be constructed by means of noncommutative Dirichlet forms[7] and the method similar to that used in [4, 10].

The paper is organized as follows: In Section 2, we give explicit expressions of Markov semigroups and state the main results. Section 3 is devoted to the proof of Theorem 2.1 and give a concrete example.

## 2. Markov semigroups and main results

Let  $\mathcal{A}$  be the  $C^*$ -algebra generated by the identity  $\mathbf{1}$  and element  $a$  satisfying the canonical anti-commutation relations(CARs):  $a^2 = 0$  and  $\{a, a^*\} := aa^* + a^*a = \mathbf{1}$ . The algebra  $\mathcal{A}$  is a linear span of four elements  $\mathbf{1}, a, a^*$  and  $aa^*$  and also is isomorphic with the  $C^*$ -algebra of  $2 \times 2$  complex matrices[6]. It is the simplest CAR algebra. For matrix representations of  $a$  and  $a^*$ , see Example 3.1 at the end of Section 3.

Define two elements  $b_+, b_-$  by

$$(2.1) \quad b_+ := 2^{-1/2}(a^* + a), \quad b_- := i2^{-1/2}(a^* - a).$$

We can write

$$a = 2^{-1/2}(b_+ + ib_-), \quad a^* = 2^{-1/2}(b_+ - ib_-).$$

From the CARs, four elements  $\mathbf{1}, b_+, b_-$  and  $b_+b_-$  also is a basis for  $\mathcal{A}$  and

$$(2.2) \quad b_+^2 = b_-^2 = 2^{-1}\mathbf{1}, \quad b_+b_- + b_-b_+ = 0.$$

Let a  $\mathbb{Z}_2$ -grading  $\gamma : \mathcal{A} \rightarrow \mathcal{A}$  be the  $*$ -automorphism defined by

$$(2.3) \quad \gamma(a) = -a, \quad \gamma(a^*) = -a^*.$$

Then  $\gamma$  is involutive :  $\gamma^2 = \text{id}$ .  $(\mathcal{A}, \gamma)$  is a  $\mathbb{Z}_2$ -graded  $C^*$ -algebra.  $\mathcal{A}$  is a direct sum of  $\mathcal{A}_e := \{A \in \mathcal{A} : \gamma(A) = A\}$  and  $\mathcal{A}_o := \{A \in \mathcal{A} : \gamma(A) = -A\}$ . The elements of  $\mathcal{A}_e$  are called *even* and those of  $\mathcal{A}_o$  *odd*. In particular,  $b_{\pm} \in \mathcal{A}_o$  and  $\mathbf{1} \in \mathcal{A}_e$ . A *superderivation*  $\delta$  on  $(\mathcal{A}, \gamma)$  is a linear map satisfying for all  $A, B \in \mathcal{A}$ ,

$$(a) \delta(AB) = \delta(A)B + \gamma(A)\delta(B),$$

$$(b) \delta(\gamma(A)) = -\gamma(\delta(A)).$$

A superderivation  $\delta$  is called inner if there is  $x \in \mathcal{A}$  such that

$$(2.4) \quad \delta(A) = xA - \gamma(A)x, \quad \forall A \in \mathcal{A}.$$

Such superderivation will be denoted by  $\delta_x$ . The element  $x$  which defines an inner superderivation as in (2.4) is odd. See Lemma 1.1 of [9].

Let  $\omega$  be a gauge invariant state on  $\mathcal{A}$  defined by

$$(2.5) \quad \omega(a) = 0, \quad \omega(a^*a) = e^{-\beta}(1 + e^{-\beta})^{-1},$$

where  $\beta > 0$ . Let  $\sigma_t : \mathcal{A} \rightarrow \mathcal{A}$  be the  $*$ -automorphisms defined by

$$(2.6) \quad \sigma_t(a) = e^{i\beta t}a$$

for any  $t \in \mathbb{R}$ . Then one can check that  $\omega$  satisfies the KMS condition[6].

Next let us describe the explicit forms of the generators of Markov semigroups on  $\mathcal{A}$  which leave  $\omega$  invariant. In [10], by modifying the form of generators made in [5](see the equations (1.5) and (1.6) of [10]), a family of symmetric Markov semigroups on the natural standard form associated to the CCR algebras over one dimensional Hilbert space was constructed. In this paper we work on simple CAR algebras by more or less similar method used in [10] and the notion of superderivation[9].

Let  $\mathcal{N}$  be a fixed von Neumann algebra with  $\mathbb{Z}_2$ -graded involution  $\gamma$  and faith, normal semi-finite trace. Assume that the trace is  $\gamma$ -invariant and  $\gamma$  satisfies  $*$ -automorphism. For  $p \in [0, \infty]$ , let  $L^p$  denote the corresponding non-commutative  $L^p$ -space in the sense of I. E. Segal[16]. If a  $*$ -superderivation  $\delta$  is a  $L^2$ -norm bounded, then  $\delta = \delta_x$  for some  $x \in L^\infty$  satisfying  $x = x^*$ ,  $\gamma(x) = -x$ . The adjoint  $\delta_x^*$  is given by for  $A \in L^2$ ,  $\delta_x^*(A) = xA + \gamma(A)x$  and

$$\delta_x^*(\delta_x(A)) = x^2A - 2x\gamma(A)x + Ax^2.$$

Moreover  $\delta_x^*\delta_x$  is a generator of a completely positive, symmetric Markov semigroup leaving any tracial state invariant. See Theorem 3.5 of [9] and Theorem 4.1 of [9].

Now, consider for  $x^* = x \in \mathcal{A}_o$ , let  $G^x$  be an operator on  $\mathcal{A}$  defined by

$$(2.7) \quad G^x(A) := \frac{1}{2}G_b(A) - i(x\delta_b(A) - \gamma(\delta_b(A))x) + \frac{1}{2}G_x(A), \quad A \in \mathcal{A},$$

where  $G_c(A) := c^2A - 2c\gamma(A)c + Ac^2$ ,  $c \in \mathcal{A}_o$ ,  $A \in \mathcal{A}$ . See (1.5) of [10]. Let  $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$  be real parameters. In (2.7), we choose  $b = b_-$  and  $x = \alpha_1b_+$ , and let

$$G_1(A) := \frac{1}{2}G_{b_-}(A) - i\alpha_1(b_+\delta_{b_-}(A) - \gamma(\delta_{b_-}(A))b_+) + \frac{1}{2}\alpha_1^2G_{b_+}(A).$$

Note that  $\delta_{\alpha_3b_+}(A) = \alpha_3\delta_{b_+}(A)$ . Next, choose  $b = \alpha_3b_+$  and  $x = \alpha_2b_-$ , and let

$$G_2(A) := \frac{1}{2}\alpha_3^2G_{b_+}(A) - i\alpha_2\alpha_3(b_-\delta_{b_+}(A) - \gamma(\delta_{b_+}(A))b_-) + \frac{1}{2}\alpha_2^2G_{b_-}(A).$$

Let  $G$  be the bounded operator on  $\mathcal{A}$  given by

$$(2.8) \quad \begin{aligned} G(A) &:= \mu\{G_1(A) + G_2(A)\} \\ &= \frac{\mu}{2}(1 + \alpha_2^2)G_{b_-}(A) + \frac{\mu}{2}(\alpha_1^2 + \alpha_3^2)G_{b_+}(A) \\ &\quad - i\mu\alpha_1(b_+\delta_{b_-}(A) - \gamma(\delta_{b_-}(A))b_+) \\ &\quad - i\mu\alpha_2\alpha_3(b_-\delta_{b_+}(A) - \gamma(\delta_{b_+}(A))b_-), \end{aligned}$$

where  $\mu$  is the normalized constant which will be chosen as  $\mu = (1 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2)^{-1}$ . Substituting the definitions of  $G_{b_{\pm}}$  and  $\delta_{b_{\pm}}$  into (2.8), and using  $\gamma(b_{\pm}) = -b_{\pm}$ ,  $\gamma^2 = \text{id}$  and (2.2) we rewrite

$$\begin{aligned}
 (2.9) \quad G(A) &= \frac{\mu}{2}(1 + \alpha_2^2)(b_-^2 A - 2b_- \gamma(A)b_- + Ab_-^2) \\
 &\quad + \frac{\mu}{2}(\alpha_1^2 + \alpha_3^2)(b_+^2 A - 2b_+ \gamma(A)b_+ + Ab_+^2) \\
 &\quad - i\mu\alpha_1(b_+ b_- A - b_+ \gamma(A)b_- + b_- \gamma(A)b_+ - Ab_- b_+) \\
 &\quad - i\mu\alpha_2\alpha_3(b_- b_+ A - b_- \gamma(A)b_+ + b_+ \gamma(A)b_- - Ab_+ b_-) \\
 &= \frac{\mu}{2}(1 + \alpha_2^2)(A - 2b_- \gamma(A)b_-) + \frac{\mu}{2}(\alpha_1^2 + \alpha_3^2)(A - 2b_+ \gamma(A)b_+) \\
 &\quad + i\mu(\alpha_2\alpha_3 - \alpha_1)(b_+ b_- A - b_+ \gamma(A)b_- + b_- \gamma(A)b_+ - Ab_- b_+).
 \end{aligned}$$

We state the main results.

**THEOREM 2.1.** *Let the parameters  $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$  satisfy the following relations*

$$\begin{aligned}
 (2.10) \quad &\frac{1}{2}(1 + \alpha_2^2)\sinh(\beta/2) = -\alpha_1\cosh(\beta/2), \\
 &\frac{1}{2}(\alpha_1^2 + \alpha_3^2)\sinh(\beta/2) = \alpha_2\alpha_3\cosh(\beta/2).
 \end{aligned}$$

Then the actions of  $G$  defined as in (2.8) and (2.9) and the semigroup  $\{S_t\}_{t \geq 0}$ ,  $S_t = e^{-tG}$  on  $\mathcal{A}$  are given as follows: for any  $A = z_1\mathbf{1} + z_2b_+ + z_3b_- + z_4b_+b_-$ ,  $z_i \in \mathbb{C}$ ,  $i = 1, 2, 3, 4$ ,

$$\begin{aligned}
 (2.11) \quad G(A) &= z_2\mu(\alpha_1^2 + \alpha_3^2)b_+ + z_3\mu(1 + \alpha_2^2)b_- \\
 &\quad + z_4(b_+b_- + i\mu(\alpha_1 - \alpha_2\alpha_3)\mathbf{1}) \\
 &= -iz_42^{-1}\tanh(\beta/2)\mathbf{1} + z_2(1 + \theta)^{-1}b_+ \\
 &\quad + z_3\theta(1 + \theta)^{-1}b_- + z_4b_+b_-, \\
 S_t(A) &= (z_1 + iz_42^{-1}\tanh(\beta/2)(1 - e^{-t}))\mathbf{1} \\
 &\quad + z_2e^{-(1+\theta)^{-1}t}b_+ + z_3e^{-\theta(1+\theta)^{-1}t}b_- + z_4e^{-t}b_+b_-,
 \end{aligned}$$

where  $\theta = -\alpha_1/(\alpha_2\alpha_3)$ . Moreover, the semigroup  $\{S_t\}_{t \geq 0}$  is a quantum Markov semigroup on  $\mathcal{A}$  which leaves  $\omega$  invariant.

A semigroup  $T = \{T_t\}_{t \geq 0}$  on  $\mathcal{A}$  is said to be *ergodic* if

$$T_t A = A, \forall A \in \mathcal{A}, \forall t \geq 0 \implies A = \alpha\mathbf{1} \text{ for some } \alpha \in \mathbb{C}.$$

A real number  $\lambda$  will be called an eigenvalue of the semigroup  $\{T_t\}_{t \geq 0}$  if there is a nonzero element  $A \in \mathcal{A}$  such that  $T_t A = e^{-t\lambda} A, \forall t \geq 0$ . Each such  $A$  is then called an eigenvector corresponding to the eigenvalue  $\lambda$ . By the (point) spectrum of  $T$ , denoted by  $\sigma(T)$ , we mean the set of all eigenvalue of  $T_t$ . To each eigenvalue  $\lambda$  we associate the eigenspace  $\mathcal{A}_\lambda = \{A \in \mathcal{A} : T_t A = e^{-t\lambda} A, \forall t \geq 0\}$  ([11]).

**THEOREM 2.2.** (a) *The algebra  $\mathcal{A}$  is the direct sum of four eigenspaces of  $S = \{S_t\}_{t \geq 0}$  in Theorem 2.1:*

$$\begin{aligned} \mathcal{A} &= \mathcal{A}_0 \oplus \mathcal{A}_{(1+\theta)^{-1}} \oplus \mathcal{A}_{\theta(1+\theta)^{-1}} \oplus \mathcal{A}_1, \\ \mathcal{A}_0 &= \mathbb{C}\mathbf{1}, \quad \mathcal{A}_{(1+\theta)^{-1}} = \mathbb{C}b_+, \\ \mathcal{A}_{\theta(1+\theta)^{-1}} &= \mathbb{C}b_-, \quad \mathcal{A}_1 = \mathbb{C}(-i2^{-1}\tanh(\beta/2)\mathbf{1} + b_+b_-). \end{aligned}$$

Moreover,  $\sigma(S) = \{0, (1 + \theta)^{-1}, \theta(1 + \theta)^{-1}, 1\}$ , and the semigroup  $\{S_t\}_{t \geq 0}$  is ergodic.

(b) *The semigroup  $\{S_t\}_{t \geq 0}$  commutes with the  $\mathbb{Z}_2$  grading  $\gamma$ : i.e.,  $S_t(\gamma(A)) = \gamma(S_t(A)) \forall A \in \mathcal{A}, t \geq 0$ , in particular,  $S_t(\mathcal{A}_o) \subset \mathcal{A}_o$  and  $S_t(\mathcal{A}_e) \subset \mathcal{A}_e$ . But except for  $\theta = 1$  it does not commute with the modular automorphisms  $\sigma_t$ .*

**PROOF.** Notice that four elements  $\mathbf{1}, b_+, b_-$  and  $b_+b_-$  is a basis for  $\mathcal{A}$ . The proof of (a) and the first part of (b) directly follows from (2.11) in Theorem 2.1.

Let us prove the other part of (b). By (2.1) and (2.6), we have

$$(2.12) \quad \begin{aligned} \sigma_t(b_\pm) &= \cos(\beta t)b_\pm \mp \sin(\beta t)b_\mp, \\ \sigma_t(b_+b_-) &= b_+b_- \end{aligned}$$

for any  $t \in \mathbb{R}$ . Using (2.11) and (2.12), a tedious but straightforward computation yields the followings: for any  $A = z_1\mathbf{1} + z_2b_+ + z_3b_- + z_4b_+b_-$ ,  $z_i \in \mathbb{C}, i = 1, 2, 3, 4, \forall t, t' \geq 0$

$$\begin{aligned} S_{t'}(\sigma_t(A)) - \sigma_t(S_{t'}(A)) &= \sin(\beta t)(z_3b_+ + z_2b_-) \left( e^{-t'\theta(1+\theta)^{-1}} - e^{-t'(1+\theta)^{-1}} \right), \end{aligned}$$

which implies that the proof is completed. □

**REMARK 2.3.** In (2.5), consider  $\beta = 0$ . In this case the state  $\omega$  is tracial. It follows from (2.10) that  $\alpha_1 = \alpha_3 = 0$  or  $\alpha_1 = \alpha_2 = 0$ , which implies

$$G(A) = \frac{1}{2}G_{b_-}(A), \quad A \in \mathcal{A}$$

or

$$G(A) = \frac{\mu}{2}G_{b_-}(A) + \frac{\mu}{2}\alpha_3^2G_{b_+}(A), \quad A \in \mathcal{A},$$

respectively. They are also the generators of quantum Markov semigroups. See the proof of Theorem 4.1 of [9]. And in case  $\beta < 0$ , by the similar method, we can also get a family of Markov semigroups.

Before closing this section, we show that distinct values of the parameters  $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$  satisfying (2.10) give distinct semigroups on  $\mathcal{A}$ . Since the range of  $\theta$  is a closed interval containing 1 with non-zero length, there exist infinitely many different semigroups. We first consider two special cases and then describe general case. See also Section 3 in [10].

EXAMPLE 2.1. Choose  $\alpha_3 = 1$  and  $-\alpha_1 = \alpha_2$ . It follows from (2.10) that  $\theta = 1$  and

$$(2.13) \quad \alpha_{2,\pm}(\beta) := \alpha_2 = (\cosh(\beta/2) \pm 1) / \sinh(\beta/2).$$

Consider  $\alpha_2 = 1$ . It follows from (2.10) that

$$\begin{aligned} \alpha_1 &= -\tanh(\beta/2), \\ \alpha_3 &= \coth(\beta/2) \pm (\coth^2(\beta/2) - \tanh^2(\beta/2))^{1/2} \\ &= \tanh^2(\beta/2) / [\coth(\beta/2) \mp (\coth^2(\beta/2) - \tanh^2(\beta/2))^{1/2}], \\ (2.14) \quad \theta &= \coth(\beta/2) [\coth(\beta/2) \mp (\coth^2(\beta/2) - \tanh^2(\beta/2))^{1/2}] \\ &\neq 1. \end{aligned}$$

Next, we consider the general case. Consider  $\alpha_2$  as a parameter. Then we get from (2.10) that

$$(2.15) \quad \alpha_1 = -\frac{1}{2}(1 + \alpha_2^2) \tanh(\beta/2),$$

$$\alpha_3 = [\alpha_2 \cosh(\beta/2) \pm (\alpha_2^2 \cosh^2(\beta/2) - \alpha_1^2 \sinh^2(\beta/2))^{1/2}] / \sinh(\beta/2).$$

For real  $\alpha_3$ , one requires that  $\alpha_2^2 \cosh^2(\beta/2) \geq \alpha_1^2 \sinh^2(\beta/2)$ . By the first equation in (2.15), positive solutions  $\alpha_2$  exist if and only if

$$2\alpha_2 \cosh^2(\beta/2) \geq (1 + \alpha_2^2) \sinh^2(\beta/2).$$

Denote

$$\begin{aligned} \kappa_{\pm}(\beta) &:= [\cosh^2(\beta/2) \pm (\cosh^4(\beta/2) - \sinh^4(\beta/2))^{1/2}] / \sinh^2(\beta/2), \\ I_{\beta} &:= [\kappa_-(\beta), \kappa_+(\beta)]. \end{aligned}$$

Thus  $\alpha_2 \in I_\beta$ . Note that

$$\begin{aligned} \kappa_+(\beta) &= [\cosh^2(\beta/2) + (\cosh^2(\beta/2) + \sinh^2(\beta/2))^{1/2}] / \sinh^2(\beta/2), \\ &\geq [\cosh^2(\beta/2) + \cosh(\beta/2)] / \sinh^2(\beta/2), \\ &= \coth(\beta/2)\alpha_{2,+}(\beta), \end{aligned}$$

where  $\alpha_{2,+}(\beta)$  has been defined in (2.13). The above implies that  $\kappa_+(\beta) > \alpha_{2,+}(\beta)$ . An argument similar to that used above yields that  $\kappa_-(\beta) < \alpha_{2,-}(\beta)$ . Thus we conclude that  $1 \in [\alpha_{2,-}(\beta), \alpha_{2,+}(\beta)] \subset I_\beta$ .

Define

$$\begin{aligned} \theta_\pm(\alpha_2) &:= -\alpha_1(\alpha_2) / \alpha_2 \alpha_{3,\pm}(\alpha_2), \\ \alpha_{3,\pm}(\alpha_2) &= \alpha_2 \cosh(\beta/2) / \sinh(\beta/2) \\ &\quad \pm (\alpha_2^2 \cosh^2(\beta/2) - \alpha_1^2 \sinh^2(\beta/2))^{1/2} / \sinh(\beta/2) \\ &= \alpha_2 \coth(\beta/2) \pm (\alpha_2^2 \coth^2(\beta/2) - \frac{1}{4}(1 + \alpha_2^2)^2 \tanh^2(\beta))^{1/2}. \end{aligned}$$

Since  $\theta_\pm$  are continuous on  $I_\beta \subset (0, \infty)$ ,  $\theta_+(I_\beta)$  and  $\theta_-(I_\beta)$  are closed intervals which have common point at  $\alpha_2 = \kappa_-(\beta)$  and  $\alpha_2 = \kappa_+(\beta)$ . Thus  $\theta_+(I_\beta) \cup \theta_-(I_\beta)$  is a closed interval containing 1 with non-zero length (by (2.14)). Therefore there exist infinitely many different semigroups.  $\square$

### 3. Proof of Theorem 2.1

In this section we produce the proof of Theorem 2.1 in Section 2 and give a concrete example. In the rest of this paper we denote by  $b$  either  $b_+$  or  $b_-$ .

By (2.6), one can check that for any  $z \in \mathbb{C}$ ,

$$(3.1) \quad \sigma_z(a) = e^{i\beta z} a, \quad \sigma_z(a^*) = e^{-i\beta z} a^*,$$

and so

$$\sigma_z(b_+) = 2^{-1/2}(e^{-i\beta z} a^* + e^{i\beta z} a), \quad \sigma_z(b_-) = i2^{-1/2}(e^{-i\beta z} a^* - e^{i\beta z} a).$$

Let  $a^\#$  denote either  $a$  or  $a^*$ . In fact, one may be able to show that by (2.5) the function  $t \mapsto \omega(\sigma_t((a^\#)^* a^\#))$  has an analytic extension on  $\mathbb{C}$ , which is denoted by  $\sigma_z(a^\#)$ , and that  $\sigma_z(a^\#)$  is equal to that of (3.1). Note that by (3.1)

$$(3.2) \quad \sigma_{\pm i/4}(a^*) = e^{\pm\beta/4} a^*, \quad \sigma_{\pm i/4}(a) = e^{\mp\beta/4} a.$$



PROOF OF THEOREM 2.1. Recall  $\theta = -\alpha_1/(\alpha_2\alpha_3)$ . By (2.10) we have

$$\begin{aligned} \mu &= (1 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2)^{-1} = (2(\alpha_2\alpha_3 - \alpha_1))^{-1}\tanh(\beta/2), \\ \mu(\alpha_1^2 + \alpha_3^2) &= (1 + (\alpha_1^2 + \alpha_3^2)^{-1}(1 + \alpha_2^2))^{-1} \\ &= (1 - \alpha_1/(\alpha_2\alpha_3))^{-1} = (1 + \theta)^{-1}. \end{aligned}$$

Notice that the set of four elements  $\mathbf{1}, b_+, b_-$  and  $b_+b_-$  is a basis for  $\mathcal{A}$ . It follows from (2.2), (2.8) and (2.9) that we can directly calculate the actions of generator  $G$  and semigroup  $S_t$  on  $\mathcal{A}$  in the first part of Theorem 2.1.

To show that  $\{S_t\}_{t \geq 0}$  is a Markov semigroup on  $\mathcal{A}$  which leaves  $\omega$  invariant, we will check that the generator  $G$  satisfies the conditions (1). By (2.11), it is easily checked the conditions (a) and (b).

Next, we will yield that  $G$  satisfies the third condition. We have that

$$\begin{aligned} b_- &= \sigma_{-i/4}(\sigma_{i/4}(b_-)) \\ &= i2^{-1/2}\sigma_{-i/4}(e^{\beta/4}a^* - e^{-\beta/4}a) \\ (3.3) \quad &= i2^{-1}e^{\beta/4}(\sigma_{-i/4}(b_+) - i\sigma_{-i/4}(b_-)) \\ &\quad - i2^{-1}e^{-\beta/4}(\sigma_{-i/4}(b_+) + i\sigma_{-i/4}(b_-)) \\ &= i \sinh(\beta/4)\sigma_{-i/4}(b_+) + \cosh(\beta/4)\sigma_{-i/4}(b_-). \end{aligned}$$

By the method used above, we also have that

$$(3.4) \quad b_+ = \cosh(\beta/4)\sigma_{-i/4}(b_+) - i \sinh(\beta/4)\sigma_{-i/4}(b_-).$$

Let  $A \in \mathcal{A}$ . It follows from (3.3) and (3.4) that

$$\begin{aligned} (3.5) \quad \delta_{b_-}(A) &= \cosh(\beta/4)\delta_{\sigma_{-i/4}(b_-)}(A) + i \sinh(\beta/4)\delta_{\sigma_{-i/4}(b_+)}(A), \\ \delta_{b_+}(A) &= \cosh(\beta/4)\delta_{\sigma_{-i/4}(b_+)}(A) - i \sinh(\beta/4)\delta_{\sigma_{-i/4}(b_-)}(A). \end{aligned}$$

By the simple calculations, we obtain two relations

$$\begin{aligned} (3.6) \quad G_b(A^*A) &= b^2A^*A - 2b\gamma(A^*)\gamma(A)b + A^*Ab^2 \\ &= G_b(A^*)A + A^*G_b(A) \\ &\quad + 2b\gamma(A^*)(bA - \gamma(A)b) - 2A^*b(bA - \gamma(A)b) \\ &= G_b(A^*)A + A^*G_b(A) - 2(\delta_b(A))^*\delta_b(A) \end{aligned}$$

and

$$\begin{aligned}
 & b_+ \delta_{b_-} (A^* A) - \gamma(\delta_{b_-} (A^* A)) b_+ \\
 &= b_+ (b_- A^* A - \gamma(A^*) \gamma(A) b_-) - \gamma(b_- A^* A - \gamma(A^*) \gamma(A) b_-) b_+ \\
 &= b_+ b_- A^* A - b_+ \gamma(A^*) \gamma(A) b_- + b_- \gamma(A^*) \gamma(A) b_+ - A^* A b_- b_+ \\
 (3.7) \quad &= (b_+ \delta_{b_-} (A^*) - \gamma(\delta_{b_-} (A^*)) b_+) A + A^* (b_+ \delta_{b_-} (A) - \gamma(\delta_{b_-} (A)) b_+) \\
 &\quad + b_+ \gamma(A^*) (b_- A - \gamma(A) b_-) - b_- \gamma(A^*) (b_+ A - \gamma(A) b_+) \\
 &\quad + A^* b_- (b_+ A - \gamma(A) b_+) - A^* b_+ (b_- A - \gamma(A) b_-) \\
 &= (b_+ \delta_{b_-} (A^*) - \gamma(\delta_{b_-} (A^*)) b_+) A + A^* (b_+ \delta_{b_-} (A) - \gamma(\delta_{b_-} (A)) b_+) \\
 &\quad - (\delta_{b_+} (A))^* \delta_{b_-} (A) + (\delta_{b_-} (A))^* \delta_{b_+} (A).
 \end{aligned}$$

Here we have used  $\gamma(b) = -b$  and  $\gamma^2 = \text{id}$ . By the method used above we also get that

$$\begin{aligned}
 b_- \delta_{b_+} (A^* A) - \gamma(\delta_{b_+} (A^* A)) b_- &= (b_- \delta_{b_+} (A^*) - \gamma(\delta_{b_+} (A^*)) b_-) A \\
 &\quad + A^* (b_- \delta_{b_+} (A) - \gamma(\delta_{b_+} (A)) b_-) \\
 &\quad - (\delta_{b_-} (A))^* \delta_{b_+} (A) + (\delta_{b_+} (A))^* \delta_{b_-} (A).
 \end{aligned}$$

Substituting (3.6), (3.7), and (3.8) into (2.8), one gets

$$G(A^* A) = G(A^*) A + A^* G(A) + R(A),$$

where

$$\begin{aligned}
 (3.8) \quad R(A) &= -\mu(1 + \alpha_2^2)(\delta_{b_-} (A))^* \delta_{b_-} (A) - \mu(\alpha_1^2 + \alpha_3^2)(\delta_{b_+} (A))^* \delta_{b_+} (A) \\
 &\quad + i\mu\alpha_1(\delta_{b_+} (A))^* \delta_{b_-} (A) - i\mu\alpha_1(\delta_{b_-} (A))^* \delta_{b_+} (A) \\
 &\quad - i\mu\alpha_2\alpha_3(\delta_{b_+} (A))^* \delta_{b_-} (A) + i\mu\alpha_2\alpha_3(\delta_{b_-} (A))^* \delta_{b_+} (A).
 \end{aligned}$$

We will show  $R(A) \leq 0$ . We substitute (3.5) into (3.8) to obtain that

$$\begin{aligned}
 (3.9) \quad R(A) &= -\mu\phi_1(\alpha_1, \alpha_2, \alpha_3)(\delta_{\sigma_{-i/4}(b_-)}(A))^* \delta_{\sigma_{-i/4}(b_-)}(A) \\
 &\quad - \mu\phi_2(\alpha_1, \alpha_2, \alpha_3)(\delta_{\sigma_{-i/4}(b_+)}(A))^* \delta_{\sigma_{-i/4}(b_+)}(A) \\
 &\quad - \mu\phi_3(\alpha_1, \alpha_2, \alpha_3)(\delta_{\sigma_{-i/4}(b_-)}(A))^* \delta_{\sigma_{-i/4}(b_+)}(A) \\
 &\quad - \mu\phi_4(\alpha_1, \alpha_2, \alpha_3)(\delta_{\sigma_{-i/4}(b_+)}(A))^* \delta_{\sigma_{-i/4}(b_-)}(A),
 \end{aligned}$$

where

$$\begin{aligned}
 & \phi_1(\alpha_1, \alpha_2, \alpha_3) \\
 &= (1 + \alpha_2^2)\cosh^2(\beta/4) + (\alpha_1^2 + \alpha_3^2)\sinh^2(\beta/4) + (\alpha_1 - \alpha_2\alpha_3)\sinh(\beta/2) \\
 &= (\cosh(\beta/4) + \alpha_1\sinh(\beta/4))^2 + (\alpha_2\cosh(\beta/4) - \alpha_3\sinh(\beta/4))^2, \\
 & \phi_2(\alpha_1, \alpha_2, \alpha_3) \\
 &= (1 + \alpha_2^2)\sinh^2(\beta/4) + (\alpha_1^2 + \alpha_3^2)\cosh^2(\beta/4) + (\alpha_1 - \alpha_2\alpha_3)\sinh(\beta/2) \\
 &= (\sinh(\beta/4) + \alpha_1\cosh(\beta/4))^2 + (\alpha_2\sinh(\beta/4) - \alpha_3\cosh(\beta/4))^2, \\
 & \phi_3(\alpha_1, \alpha_2, \alpha_3) = -\phi_4(\alpha_1, \alpha_2, \alpha_3) \\
 &= i \left[ \frac{1}{2}(1 + \alpha_2^2)\sinh(\beta/2) + \alpha_1\cosh(\beta/2) \right] \\
 & \quad + i \left[ \frac{1}{2}(\alpha_1^2 + \alpha_3^2)\sinh(\beta/2) - \alpha_2\alpha_3\cosh(\beta/2) \right].
 \end{aligned}$$

We have used that  $2 \sinh(\beta/4) \cosh(\beta/4) = \sinh(\beta/2)$  and  $\cosh^2(\beta/4) + \sinh^2(\beta/4) = \cosh(\beta/2)$ . By (2.10) and the above relations, we conclude  $\phi_i(\alpha_1, \alpha_2, \alpha_3) \geq 0, i = 1, 2$  and  $\phi_i(\alpha_1, \alpha_2, \alpha_3) = 0, i = 3, 4$ , which implies  $R(A) \leq 0$ .

Let us to show the final condition (d) in (1). For any  $A \in \mathcal{A}$ , we can write  $A = z_1\mathbf{1} + z_2b_+ + z_3b_- + z_4b_+b_-$ ,  $z_i \in \mathbb{C}, i = 1, 2, 3, 4$ . By (2.1), we have  $b_+b_- = i(2^{-1}\mathbf{1} - a^*a)$ . It follows from (2.5) and (2.11) that

$$\begin{aligned}
 \omega(G(A)) &= \omega(-iz_42^{-1}\tanh(\beta/2)\mathbf{1} + z_2(1 + \theta)^{-1}b_+ \\
 & \quad + z_3\theta(1 + \theta)^{-1}b_- + z_4b_+b_-) \\
 &= -iz_42^{-1}\tanh(\beta/2) + z_4\omega(b_+b_-), \\
 &= iz_42^{-1}\tanh(\beta/2) + iz_4(2^{-1} - e^{-\beta}(1 + e^{-\beta})^{-1}) = 0.
 \end{aligned}$$

The proof is completed. □

**EXAMPLE 3.1.** Consider a quantum dynamical system of the single two-level atom. Let  $\mathcal{A}$  be the  $C^*$ -algebra  $\mathcal{M}_2$  of  $2 \times 2$  matrices with complex entries. Then it is the linear span of the Pauli matrices  $(\sigma_x, \sigma_y, \sigma_z)$  and identity  $I$ :

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We define the two matrices

$$a = 2^{-1}(\sigma_x + i\sigma_y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a^* = 2^{-1}(\sigma_x - i\sigma_y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

called the annihilation (spin raising) and creation (spin raising) operators respectively, and  $b_+ = 2^{-1/2}\sigma_x$ ,  $b_- = 2^{-1/2}\sigma_y$ . Two matrices  $a, a^*$  satisfy the anti-commutation relations:  $a^2 = 0$ ,  $\{a, a^*\} = I$  and  $\mathcal{A}$  is a  $C^*$  algebra generated by  $I$  and  $a$ .  $\mathcal{A}$  is an example of the simplest CAR algebra.

Let  $\omega_\rho$  be the faithful normal state on  $\mathcal{A}$  defined by

$$\omega_\rho(A) = \text{Tr}(\rho A), \quad \rho = \begin{pmatrix} \nu & 0 \\ 0 & 1 - \nu \end{pmatrix},$$

where  $A \in \mathcal{A}$ ,  $\nu = (1 + e^{-\beta})^{-1}$ ,  $\beta > 0$ . Then  $\omega_\rho$  is a gauge invariant state satisfying the condition (2.5), and also the  $*$ -(modular) automorphism  $\sigma_t$  associated to  $\omega_\rho$  is given by  $\sigma_t(a) = \rho^{it} a \rho^{-it} = e^{i\beta t} a$ . By Theorem 2.1 we have the family of semigroups  $\{S_t\}_{t \geq 0}$  on  $\mathcal{A}$  leaving  $\omega_\rho$  invariant and their generators  $G$ . The concrete action of  $G$  on  $\mathcal{A}$  is given by for  $z_i \in \mathbb{C}$ ,  $i = 1, 2, 3, 4$ ,

$$A = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} = 2^{-1}(z_1 + z_4)I + (z_2 + z_3)2^{-1/2}b_+ \\ + i(z_2 - z_3)2^{-1/2}b_- + i(z_4 - z_1)b_+b_-, \\ G(A) = \begin{pmatrix} (z_1 - z_4)e^{-\beta}(1 + e^{-\beta})^{-1} & 2^{-1}(z_2(1 + \theta)^{-1}(1 - \theta) + z_3) \\ 2^{-1}(z_2 + z_3(1 + \theta)^{-1}(1 - \theta)) & (z_4 - z_1)(1 + e^{-\beta})^{-1} \end{pmatrix},$$

where  $\theta$  is defined in Theorem 2.1. In case  $\theta = 1$ , this structure is equal to one of the type introduced by G. Alli and E. L. Sewell[1].  $\square$

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