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ON A GENERALIZED DIFFERENCE SEQUENCE SPACES OVER NON-ARCHIMEDIAN FIELDS AND RELATED MATRIX TRANSFORMATIONS

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Abstract. Let F be a non-trivial non-Archimedian field. The sequence spaces $\Gamma(F)$ and $\Gamma^*(F)$ were defined and studied by Somasundaram[4], where these spaces denote the spaces of entire and analytic sequences defined over F, respectively. In 1997, these spaces were generalized by Mursaleen and Qamaruddin[1] by considering an arbitrary sequence $u = (u_k)$, $u_k \neq 0$ ($k = 1, 2, 3, \cdots$). They characterized some classes of infinite matrices considering these new classes of sequences. In this paper, we further generalize the above mentioned spaces and define the spaces $\Gamma(u, F, \Delta), \Gamma^*(u, F, \Delta), l_p(u, F, \Delta)$, and $bv(u, F, \Delta)$. We also study some matrix transformations on these new spaces.

1. Introduction

In 1970, Somasundaram[4] introduced and studied the following sequence spaces defined over non-trivial non-Archimedian field F :

$$\Gamma(F) = \{x = (x_k) \text{ defined over } F : \ |x_k|^{\frac{1}{k}} \to 0 \text{ as } k \to \infty\},$$

$$\Gamma^*(F) = \{x = (x_k) \text{ defined over } F : \ \sup_k |x_k|^{\frac{1}{k}} < \infty\},$$

$$l_p(F) = \{x = (x_k) \text{ defined over } F : \ \sum_k |x_k|^p < \infty, \ p \geq 1\},$$

and

$$bv(F) = \{x = (x_k) \text{ defined over } F : \ \sum_k |x_k - x_{k-1}| < \infty, \ x_0 = 0\}.$$

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These spaces were also studied by Nanda[2] in 1981. It was also noted that $\Gamma(F)$ is a non-Archimedian complete linear metric space with the metric $d(x,y)=|x-y|$, where $|x|=\sup|x_k|^{\frac{1}{k}}$ and $\Gamma^*(F)$ is a conjugate space of $\Gamma(F)$. $l_p(F)$ $(p\geq 1)$ and $bv(F)$ are non-Archimedian Banach spaces with respect to the norms
$\|x\|_p = (\sum_k |x_k|^p)^{\frac{1}{p}}$ and $\|x\|_{bv} = \|x_1\| + \sum_{k=2}^{\infty} |x_k - x_{k-1}|$ respectively.

In 1995, Sirajudeen and Sulaiman[3] considered matrix transformations of $l_p(F)$ and $bv(F)$ into $\Gamma(F)$ and $\Gamma^*(F)$.

If $X$and Y are any two sequence spaces, then (X,Y) denote the class of all infinite matrices $A=(a_{nk})$ $(n,k=1,\,2,\cdots)$ that maps X into Y, i.e. for which the series $A_n(x)=\sum_k a_{nk}\,x_k$ converges for all $x\in X$ and all $n=1,\,2,\cdots$, and such that $A(x)=A_n(x)\in Y$ for all $x\in X$.

For any sequence $x=(x_k)$, the difference sequence Δx is defined by $\Delta x=(\Delta x_k)_{k=1}^{\infty}=(x_k-x_{k+1})_{k=1}^{\infty}$.

In this paper, we define the following difference sequence spaces which are a generalization of those defined and studied by Mursaleen and Qamaruddin[1] :

$$\Gamma(u,F,\Delta)=\left\{x=(x_k) \text{ defined over } F:\ |u_k\Delta x_k|^{\frac{1}{k}}\to 0 \text{ as } k\to\infty\right\},$$

$$\Gamma^*(u,F,\Delta)=\left\{x=(x_k) \text{ defined over } F:\ \sup_k |u_k\Delta x_k|^{\frac{1}{k}}<\infty\right\},$$

$$l_p(u,F,\Delta)=\left\{x=(x_k) \text{ defined over } F:\ \sum_k |u_k\Delta x_k|^{p}<\infty,\ p\geq 1\right\},$$

and

$$bv(u,F,\Delta)=\left\{x=(x_k) \text{ defined over } F:\ \sum_k |u_k\Delta x_k-u_{k-1}\Delta x_{k-1}|<\infty,\ u_0x_0=0\right\}.$$

If $\Delta x_k=x_k$ for all k, then the above spaces reduce to those defined by Mursaleen and Qamaruddin[1].

If $\Delta x_k=x_k$ for all k and $u=e=(1,1,1,\cdots)$, then $\Gamma(u,F,\Delta)=\Gamma(F)$, $\Gamma^*(u,F,\Delta)=\Gamma^*(F)$, $l_p(u,F,\Delta)=l_p(F)$ and $bv(u,F,\Delta)=bv(F)$.

The following results are the trivial extensions of the results due to Sirajudeen and Sulaiman[3] :

THEOREM 1.1. *$l_p(u,F,\Delta)$ $(p \geq 1)$ and $bv(u,F,\Delta)$ are non-Archimedian Banach spaces with respect to the norms $\|x\| = \|u\Delta x\|_p$ and $\|x\| = \|u\Delta x\|_{bv}$ respectively.*

THEOREM 1.2. *If $\sum_k a_k \Delta x_k$ converges for every $x \in l_p(u,F,\Delta)$, then $\sum_k \left|\frac{a_k}{u_k}\right|$ is convergent, where $p > 1$ and $p^{-1} + q^{-1} = 1$.*

2. Main results

In this section, we characterize the infinite matrices which transform $l_p(u,F,\Delta)$ and $bv(u,F,\Delta)$ into $\Gamma(F)$ and $\Gamma^*(F)$. We prove the following theorems :

THEOREM 2.1. *$A \in (l_p(u,F,\Delta), \Gamma(F))$ if and only if*

$$\sup_k \left|\frac{a_{nk}}{u_k}\right| \to 0 \text{ as } n \to \infty, \text{ where } p > 1 \text{ and } p^{-1} + q^{-1} = 1 \tag{2.1}$$

Proof. Sufficiency : Let the condition (2.1) holds and $(x_k) \in l_p(u,F,\Delta)$, so that $\sum_k \mid u_k \Delta x_k \mid^p$ converges to L, say. Then

$$\begin{aligned}
&|y_n|^{\frac{1}{n}} \\
&= \left|\sum_k \frac{a_{nk}}{u_k} u_k \Delta x_k\right|^{\frac{1}{n}} \\
&= \left|\sum_k \frac{a_{nk}}{u_k} z_k\right|^{\frac{1}{n}}, \quad \text{where } z_k = u_k \Delta x_k, \ (x \in l_p(u,F,\Delta) \Leftrightarrow z \in l_p(F,\Delta)) \\
&\leq \left(\sum_k \left|\frac{a_{nk}}{u_k}\right|^q\right)^{1/nq} \left(\sum_k \mid z_k \mid^p\right)^{1/p}, \quad (\text{ by Hölder's inequality }) \\
&\leq \sup_{1 \leq k < \infty} \left(\left|\frac{a_{nk}}{u_k}\right|^q\right)^{1/nq} L^{1/nq} \\
&\leq C^{1/nq} \sup_{1 \leq k < \infty} \left(\left|\frac{a_{nk}}{u_k}\right|^q\right)^{1/nq}, \quad \text{where } C = \max(1, L).
\end{aligned}$$

Hence, we find that

$$|y_n|^{\frac{1}{n}} \to 0 \text{ as } n \to \infty,$$

so $(y_n) \in \Gamma(F)$.

Necessity : Let $A \in (l_p(u, F, \Delta), \Gamma(F))$. If (2.1) does not hold, then for some $\epsilon > 0$,

$$\sup_{1 \le k < \infty} \left| \frac{a_{nk}}{u_k} \right|^{q/n} > \epsilon \text{ for sufficiently large } n. \tag{2.2}$$

Hence $y_n = \sum a_{nk} \Delta x_k$ is defined for all $(x_k) \in l_p(u, F, \Delta)$. By Theorem 1.2, we see that $\sum_k \left| \frac{a_{nk}}{u_k} \right|^q$ is convergent so that $\left| \frac{a_{nk}}{u_k} \right| \to 0$ as $k \to \infty$ for every fixed n. Hence

$$\left| \frac{a_{nk}}{u_k} \right|^{q/n} \to 0 \text{ as } k \to \infty \text{ for every fixed } n. \tag{2.3}$$

Since $e_k \in l_p(u, F, \Delta)$, we get $(y_n) = (\frac{a_{nk}}{u_k}) \in \Gamma(F)$, so that $\left| \frac{a_{nk}}{u_k} \right|^{1/n} \to 0$ as $n \to \infty$ for every fixed k. Hence

$$\left| \frac{a_{nk}}{u_k} \right|^{q/n} < \epsilon \text{ for } n \ge n_k \text{ for every fixed } k. \tag{2.4}$$

Proceeding as in Somasundaram[4] and taking

$$u_k \Delta x_k = \begin{cases} \left| \frac{a_{nk}}{u_k} \right|^{q-1}, & k = k_1, k_2, k_3, \cdots, \\ 0, & k \ne k_1, k_2, k_3, \cdots, \end{cases}$$

$$|u_k \Delta x_k|^p = \left| \frac{a_{nk}}{u_k} \right|^{p(q-1)} = \left| \frac{a_{nk}}{u_k} \right|^q,$$

we see that

$$\sum |u_k \Delta x_k|^p = \sum \left| \frac{a_{nk}}{u_k} \right|^q < \infty.$$

This implies that $x \in l_p(u, F, \Delta)$ and

$$y_n = \sum_k \frac{a_{nk}}{u_k} . u_k \Delta x_k = \sum_k \frac{a_{nk}}{u_k} \left| \frac{a_{nk}}{u_k} \right|^{q-1}.$$

Thus

$$|y_n| = \sum_k \left| \frac{a_{nk}}{u_k} \right| . \left| \frac{a_{nk}}{u_k} \right|^{q-1} = \sum_k \left| \frac{a_{nk}}{u_k} \right|^q.$$

Hence

$$|y_n|^{1/n} = \sum_k \left| \frac{a_{nk}}{u_k} \right|^{q/n}$$

By using (2.2), (2.3), and (2.4), we can see that $|y_n|^{1/n} \nrightarrow 0$ as $n \to \infty$, which contradicts that $A \in (l_p(u, F, \Delta), \Gamma(F))$. Hence (2.1) must hold. This completes the proof. □

THEOREM 2.2. *$A \in (l_p(u, F, \Delta), \Gamma^*(F))$ if and only if*

$$\sup_{1 \le k < \infty, 1 \le n < \infty} \left| \frac{a_{nk}}{u_k} \right|^{q/n} \le M,$$

for some constant M, and $p > 1$.

This can be proved in a similar way as in Theorem 2.1.

For the next results we will use the following lemmas (see Somasundaram[5]):

LEMMA A. *$A \in (l(F, \Delta), \Gamma(F))$ if and only if*

$$\sup_{1 \le k < \infty} |a_{nk}|^{1/n} \to 0 \text{ as } n \to \infty.$$

LEMMA B. *$A \in (l(F, \Delta), \Gamma^*(F))$ if and only if*

$$\sup_{1 \le k < \infty, 1 \le n < \infty} |a_{nk}|^{1/n} < M \text{ for some constant } M.$$

THEOREM 2.3. *$A \in (bv(u, F, \Delta), \Gamma(F))$ if and only if*

$$\sup_{1 \le j < \infty} \left| \sum_{k=j}^{\infty} \frac{a_{nk}}{u_k} \right|^{1/n} \to 0 \text{ as } n \to \infty.$$

Proof. Let $b_{nj} = \sum_{k=j}^{\infty} \frac{a_{nk}}{u_k}$, so that $b_{nj} \to 0$ as $j \to \infty$. Then $(b_{nj})_{j=1}^{\infty}$ is well defined.

Hence

$$
\begin{aligned}
y_n &= \sum_{k=1}^{\infty} a_{nk} x_k \\
&= \sum_{k=1}^{\infty} \frac{a_{nk}}{u_k} u_k x_k \\
&= \sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{a_{nk}}{u_k} \right) \Delta y_j \text{ (by Abel's partial summation)} \\
&= \sum_{j=1}^{\infty} b_{nj} \Delta y_j, \text{ where } y_j = u_j x_j.
\end{aligned}
$$

Hence $(x_j) \in bv(u, F, \Delta)$ implies $(\Delta y_j) \in l(F)$, so that $A = (a_{nk}) \in (bv(u, F, \Delta), \Gamma(F))$ if and only if $B = (b_{nj}) \in (l(F, \Delta), \Gamma(F))$. By Lemma A, we have

$$B \in (l(F, \Delta), \Gamma(F)) \text{ if and only if } \sup_{1 \le j < \infty} |b_{nj}|^{1/n} \to 0 \text{ as } n \to \infty,$$

that is,

$$\sup_{1 \le j < \infty} \left| \sum_{k=j}^{\infty} \frac{a_{nk}}{u_k} \right|^{1/n} \to 0 \text{ as } n \to \infty.$$

Similarly, we can prove the following theorem :

THEOREM 2.4. *$A \in (bv(u, F, \Delta), \Gamma^*(F))$ if and only if*

$$\sup_{1 \le j < \infty, 1 \le n < \infty} \left| \sum_{k=j}^{\infty} \frac{a_{nk}}{u_k} \right|^{1/n} \le M \text{ for some constant } M.$$

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