Commun. Korean Math. Soc. **20** (2005), No. 4, pp. 695–702

SIMPLE APPROACH TO MULTIFRACTAL SPECTRUM OF A SELF-SIMILAR CANTOR SET

In-Soo Baek

Abstract. We study the transformed measures with respect to the real parameters of a self-similar measure on a self-similar Cantor set to give a simple proof for some result of its multifractal spectrum. A transformed measure with respect to a real parameter of a self-similar measure on a self-similar Cantor set is also a self-similar measure on the self-similar Cantor set and it gives a better information for multifractals than the original self-similar measure. A transformed measure with respect to an optimal parameter determines Hausdorff and packing dimensions of a set of the points which has same local dimension for a self-similar measure. We compute the values of the transformed measures with respect to the real parameters for a set of the points which has same local dimension for a self-similar measure. Finally we investigate the magnitude of the local dimensions of a self-similar measure and give some correlation between the local dimensions.

1. Introduction

Recently the Hausdorff and packing dimensions of multifractal subsets by a self-similar measure on a self-similar Cantor set(cf. [18]) were studied([14, 16, 17, 20]) for the investigation of the sizes of subsets of fixed local dimension. We note that some authors([21, 22]) also investigated the Hausdorff and packing dimensions of multifractal subsets by a self-conformal measure on a self-conformal set as its general case. In their cases, the proof of some result is a little complicated. In this paper, we give a simpler proof to get such a result and an easier method to find the dimensions.

Received November 6, 2004.
2000 Mathematics Subject Classification: 28A78.
Key words and phrases: Hausdorff dimension, packing dimension, Cantor set, self-similar measure, distribution set.
This paper was supported by the research grants of Pusan University of Foreign Studies in 2004.

We define a transformed measure with respect to a real parameter of a self-similar measure on a self-similar Cantor set. We show that the transformed measures with respect to the real parameters are also self-similar measures on the self-similar Cantor set. It gives a better information of upper bounds of dimensions for multifractals than the original self-similar measure. Recently we([9, 10]) found the relation between a multifractal subset by a self-similar measure on a self-similar Cantor set and a distribution set([15, 19]) of the self-similar Cantor set. Using this, we find that the values of the transformed measures with respect to the real parameters for a set of the points which has same local dimension for a self-similar measure are 0 or 1. A transformed measure with respect to an optimal parameter which gives its transformed measure value 1 determines Hausdorff and packing dimensions of a set of the points of the same local dimension for a self-similar measure since the transformed measure value 1 gives a lower bound for its dimensions. We check the range of the local dimensions of a self-similar measure and compare their magnitudes of the local dimensions of a self-similar measure with some particular numbers related to the self-similar Cantor set and the self-similar measure, which gives more information of correlation between local dimensions.

Recently we([2, 3, 8]) studied a deranged Cantor set which is the most generalized Cantor set which has a local structure of a perturbed Cantor set([1, 6, 11, 13]), which is also a generalized form of self-similar Cantor set. Further we also introduced a quasi-self-similar measure([7, 10, 12]) on it. We note that a transformed measure with respect to a real parameter of a quasi-self-similar measure on a deranged Cantor set plays an important role to give an information of dimensions of a set of the points which has same local dimension for the quasi-self-similar measure. However we don't have such information of a transformed measure on a deranged Cantor set as that on a self-similar Cantor set since there is no result on a deranged Cantor set like the relation between a multifractal subset by a self-similar measure on a self-similar Cantor set and a distribution set of the self-similar Cantor set. We correlate $\alpha q + \beta(q)$ which appears in the formula $f(\alpha) = \inf_{q \in \mathbb{R}}\{\alpha q + \beta(q)\}$ to compute dimensions of multifractal in the result of Olsen([17, 20]) with a transformed measure with respect to a parameter q and give a hint to make a generalization of transformed measure on a deranged Cantor set to get a better information of dimensions of multifractal on a deranged Cantor set.

2. Preliminaries

We recall the definition of a deranged Cantor set([3]). Let $X_\phi = [0,1]$. We obtain the left subinterval $X_{\mathfrak{i},1}$ and the right subinterval $X_{\mathfrak{i},2}$ of $X_{\mathfrak{i}}$ by deleting a middle open subinterval of $X_{\mathfrak{i}}$ inductively for each $\mathfrak{i} \in \{1,2\}^n$, where $n = 0,1,2,\ldots$. Let $E_n = \cup_{\mathfrak{i}\in\{1,2\}^n} X_{\mathfrak{i}}$. Then E_n is a decreasing sequence of closed sets. For each n, we set $|X_{\mathfrak{i},1}|/|X_{\mathfrak{i}}| = c_{\mathfrak{i},1}$ and $|X_{\mathfrak{i},2}|/|X_{\mathfrak{i}}| = c_{\mathfrak{i},2}$ for every $\mathfrak{i} \in \{1,2\}^n$, where $n = 0,1,2,\cdots$ where $|X|$ denotes the length of X. We assume that the contraction ratios $c_{\mathfrak{i}}$ and gap ratios $1-(c_{\mathfrak{i},1}+c_{\mathfrak{i},2})$ are uniformly bounded away from 0. We call $F = \cap_{n=0}^{\infty} E_n$ a deranged Cantor set([3]). We note that a deranged Cantor set satisfying $c_{\mathfrak{i},1} = a_{n+1}$ and $c_{\mathfrak{i},2} = b_{n+1}$ for all $\mathfrak{i} \in \{1,2\}^n$, for each $n = 0,1,2,\cdots$ is called a perturbed Cantor set([1]). Further a perturbed Cantor set with $a_{n+1} = a$ and $b_{n+1} = b$ for all $n = 0,1,2,\cdots$ is called a self-similar Cantor set([17]).

For $\mathfrak{i} \in \{1,2\}^n$, $X_{\mathfrak{i}}$ denotes a fundamental interval of the n-stage of construction of a deranged Cantor set. Let $\mathbb{R}$ be the set of all real numbers and $\mathbb{N}$ be the set of all natural numbers.

For $x \in F$, we write $X_n(x)$ for the n-th level set $X_{i_1\cdots i_n}$ that contains x. We also note that if $x \in F$, then there is $\sigma \in \{1,2\}^{\mathbb{N}}$ such that $\bigcap_{n=0}^{\infty} X_{\sigma|n} = \{x\}$ (Here $\sigma|n = i_1, i_2, \cdots, i_n$, where $\sigma = i_1, i_2, \cdots, i_n, i_{n+1} \cdots$). Hereafter, we use $\sigma \in \{1,2\}^{\mathbb{N}}$ and $x \in F$ as the same identity freely. In a self-similar Cantor set F, we can consider a generalized expansion of x from σ, that is if $\sigma = i_1, i_2, \cdots, i_k, i_{k+1}, \cdots$ then the expansion of x is $0.j_1, j_2, \cdots, j_k, j_{k+1}, \cdots$ where $j_k = 0$ if $i_k = 1$ and $j_k = 2$ if $i_k = 2$. We denote $n_0(x|k)$ the number of times the digit 0 occurs in the first k places of the generalized expansion of x([19]).

For $r \in [0,1]$, we define a distribution set $F(r)$ containing the digit 0 in proportion r by

$$F(r) = \left\{ x \in F | \lim_{k\to\infty} \frac{n_0(x|k)}{k} = r \right\}.$$

The *lower* and *upper local dimension* of a finite measure μ at $x \in \mathbb{R}$ are defined([17]) by

$$\underline{\dim}_{loc}\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r},$$

$$\overline{\dim}_{loc}\mu(x) = \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r},$$

where $B(x,r)$ is the closed ball with center $x \in \mathbb{R}$ and radius $r > 0$.

If $\underline{\dim}_{loc}\mu(x) = \overline{\dim}_{loc}\mu(x)$, we call it *the local dimension of* μ *at* x and write it as $\dim_{loc}\mu(x)$. These local dimensions express the power law behaviour of $\mu(B(x,r))$ for some $r > 0$.

We recall a self-similar measure γ_p([17]) on a self-similar Cantor set where $p \in (0,1)$ induced by $\gamma_p(X_\mathrm{i}) = p_{i_1}p_{i_1,i_2}\cdots p_{i_1,i_2,\cdots,i_n}$ for $\mathrm{i} \in \{1,2\}^n$ where $p_{i_1,\cdots,i_k,1} = p$ and $p_{i_1,\cdots,i_k,2} = 1-p$ where $k = 0,1,\cdots,n-1$.

For $\alpha \geq 0$ define for a self-similar measure γ_p

$$\begin{aligned} E_\alpha^{(p)} &= \{x \in \mathbb{R} \,|\, \dim_{loc}\gamma_p(x) = \alpha\} \\ &= \left\{x \in \mathbb{R} \,|\, \lim_{r\to 0} \frac{\log \gamma_p(B(x,r))}{\log r} = \alpha\right\}. \end{aligned}$$

From now on, $\dim(E)$ denotes the Hausdorff dimension of $E \subset \mathbb{R}$ and $\mathrm{Dim}(E)$ denotes the packing dimension of E.

The following Proposition is fundamental to give informations of dimensions of a subset of a self-similar Cantor set.

PROPOSITION 1. [17] Let $E \subset \mathbb{R}$ be a Borel set and let μ be a finite measure.

(a) If $\underline{\dim}_{loc}\mu(x) \geq s$ for all $x \in E$ and $\mu(E) > 0$ then $\dim(E) \geq s$.
(b) If $\underline{\dim}_{loc}\mu(x) \leq s$ for all $x \in E$ then $\dim(E) \leq s$.
(c) If $\overline{\dim}_{loc}\mu(x) \geq s$ for all $x \in E$ and $\mu(E) > 0$ then $\mathrm{Dim}(E) \geq s$.
(d) If $\overline{\dim}_{loc}\mu(x) \leq s$ for all $x \in E$ then $\mathrm{Dim}(E) \leq s$.

In this paper, we assume that $0\log 0 = 0$ for convenience.

3. Main results

Consider a self-similar Cantor set F with two contraction ratios a and b. Fix $p \in (0,1)$ and let $q \in \mathbb{R}$ and $\beta(q)$ satisfy $p^q a^{\beta(q)} + (1-p)^q b^{\beta(q)} = 1$.

For positive α between $\frac{\log(1-p)}{\log b}$ and $\frac{\log p}{\log a}$, the Legendre transform $f(\alpha)$ of the function β is defined by

$$f(\alpha) = \inf_{-\infty<q<\infty}\{\alpha q + \beta(q)\}.$$

We define *a transformed measure* $\gamma_{t(q)}$ *with respect to a real* q *of a self-similar measure* γ_p to be $\gamma_{p^q a^{\beta(q)}}$.

LEMMA 2. [5, 9] Let $p \in (0,1)$. Consider a self-similar measure γ_p on a self-similar Cantor set F and let $r \in [0,1]$ and $g(r,p) = \frac{r\log p + (1-r)\log(1-p)}{r\log a + (1-r)\log b}$ Then $\dim(E_\alpha^{(p)}) = \mathrm{Dim}(E_\alpha^{(p)}) = g(r,r)$ for $\alpha = g(r,p)$.

LEMMA 3. *Fix* $p \in (0,1)$ *and* $\frac{\log(1-p)}{\log b} < \alpha < \frac{\log p}{\log a}$ *or* $\frac{\log p}{\log a} < \alpha < \frac{\log(1-p)}{\log b}$. *Let* $\alpha = g(r_0, p)$. *Then there is* $q_0 \in \mathbb{R}$ *such that* $\alpha q_0 + \beta(q_0) = g(r_0, r_0)$.

PROOF. Since $\dim(E_\alpha^{(p)}) = \mathrm{Dim}(E_\alpha^{(p)}) = g(r_0, r_0)$ where $\alpha = g(r_0, p)$, we easily see that $f(\alpha) = \inf_{-\infty<q<\infty}\{\alpha q + \beta(q)\} = g(r_0, r_0)$([17, 20]). Noting $\frac{\log(1-p)}{\log b} \neq \alpha \neq \frac{\log p}{\log a}$, $0 \neq r_0 \neq 1$. Since $g(0,0)$ or $g(1,1)$ is $\lim_{q\to-\infty}(\alpha q + \beta(q))$ or $\lim_{q\to\infty}(\alpha q + \beta(q))$, it follows. □

REMARK 1. From now on, we fix $p \in (0,1)$ and $\frac{\log(1-p)}{\log b} < \alpha < \frac{\log p}{\log a}$ or $\frac{\log p}{\log a} < \alpha < \frac{\log(1-p)}{\log b}$ if there is no mention of α.

LEMMA 4. *For* q_0 *such that* $\alpha q_0 + \beta(q_0) = g(r_0, r_0)$ *where* $\alpha = g(r_0, p)$, $p^{q_0} a^{\beta(q_0)} = r_0$.

PROOF. q_0 such that $\alpha q_0 + \beta(q_0) = g(r_0, r_0)$ where $\alpha = g(r_0, p)$ satisfies $\beta'(q_0) = -\alpha$([17]). We note that $\beta'(q_0) = -\alpha \iff \alpha = g(p^{q_0} a^{\beta(q_0)}, p)$([17]). Further $\alpha = g(r_0, p)$ and $\alpha = g(p^{q_0} a^{\beta(q_0)}, p)$. Since $g(r, p)$ is one to one(strictly increasing) function for $r \in (0,1)$, $p^{q_0} a^{\beta(q_0)} = r_0$. □

LEMMA 5. *Let* $a^s + b^s = 1$ *and* $p(\neq a^s) \in (0,1)$. *Let* $q \in \mathbb{R}$ *and* $\beta(q)$ *satisfy* $p^q a^{\beta(q)} + (1-p)^q b^{\beta(q)} = 1$. *If* $q \neq 0$, $E_\alpha^{(p)} = E_{\alpha q+\beta(q)}^{(p^q a^{\beta(q)})}$. *If* $q = 0$, $E_\alpha^{(p)} \subset E_{\alpha q+\beta(q)}^{(p^q a^{\beta(q)})}$.

PROOF. $E_\alpha^{(p)} = F(r_0)$, where $\alpha = g(r_0, p)$([9]). Let $q(\neq 0) \in \mathbb{R}$. We only need to show that $E_{\alpha q+\beta(q)}^{(p^q a^{\beta(q)})} = F(r_0)$. For this, we only need to show that $\alpha q + \beta(q) = g(r_0, r)$ where $p^q a^{\beta(q)} = r$ by Lemma 2 and [9]. We note that $\beta(q)\log a = \log r - q\log p$ from $p^q a^{\beta(q)} = r$ and $\beta(q)\log b = \log(1-r) - q\log(1-p)$ from $(1-p)^q b^{\beta(q)} = 1-r$. Hence we get

$$\beta(q) = \frac{r_0 \log r + (1-r_0)\log(1-r) - r_0 q \log p - (1-r_0) q \log(1-p)}{r_0 \log a + (1-r_0)\log b}.$$

Noting $\alpha = g(r_0, p)$, we see that $\alpha q + \beta(q) = g(r_0, r_0)$ by cancellation.

If $q = 0$ then $\beta(0) = s$. So $E_{\alpha 0+\beta(0)}^{(a^s)} = F$. □

We get information of a transformed measure $\gamma_{p^q a^{\beta(q)}}$ with respect to a parameter q of a self-similar measure γ_p. We note that the following Theorem is essentially similar with the lemma 4.5 in [21] for the case of a self-similar measure on a self-similar Cantor set but our proof is much simpler.

THEOREM 6. *Let* $p(\neq a^s) \in (0,1)$, *where* $a^s + b^s = 1$. *Consider* $E_\alpha^{(p)}$ *and* q_0 *such that* $\alpha q_0 + \beta(q_0) = g(r_0, r_0)$, *where* $\alpha = g(r_0, p)$ *and* $p^q a^{\beta(q)} + (1-p)^q b^{\beta(q)} = 1$. *For* $q(\neq 0) \in \mathbb{R}$,

$$\gamma_{p^q a^{\beta(q)}}(E_\alpha^{(p)}) = \gamma_{p^q a^{\beta(q)}}(E_{\alpha q+\beta(q)}^{(p^q a^{\beta(q)})}) = \begin{cases} 0 & \text{if } q \neq q_0, \\ 1 & \text{if } q = q_0. \end{cases}$$

PROOF. Since $\gamma_r(F(r)) = 1$ by the strong law of large numbers, if $r_0 \neq r$ then $\gamma_r(F(r_0)) = 0$. We note that $E_\alpha^{(p)} = F(r_0)$ where $\alpha = g(r_0, p)$. We can assume that $p^q a^{\beta(q)} \neq a^s$ since $q \neq 0$. If $q \neq q_0$, by the above Lemma $\gamma_{p^q a^{\beta(q)}}(E_{\alpha q+\beta(q)}^{(p^q a^{\beta(q)})}) = \gamma_{p^q a^{\beta(q)}}(E_\alpha^{(p)}) = 0$ since $p^q a^{\beta(q)} \neq r_0$ by Lemma 4. If $q = q_0$, then $p^q a^{\beta(q)} = r_0$ by Lemma 4, so $\gamma_{p^q a^{\beta(q)}}(E_{\alpha q+\beta(q)}^{(p^q a^{\beta(q)})}) = \gamma_{p^q a^{\beta(q)}}(E_\alpha^{(p)}) = 1$. □

REMARK 2. In the above Theorem, if $q = 0$, then $p^q a^{\beta(q)} = a^s$. So $\gamma_{p^q a^{\beta(q)}}(E_{\alpha q+\beta(q)}^{(p^q a^{\beta(q)})}) = \gamma_{a^s}(F) = 1$ for $q = 0$. But $E_\alpha^{(p)} = F(r_0)$ where $\alpha = g(r_0, p)$. Hence

$$\gamma_{a^s}(E_\alpha^{(p)}) = \begin{cases} 0 & \text{if } 0 \neq q_0 (\iff r_0 \neq a^s), \\ 1 & \text{if } 0 = q_0 (\iff r_0 = a^s). \end{cases}$$

REMARK 3. A transformed measure $\gamma_{t(q)}(= \gamma_{p^q a^{\beta(q)}})$ with respect to a real q of a self-similar measure γ_p gives information for Hausdorff and packing dimensions $\alpha q_0 + \beta(q_0)$ of $E_\alpha^{(p)} = E_{\alpha q_0+\beta(q_0)}^{(p^{q_0} a^{\beta(q_0)})}$ since $\gamma_{t(q_0)}(E_\alpha^{(p)}) = \gamma_{p^{q_0} a^{\beta(q_0)}}(E_{\alpha q_0+\beta(q_0)}^{(p^{q_0} a^{\beta(q_0)})}) = 1$ if $p^{q_0} a^{\beta(q_0)} \neq a^s$(if $p^{q_0} a^{\beta(q_0)} = a^s(\iff q_0 = 0)$ then its Hausdorff and packing dimensions are also $\alpha q_0 + \beta(q_0)(= s)$ since $E_\alpha^{(p)} = F(a^s) \subset E_s^{(a^s)} = F$ and $\gamma_{a^s}(E_\alpha^{(p)}) = 1$). By Proposition 1, this means that a transformed measure $\gamma_{t(q)}$ with respect to an optimal parameter to find the Hausdorff and packing dimensions of $E_\alpha^{(p)}$ is $\gamma_{t(q_0)}$ with respect to an optimal parameter q_0 where $\alpha q_0 + \beta(q_0) = g(r_0, r_0)$ where $\alpha = g(r_0, p)$. We also note that $\gamma_{t(q_0)} = \gamma_{p^{q_0} a^{\beta(q_0)}}$ and it is also a self-similar measure.

REMARK 4. The transformed measures with respect to real parameters of a quasi-self-similar measure([7, 10, 12]) on a deranged Cantor set can also be defined and give a better information for dimensions, but are not quasi-self-similar measures([12]).

REMARK 5. In the above Theorem, we consider $E_\alpha^{(p)}$ together with q_0 such that $\alpha q_0 + \beta(q_0) = g(r_0, r_0)$ where $\alpha = g(r_0, p)$, the range of α is $(\frac{\log p}{\log a}, \frac{\log(1-p)}{\log b})$ or $(\frac{\log(1-p)}{\log b}, \frac{\log p}{\log a})$. But when we just consider $E_\alpha^{(p)}$, the range of α is $[\frac{\log p}{\log a}, \frac{\log(1-p)}{\log b}]$ or $[\frac{\log(1-p)}{\log b}, \frac{\log p}{\log a}]$ which has non-empty interior if $\frac{\log p}{\log a} \neq \frac{\log(1-p)}{\log b}$. A singular behaviour is observed when $p = a^s$ where $a^s + b^s = 1$.

We have more information of the values of α from the following Theorems.

THEOREM 7. *Let $0 < p < a^s$ and consider $E_\alpha^{(p)}$. Then we have $\frac{\log(1-p)}{\log b} < g(p,p) < s < g(a^s, p) < \frac{\log p}{\log a}$, where $a^s + b^s = 1$.*

PROOF. Putting $g(a^s, p) = h(p)$, where $0 < p < 1$, we have $h'(p) < 0$ where $0 < p < a^s$ and $h'(a^s) = 0$. So we have $h(p) > h(a^s) = g(a^s, a^s) = s$ where $0 < p < a^s$. Since $\frac{\log(1-p)}{\log b} < \frac{\log p}{\log a}$, $k'(r) > 0$ where $0 < r < a^s$ and $k(r) = g(r,r)$. So $g(p,p) < g(a^s, a^s) = s$. Since $g(r,p)$ is a strictly increasing function for $r \in [0,1]$, we see that $g(0,p) = \frac{\log(1-p)}{\log b} \leq g(r,p) \leq \frac{\log p}{\log a} = g(1,p)$. □

THEOREM 8. *Let $a^s < p < 1$ and consider $E_\alpha^{(p)}$. Then we have $\frac{\log p}{\log a} < g(a^s, p) < s < g(p,p) < \frac{\log(1-p)}{\log b}$, where $a^s + b^s = 1$.*

PROOF. It follows from the dual arguments of the proof of the above Theorem. □

References

[1] I. S. Baek, *Dimension of the perturbed Cantor set*, Real Anal. Exchange **19** (1994), no. 1, 269–273.

[2] ———, *Dimensions of weakly convergent deranged Cantor sets*, Real Anal. Exchange **23** (1998), no. 2, 689–696.

[3] ———, *Weak local dimension on deranged Cantor sets*, Real Anal. Exchange **26** (2001), no. 2, 553–558.

[4] ———, *Hausdorff dimension of perturbed Cantor sets without some boundedness condition*, Acta Math. Hungar. **99** (2003), no. 4, 279–283.

[5] ———, *On a self-similar measure on a self-similar Cantor set*, J. Chungcheong Math. Soc. **16** (2003), no. 2, 1–10.

[6] ———, *Dimensions of measures on perturbed Cantor set*, J. Appl. Math. Comput. **14** (2004), no. 1-2, 397–403.

[7] ———, *Multifractal spectra by quasi-self-similar measures on a perturbed Cantor set*, J. Appl. Math. Comput. **14** (2004), no. 1-2, 447–454.

[8] ———, *Cantor dimension and its application*, Bull. Korean Math. Soc. **41** (2004), no. 1, 13–18.

[9] ———, *Relation between spectral classes of a self-similar Cantor set*, J. Math. Anal. Appl. **292** (2004), no. 1, 294–302.

[10] ———, *On a quasi-self-similar measure on a self-similar set on the way to a perturbed Cantor set*, J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math. **11** (2004), no. 1, 51–61.

[11] ———, *Spectra of deranged Cantor set by weak local dimension*, J. Math. Kyoto Univ. **44** (2004), no. 3, 493–500.

[12] ———, *On transformed dimension*, preprint.

[13] I. S. Baek and H. H. Lee, *Perturbed type random Cantor set*, Real Anal. Exchange **23** (1998), no. 1, 223–224.

[14] L. Barreira and B. Saussol, *Variational principles and mixed multifractal spectra*, Trans. Amer. Math. Soc. **353** (2001), 3919–3944.

[15] L. Barreira, B. Saussol, and J. Schmeling, *Distribution of frequencies of digits via multifractal analysis*, J. Number Theory **97** (2002), 410–438.

[16] R. Cawley and R. D Mauldin, *Multifractal decompositions of Moran fractals*, Adv. Math. **92** (1992), 196–236.

[17] K. J. Falconer, *Techniques in fractal geometry*, John Wiley and Sons, 1997.

[18] T. H. Kim, S. P. Hong, and H. H. Lee, *The Hausdorff dimension of deformed self-similar sets*, Hiroshima Math. J. **32** (2002), no. 1, 1–6.

[19] H. H. Lee and I. S. Baek, *Dimensions of a Cantor type set and its distribution sets*, Kyungpook Math. J. **32** (1992), no. 2, 149–152.

[20] L. Olsen, *A multifractal formalism*, Adv. Math. **116** (1995), 82–196.

[21] N. Patzschke, *Self-conformal multifractal measure*, Adv. Math. Sci. Appl. **19** (1997), no. 4, 486–513.

[22] D. Rand, *The singularity spectrum $f(\alpha)$ for cookie-cutters*, Ergodic Theory Dynam. Systems **9** (1989), 527–541.

Department of Mathematics
Pusan University of Foreign Studies
Pusan 608-738, Korea
E-mail: isbaek@pufs.ac.kr