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GREEN FUNCTIONS ON THE p-ADIC VECTOR SPACE

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ABSTRACT. Calculations of some integrals on the n-dimensional vector space over $\mathbb{Q}_p$ are useful in getting some other formulations of quantum mechanics and the field theory of p-adic mathematical physics. For reasons of these, we estimate several integrals. As an application, we derive some properties for the p-adic Green functions.

1. Introduction

Let $\mathbb{Z}$, $\mathbb{C}$ and $\mathbb{Q}_p$ denote the ring of integers, the field of complex numbers, the field of p-adic rational numbers, respectively. Let ord_p denote the p-adic ordinal over $\mathbb{Q}_p$ such that $\mathrm{ord}_p(p) = 1$, hence ord_p is determined uniquely. The corresponding non-Archimedian absolute value is $|x|_p = p^{-\mathrm{ord}_p(x)}$. Moreover, let us note that any p-adic number $x \neq 0$ is uniquely represented in the canonical form $x = \sum_{k=\gamma}^{\infty} x_k p^k$, where $\gamma = \mathrm{ord}_p(x) \in \mathbb{Z}$ and x_k are integers such that $0 \leq x_k \leq p-1$. The fractional part $\{x\}_p$ of a number $x \in \mathbb{Q}_p$ is defined by

$$\{x\}_p = \begin{cases} 0 & \text{if } \gamma \geq 0 \text{ or } x = 0, \\ \sum_{k=\gamma}^{-1} x_k p^k & \text{if } \gamma < 0. \end{cases}$$

The function $\chi(\xi x) = \exp(2\pi i \{\xi x\}_p)$ for every fixed $\xi \in \mathbb{Q}_p$ is an additive character of $\mathbb{Q}_p$. It is easy to see that $\chi(x) = 1$ if $|x|_p \leq 1$.

Let $\mathbb{Q}_p^n$ be the n-dimensional vector space over $\mathbb{Q}_p$, which contains all n-tuples of $\mathbb{Q}_p$, is defined with the norm $\|x\| = \max_{1 \leq i \leq n} |x_i|_p, x =$

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$(x_1, \cdots, x_n) \in \mathbb{Q}_p^n$. Put $\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n$ for $x, y \in \mathbb{Q}_p^n$. Also, we say that $x, y \in \mathbb{Q}_p^n$ are congruent modulo p^m, and write $\|x - y\| \leq p^m$, if $\max_{1\leq i\leq n} |x_i - y_i|_p \leq p^m$. Let us denote by $B_\gamma(a)$ the ball of radius p^γ with the center a and by $S_\gamma(a)$ its boundary (a sphere) i.e., $B_\gamma(a) = \{x \in \mathbb{Q}_p^n \mid \|x - a\| \leq p^\gamma\}$ and $S_\gamma(a) = B_\gamma(a) \setminus B_{\gamma-1}(a)$. For the notational convenience, let $B_\gamma(0) = B_\gamma$ and $S_\gamma(0) = S_\gamma$. The Haar measure dx on the vector space $\mathbb{Q}_p^n$ is given by the equality $dx = dx_1dx_2\cdots dx_n$ (dx_i = the normalized Haar measure on $\mathbb{Q}_p$). Note that $\text{vol}(B_0) = 1$. It is now straightforward to calculate the measure of any n-ball and also of n-sphere: $\text{vol}(B_\gamma) = p^{n\gamma}$ and $\text{vol}(S_\gamma) = p^{n\gamma}(1 - p^{-n})$. As main references, refer to [8] or [12].

Note that $\lesssim$ is an abbreviation of $\leq$ with the multiple of a constant depending on p, n and α. We denote $\approx$ when both $\lesssim$ and $\gtrsim$ hold.

2. Preliminaries

We say that $f \in L^1_{\text{loc}}(\mathbb{Q}_p^n)$ is integrable on $\mathbb{Q}_p^n$ (improper integral) if there exists

$$\lim_{N\to\infty} \int_{B_N} f(x)dx = \lim_{N\to\infty} \sum_{-\infty<\gamma\leq N} \int_{S_\gamma} f(x)dx.$$

This limit is called the integral of f on $\mathbb{Q}_p^n$, and it is denoted by $\int_{\mathbb{Q}_p^n} f(x)dx$ so that $\int_{\mathbb{Q}_p^n} f(x)dx = \sum_{-\infty<\gamma<\infty} \int_{S_\gamma} f(x)dx$.

Moreover, we introduce the following notations for it later ([12]). Let $x \in \mathbb{Q}_p^n$ and let $\gamma \in \mathbb{Z}$. Then

- $\Omega(t) = \begin{cases} 1, & 0 \leq t \leq 1, \\ 0, & 1 < t. \end{cases}$
- $\delta(\|x\| - p^\gamma) = \begin{cases} 1, & x \in S_\gamma, \\ 0, & x \notin S_\gamma. \end{cases}$
- $\triangle_\gamma(x) = \Omega(p^{-\gamma}\|x\|)$.
- $\delta_\gamma(x) = p^{n\gamma}\Omega(p^\gamma\|x\|)$.

Now we compute the measure of several shells of $\mathbb{Q}_p^n$. The following examples are simple reformulations of the well-known results ([4, 9, 12]).

Let $A = \{0, 1, \cdots, p-1\}$. For a fixed $c^v = (c_1^v, \cdots, c_n^v) \in A^n$ and $1 \leq v \leq r$, we put

$$D_{c^1} = \left\{x \in B_0 \mid x_i \equiv c_i^1 \pmod{p}, i = 1, \cdots, n\right\}.$$

By a translation $x \mapsto x + a$ with $a = c^2 - c^1$, one can shift D_{c^1} into D_{c^2}. It is clear that $B_0 = \cup_{c \in A^n} D_c$. Since the measure dx is invariant with respect to a translation, it follows that $\int_{D_{c^1}} dx = \int_{D_{c^2}} dx$ for all $c^1, c^2 \in A^n$. Using the identities $\int_{B_0} dx = \sum_{c \in A^n} \int_{D_c} dx = p^n \int_{D_c} dx$, we have

$$\int_{D_c} dx = p^{-n} \quad \text{for } c \in A^n.$$

Next put

$$D_{c^1,c^2} = \left\{ x \in D_{c^1} \mid x_i \equiv c_i^1 + c_i^2 p \pmod{p^2}, i = 1, \cdots, n \right\}.$$

Then $D_{c^1} = \bigcup_{c^2 \in A^n} D_{c^1,c^2}$. Since the Haar measure is translation-invariant we have $\int_{D_{c^1,c^2}} dx = \int_{D_{c^1,c^3}} dx$ for all $c^1, c^2, c^3 \in A^n$. As the above, we have

$$\int_{D_{c^1,c^2}} dx = p^{-2n} \quad \text{for } c^1, c^2 \in A^n.$$

We put

$$D_{c^1,\cdots,c^r} = \left\{ x \in D_{c^1,\cdots,c^{r-1}} \mid x_i \equiv c_i^1 + \cdots + c_i^r p^{r-1} \pmod{p^r} \right\}$$

for $i = 1, \cdots, n$. By the mathematical induction, we have

$$\int_{D_{c^1,\cdots,c^r}} dx = p^{-rn} \quad \text{for } c^1, \ldots, c^r \in A^n.$$

PROPOSITION 1. (cf. [8, 12]) *Let $s \in \mathbb{C}$ with $\operatorname{Re} s > -n$. Then*
(1) $\int_{B_0} \|x\|^s dx = (1 - p^{-n})(1 - p^{-s-n})^{-1}$.
(2) $\int_{B_\gamma} \|x\|^s dx = p^{\gamma(s+n)}(1 - p^{-n})(1 - p^{-s-n})^{-1}$.

PROOF. (1) First, we prove (1). Since $B_0 = \bigcup_{c \in A^n} D_c$, we have

$$\int_{B_0} \|x\|^s dx = \sum_{c \in A^n} \int_{D_c} \|x\|^s dx.$$

Let $c \in A^n$. For $c \neq 0$ and for $\|x\| = 1$, we have $\int_{D_c} \|x\|^s dx = p^{-n}$. Hence

$$\int_{B_0} \|x\|^s dx = p^{-n}(p^n - 1) + \int_{D_0} \|x\|^s dx.$$

Now we decompose D_0 into $\bigcup_{c\in A^n} D_{0,c}$. Then for $c \neq 0$ and for $\|x\| = p^{-1}$, we have $\int_{D_{0,c}} \|x\|^s dx = p^{-s-2n}$. Therefore we obtain

$$\int_{B_0} \|x\|^s dx = p^{-n}(p^n - 1) + p^{-s-2n}(p^n - 1) + \int_{D_{0,0}} \|x\|^s dx.$$

Continuing this process, we obtain (1).

To prove (2), we can write the integral of (2) as

$$\int_{B_\gamma} \|x\|^s dx = \int_{(p\widetilde{B}_\gamma)^n} \|x\|^s dx + \int_{\{x\in B_\gamma \mid |x_i|_p = p^\gamma \text{ for some } i\}} \|x\|^s dx,$$

where $\widetilde{B}_\gamma = \{x \in \mathbb{Q}_p \mid |x|_p \leq p^\gamma\}$. Now

$$\begin{aligned}
&\int_{\{x\in B_\gamma \mid |x_i|_p = p^\gamma \text{ for some } i\}} \|x\|^s dx \\
&= p^{\gamma s} \left(\operatorname{vol}(\{x \in B_\gamma \mid |x_i|_p = p^\gamma \text{ for some } i\})\right) \\
&= p^{\gamma s} \left(\operatorname{vol}(B_\gamma) - \operatorname{vol}((p\widetilde{B}_\gamma)^n)\right)
\end{aligned}$$

and $\int_{(p\widetilde{B}_\gamma)^n} \|x\|^s dx = p^{-(n+s)} \int_{B_\gamma} \|x\|^s dx$. So we have

$$(1 - p^{-(n+s)}) \int_{B_\gamma} \|x\|^s dx = p^{\gamma(s+n)}(1 - p^{-n}),$$

which yields (2). □

Proposition 1 says that (2) is the generalization of (1).

3. Results

Now we start with several terminologies.

DEFINITION 2. ([8, 12]) (a) Let $f(x)$ be a complex-valued function on $\mathbb{Q}_p^n$. A function f is called locally-constant if for any point $x \in \mathbb{Q}_p^n$ there exists $l(x) \in \mathbb{Z}$ such that

$$f(x + x') = f(x), \quad \|x'\| \leq p^{l(x)}.$$

We denote by $\mathcal{E} = \mathcal{E}(\mathbb{Q}_p^n)$ the set of all locally-constant functions on $\mathbb{Q}_p^n$.

(b) A function $f \in \mathcal{E}$ is called test function on $\mathbb{Q}_p^n$ if its support is compact. Let us denote by $\mathcal{D} = \mathcal{D}(\mathbb{Q}_p^n)$ the set of test functions on $\mathbb{Q}_p^n$.

Let $\varphi \in \mathcal{D}$. The p-adic Fourier-transform $\mathcal{F}[\varphi]$ is defined by (cf. [1-12])

$$\mathcal{F}[\varphi](\xi) = \int_{\mathbb{Q}_p^n} \chi_p(\langle \xi, x\rangle)\varphi(x)dx \quad \text{for } \xi \in \mathbb{Q}_p^n.$$

Note that $\mathcal{F}$ is the linear isomorphism from $\mathcal{D}$ onto $\mathcal{D}$, and also $\mathcal{F}$ has the inversion formula

$$f(x) = \int_{\mathbb{Q}_p^n} \chi_p(-\langle \xi, x\rangle)\mathcal{F}[f](\xi)d\xi \quad \text{for } \mathcal{F}[\varphi] \in \mathcal{D}$$

for f whose p-adic Fourier transform is valid. Let us note that on the space $\mathcal{D}'$ of distributions, there is the inversion formula.

LEMMA 3. *Let χ be a nontrivial additive character of the field $\mathbb{Q}_p$. Suppose that $f^s(x) = f(\|x\|)\|x\|^s \in \mathcal{D}$ with $\sum_{k=0}^{\infty} |f(p^{-k})|p^{-k(s+n)} < \infty$. Then*

$$\mathcal{F}[f^s](\xi) = \sum_{k=0}^{\infty} \frac{f(p^{-k}\|\xi\|^{-1})}{(p^k\|\xi\|)^{s+n}}\left(1 - \frac{1}{p^n}\right) - f(p\|\xi\|^{-1})\|\xi\|^{-(s+n)}p^s.$$

PROOF. Suppose that $f \in L^1_{\text{loc}}(\mathbb{Q}_p^n)$. By the definition of the improper integral, we get

$$\int_{\mathbb{Q}_p^n} f(\|x\|)\|x\|^s\chi(\langle \xi, x\rangle)dx = \sum_{\gamma=-\infty}^{\infty} \int_{S_\gamma} f(\|x\|)\|x\|^s\chi(\langle \xi, x\rangle)dx.$$

We also find that (cf. [2, 3, 12])

$$\int_{S_\gamma} \chi(\langle \xi, x\rangle)dx = p^{n\gamma}(1 - p^{-n})\Omega(\|\xi p^{-\gamma}\|) - p^{n(\gamma-1)}\delta(\|\xi\| - p^{-\gamma+1}),$$

since

$$\int_{B_\gamma} \chi(\langle \xi, x\rangle)dx = p^{\gamma n}\Omega(\|\xi p^{-\gamma}\|)$$

and since $S_\gamma = B_\gamma \backslash B_{\gamma-1}$. Now set $\|\xi\| = p^N$. Then $\gamma \le -N$ if $p^N \le p^{-\gamma}$, $\gamma = -N+1$ if $p^N = p^{-\gamma+1}$, and $\gamma \ge -N+2$ if $p^N \ge p^{-\gamma+2}$. As a result, we obtain

$$\begin{aligned}
&\sum_{\gamma=-\infty}^{\infty} \int_{S_\gamma} f(p^\gamma)\|x\|^s \chi(\langle \xi, x\rangle)dx \\
&= \sum_{\gamma=-\infty}^{\infty} f(p^\gamma) \cdot \begin{cases} p^{(s+n)\gamma}\left(1-\frac{1}{p^n}\right), & \gamma \le -N \\ -p^{\gamma(s+n)-n}, & \gamma = -N+1 \\ 0, & \gamma \ge -N+2 \end{cases} \\
&= \sum_{\gamma=-\infty}^{-N} f(p^\gamma)p^{(s+n)\gamma}\left(1-\frac{1}{p^n}\right) - f(p^{-N+1})p^{-(s+n)N+s}.
\end{aligned}$$

Setting $k = -\gamma - N$ and substituting $\|\xi\| = p^N$, we complete the proof.□

The Gelfand-Graev n-dimensional p-adic Γ-function Γ_n on $\mathbb{C}$ is defined by

$$\Gamma_n(s) = (1-p^{s-n})/(1-p^{-s}).$$

COROLLARY 4. (1) *Let* $s \in \mathbb{C}, s \neq (2k\pi i)/\ln p$ *for* $k \in \mathbb{Z}$ *and* $Re(s) > 0$. *Then*

$$\mathcal{F}[\|x\|^{s-n}](\xi) = \|\xi\|^{-s}\Gamma_n(s).$$

(2) *Let* $Re\, s > 0$ *and* $a > 0, t > 0$. *Then*

$$\begin{aligned}
&\mathcal{F}[\exp(-at\|x\|^s)](\xi) \\
&= \frac{1-p^{-n}}{\|\xi\|^n}\left(\sum_{k=0}^{\infty} p^{-kn}\exp(-atp^{-sk}\|\xi\|^{-s}) - \exp(-atp^s\|\xi\|^{-s})\right).
\end{aligned}$$

(3) *Let* $k \in \mathbb{Z}$. *Then*

$$\mathcal{F}[\Delta_k](\xi) = \delta_k(\xi).$$

(4) *Let* $k \in \mathbb{Z}$. *Then*

$$\mathcal{F}[\delta(\|x\| - p^k)](\xi) = \left(1-\frac{1}{p^n}\right)\delta_k(\xi) - p^{n(k-1)}\delta(\|\xi\| - p^{1-k}).$$

COROLLARY 5. [2, 8] *Let $\xi \in \mathbb{Q}_p^n$ with $\|\xi\| \leq 1$ and let $Re(s) < 0$. Then*

$$\int_{\mathbb{Q}_p^n} \|\xi\|^{s-n}[\chi(\langle \xi, x\rangle) - 1]dx = \frac{(1-p^{s-n})}{\|\xi\|^s(1-p^{-s})}.$$

REMARK. Let $f_s^n = \|x\|^{s-n}/\Gamma_n(s)$ be the generalized function, where $1 \leq n < \infty$, $s \in \mathbb{C}$ and $x \in \mathbb{Q}_p^n$. Then f_s^n is holomorphic on the complex plane except the points $n + s_k$, where $s_k = (2\pi k i)/\ln p$ for $k \in \mathbb{Z}$. In particular,

$$\begin{aligned}\operatorname*{Res}_{s=n+s_k} f_s^n &= \lim_{s \to n+s_k} (s - n - s_k) f_s^n \\ &= \lim_{\epsilon \to 0^+} \frac{\epsilon(1-p^{-\epsilon-n-s_k})\|x\|^{\epsilon+s_k}}{1-p^{\epsilon+s_k}}\end{aligned}$$

and this is true due to the relations

$$\begin{aligned} p^s &= e^{(x+iy)\ln p} \quad (s \overset{\text{put}}{=} x + iy) \\ &= e^{x \ln p}(\cos(y \ln p) + i \sin(y \ln p)) = 1 \end{aligned}$$

and second equality give $s = (2\pi k i)/\ln p$ for $k \in \mathbb{Z}$. Thus it has simple poles with the residue

$$\|x\|^{s_k}(1-p^n)/(p^n \ln p).$$

From (1) of Corollary 4, we see that

$$\mathcal{F}[f_{-s}^n] = \frac{1}{\Gamma_n(-s)}\mathcal{F}[\|x\|^{-s-n}] = \|\xi\|^s$$

for $x, \xi \in \mathbb{Q}_p^n$ (cf. [8, 9, 12]).

THEOREM 6. *Let f be a real-valued function defined on the sequence $\{p^h \mid h \in \mathbb{Z} \cup \{-\infty\}\}$ and let $1/(f(\|x\|) + m^2) \in \mathcal{D}$. The Green function $G(z, f(\|x\|))$ can be defined as*

$$G(z, f(\|x\|)) = \mathcal{F}\left[\frac{1}{f(\|x\|) + m^2}\right](z), \quad m \in \mathbb{R}^+,$$

where $\mathbb{R}^+$ denotes the set of all positive real numbers.

(1) *For* $m \in \mathbb{R}^+$ *and* $n = 1, 2, \cdots$,

$$G(z, f(\|x\|)) = \frac{(1-p^{-n})\|z\|^{2-n}}{m^2\|z\|^2 + p^2} \sum_{k=0}^{\infty} \frac{p^{-nk}(p^2 - \|z\|^2 f(\|p^k z^{-1}\|))}{\|z\|^2(f(\|p^k z^{-1}\|) + m^2)}.$$

(2) *Suppose that* $\|z\|^2 f(\|p^k z^{-1}\|) \to c_{p,k}$ (uniformly convergent) *as* $\|z\| \to \infty$, *where* $\{c_{p,k}\}, k = 0, 1, \cdots$ *is a geometric sequence such that each term is given by a multiple* r *of the previous one. The asymptotic expansion of the Green function has the following behavior*

$$G(z, f(\|x\|)) \sim \frac{p^{n+2} - rp^2 - c_{p,0}(p^n - 1)}{p^n - r} \frac{1}{m^4} \frac{1}{\|z\|^{n+2}}, \quad \|z\| \to \infty,$$

where $m \in \mathbb{R}^+$ *and* $n = 1, 2, \cdots$.
(3) *The p-adic Green function* $G(z, f(\|x\|))$ *is positive.*

Proof. Since $G(z, f(\|x\|)) = \mathcal{F}[1/(f(\|x\|) + m^2)](z)$, by Lemma 3, we have

$$\begin{aligned} &G(z, f(\|x\|)) \\ &= \left(1 - \frac{1}{p^n}\right) \frac{1}{\|z\|^n} \sum_{k=0}^{\infty} \frac{p^{-nk}}{f(p^{-k}\|z\|^{-1}) + m^2} - \frac{1}{p^2\|z\|^{n-2} + m^2\|z\|^n} \\ &= \left(1 - \frac{1}{p^n}\right) \frac{1}{\|z\|^{n-2}} \sum_{k=0}^{\infty} \left(\frac{p^{-nk}}{\|z\|^2(f(\|p^k z^{-1}\|) + m^2)} - \frac{p^{-nk}}{p^2 + m^2\|z\|^2} \right). \end{aligned}$$

This gives the (1).

To see (2), using the result of (1), we write the asymptotic expansion of the Green function $G(z, f(\|x\|))$ in the form

$$\begin{aligned} &\lim_{\|z\| \to \infty} \|z\|^{n+2}\, G(z, f(\|x\|)) \\ &= \lim_{\|z\| \to \infty} \left(1 - \frac{1}{p^n}\right) \frac{1}{m^4} \frac{m^4\|z\|^4}{m^2\|z\|^2 + p^2} \sum_{k=0}^{\infty} \frac{p^{-nk}(p^2 - \|z\|^2 f(\|p^k z^{-1}\|))}{\|z\|^2(f(\|p^k z^{-1}\|) + m^2)} \\ &= \left(1 - \frac{1}{p^n}\right) \frac{1}{m^4} \lim_{\|z\| \to \infty} \sum_{k=0}^{\infty} \frac{p^2 - \|z\|^2 f(\|p^k z^{-1}\|)}{p^{nk}} \frac{1}{\frac{p^2}{m^2\|z\|^2} + 1} \\ &\times \left(1 - \frac{\|z\|^2 f(\|p^k z^{-1}\|)}{\|z\|^2(f(\|p^k z^{-1}\|) + m^2)}\right). \end{aligned}$$

On the other hand, since the resulting series is uniformly convergent, so that passage to limit, we obtain

$$\lim_{\|z\|\to\infty} \|z\|^{n+2} G(z, f(\|x\|)) = \left(1 - \frac{1}{p^n}\right) \frac{1}{m^4} \sum_{k=0}^{\infty} (p^{2-nk} - p^{-nk} c_{p,k})$$
$$= \frac{1}{m^4} \frac{p^{n+2} - rp^2 - c_{p,0}(p^n - 1)}{p^n - r}.$$

Consequently, we obtain the desired asymptotic expansion of the Green function $G(z, f(\|x\|))$ as the following

$$G(z, f(\|x\|)) \sim \frac{p^{n+2} - rp^2 - c_{p,0}(p^n - 1)}{p^n - r} \frac{1}{m^4} \frac{1}{\|z\|^{n+2}}, \quad \|z\| \to \infty.$$

This completes the proof of (2). □

REMARK. In Theorem 6, set $n = 1$ and set $f(|x|_p) = |x|_p^2$. Then we have

$$G(z, |x|_p^2) \sim \frac{p^3(p^2 - 1)}{p^3 - 1} \frac{1}{m^4} \frac{1}{|z|_p - p^3}, \quad |z|_p \to \infty,$$

where $z \in \mathbb{Q}_p$. This asymptotic expansion of the Green function G on $\mathbb{Q}_p$ was studied in [1, 3, 5, 7, 8, 10, 12].

To study the estimate of the Green function $G(z, \|x\|^2)$, we prove the following lemma. The following result is a generalization of Lemma 1 of Bikulov[1] for all $n = 1, 2, \cdots$.

THEOREM 7. *Assume $S_q(\ell, n)$ is presented in the form*

$$S_q(\ell, n) = \sum_{k=0}^{\infty} \frac{\sqrt{q}^{nk}}{\ell + q^k}, \quad n = 1, 2, \cdots$$

for all $\ell \in \mathbb{R}^+$ and $0 < q < 1$. Then the estimate is the following:

$$\frac{1}{\ell + 1} + \frac{\sqrt{q}^n}{1 - \sqrt{q}^n} I_n(\ell) \le S_q(\ell, n) \le \frac{1}{1 - \sqrt{q}^n} I_n(\ell),$$

where $I_n(\ell) = n \int_0^1 t^{n-1}/(\ell + t^2) dt$.

PROOF. For all $\ell \in \mathbb{R}^+$ and $0 < q < 1$, we obtain

$$\begin{aligned} &S_q(\ell, n) \\ &= \frac{1}{1-\sqrt{q}^n}\left[\frac{1-\sqrt{q}^n}{\ell+1} + \frac{\sqrt{q}^n(1-\sqrt{q}^n)}{\ell+q} + \cdots + \frac{\sqrt{q}^{nk}(1-\sqrt{q}^n)}{\ell+q^k} + \cdots\right] \\ &= \frac{1}{1-\sqrt{q}^n}\left[\frac{1}{\ell+1} + \sqrt{q}^n \sum_{k=0}^{\infty} \sqrt{q}^{nk}\left(\frac{1}{\ell+q^{k+1}} - \frac{1}{\ell+q^k}\right)\right]. \end{aligned}$$

If we use the identity $\int_k^{k+1} d\left(\frac{1}{\ell+q^x}\right) = 1/(\ell+q^{k+1}) - 1/(\ell+q^k)$ we may write $S_q(\ell,n)$ as

$$S_q(\ell, n) = \frac{1}{1-\sqrt{q}^n}\left[\frac{1}{\ell+1} + \sqrt{q}^n \sum_{k=0}^{\infty} \sqrt{q}^{nk} \int_k^{k+1} d\left(\frac{1}{\ell+q^x}\right)\right].$$

We now consider the auxiliary series

$$\Phi_q(\ell, n) = \sum_{k=0}^{\infty} \int_k^{k+1} \sqrt{q}^{nx} d\left(\frac{1}{\ell+q^x}\right) = \int_0^{\infty} \sqrt{q}^{nx} d\left(\frac{1}{\ell+q^x}\right).$$

From the method of integration by parts and integration by substitution, one sees without difficulty that

$$\Phi_q(\ell, n) = -\frac{1}{\ell+1} - \frac{n}{2}\ln q \int_0^{\infty} \frac{\sqrt{q}^{nx}}{\ell+q^x} dx = -\frac{1}{\ell+1} + n\int_0^1 \frac{t^{n-1}}{\ell+t^2} dt.$$

Then

$$\Phi_q(\ell, n) = \sum_{k=0}^{\infty} \int_k^{k+1} \sqrt{q}^{nx} d\left(\frac{1}{\ell+q^x}\right).$$

We also can readily show that by the mean-value theorem

$$\begin{aligned} \int_k^{k+1} \sqrt{q}^{nx} d\left(\frac{1}{\ell+q^x}\right) &\le \sqrt{q}^{nk} \int_k^{k+1} d\left(\frac{1}{\ell+q^x}\right) \\ &\le \frac{1}{\sqrt{q}^n} \int_k^{k+1} \sqrt{q}^{nx} d\left(\frac{1}{\ell+q^x}\right). \end{aligned}$$

Let $\overline{S}_q(\ell,n)$ and $\underline{S}_q(\ell,n)$ be the upper and lower bound of $S_q(\ell,n)$, respectively, then we can find

$$\overline{S}_q(\ell, n) = \frac{1}{1-\sqrt{q}^n}\left[\frac{1}{\ell+1} + \Phi_q(\ell, n)\right]$$

and

$$\underline{S}_q(\ell,n) = \frac{1}{1-\sqrt{q}^n}\left[\frac{1}{\ell+1} + \frac{1}{\sqrt{q}^n}\Phi_q(\ell,n)\right].$$

Therefore

$$\underline{S}_q(\ell,n) \le S_q(\ell,n) \le \overline{S}_q(\ell,n).$$

For any integer $n \ge 1$, let us put

$$I_n(\ell) = n\int_0^1 \frac{t^{n-1}}{\ell+t^2}dt, \quad \ell \in \mathbb{R}_{>0}.$$

Then we get

$$\underline{S}_q(\ell,n) = \frac{1}{\ell+1} + \frac{\sqrt{q}^n}{1-\sqrt{q}^n}I_n(\ell), \quad \overline{S}_q(\ell,n) = \frac{1}{1-\sqrt{q}^n}I_n(\ell).$$

This completes the proof of the theorem. □

REMARK. [3] It is easy to verify that for $\ell \in \mathbb{R}^+$

$$\int_0^1 \frac{t^{n-1}}{\ell+t^2}dt = \begin{cases} \frac{1}{n}\frac{1}{\sqrt{\ell}}\tan^{-1}\frac{1}{\sqrt{\ell}} & \text{if } n=1, \\ \frac{1}{n}\ln\frac{\ell+1}{\ell} & \text{if } n=2, \\ \frac{1}{n-2} + \frac{(-\ell)^1}{n-4} + \cdots + (-\ell)^{k-1} + (-\ell)^k\frac{1}{\sqrt{\ell}}\tan^{-1}\frac{1}{\sqrt{\ell}} & \text{if } n=2k+1, \\ \frac{1}{n-2} + \frac{(-\ell)^1}{n-4} + \cdots + \frac{(-\ell)^{k-1}}{2} + (-\ell)^k\ln\frac{\ell+1}{\ell} & \text{if } n=2k+2, \end{cases}$$

where $k = 1, 2, \cdots$.

As a consequence of Theorem 7 we obtain the following:

COROLLARY 8. *The Green function $G(z, \|x\|^2)$ has the following estimate,*

$$\frac{1}{\|z\|^{n-2}}\left(\frac{p^n-1}{p^n}\frac{1}{m^2\|z\|^2+1} - \frac{1}{p^2+m^2\|z\|^2} + \frac{1}{p^n}I_n(\ell)\right)$$
$$\le G(z,\|x\|^2) \le \frac{1}{\|z\|^{n-2}}\left(I_n(\ell) - \frac{1}{p^2+m^2\|z\|^2}\right).$$

PROOF. Let $f(\|x\|) = \|x\|^2$. By Theorem 6, we have

$$G(z, \|x\|^2) = \frac{1}{\|z\|^n}\left[\left(1-\frac{1}{p^n}\right)\sum_{k=0}^{\infty}\frac{p^{-nk}}{p^{-2k}\|z\|^{-2}+m^2} - \frac{1}{p^2\|z\|^{-2}+m^2}\right].$$

If we substitute $\ell = m^2\|z\|^2$ and $q = p^{-2}$ for ℓ and q in Theorem 7, then we have

$$\frac{1}{\|z\|^{n-2}}\left[\left(1-\frac{1}{p^n}\right)\underline{S}_{p^{-2}}(m^2\|z\|^2, n) - \frac{1}{p^2+m^2\|z\|^2}\right]$$
$$\leq G(z, \|x\|^2) \leq \frac{1}{\|z\|^{n-2}}\left[\left(1-\frac{1}{p^n}\right)\overline{S}_{p^{-2}}(m^2\|z\|^2, n) - \frac{1}{p^2+m^2\|z\|^2}\right].$$

This completes the proof. □

REMARK.

(a) Note that Corollary 8 gives the same description to the estimate for the p-adic Green function of Theorem 4 in [3].
(b) In recent years, Sato[11] studied Green functions of pseudodifferential operators over $\mathbb{Q}_p^n$ with symbols of the form $|P^*|_p$ where P^* is a polynomial cut off near zero.
(c) Kochubei[6, 7, 8] gave the asymptotic expansion of the Green functions. He also showed that the n-dimensional Green functions could be reduced to the investigations of the one-dimensional Green functions over the unramified nth degree extension of $\mathbb{Q}_p$. On the other hand, the asymptote of one-dimensional Green functions over arbitrary local fields is found in [7] by Kochubei.
(d) The asymptotic expansion of the p-adic Green functions defined on the even dimensional space of p-adic numbers $\mathbb{Q}_p$ using the functional equation of the local zeta function was established in [5] by Jang.
(e) Some properties of the multidimensional p-adic Green functions were studied in [3] by Chuong and Co.

References

[1] A. Kh. Bikulov, *Investigation of the p-adic Green function*, Teoret. Mat. Fiz. (in Russian) **87** (1991), 376–390.
[2] W. Chong, M.-S. Kim, T. Kim, and J.-W. Son, *On the p-adic spectral analysis and multiwavelet on* $L^2(\mathbb{Q}_p^n)$, Int. J. Math. Math. Sci. **2003** (2003), 2619–2633.

[3] N. M. Chuong and N. V. Co, *The multidimensional p-adic Green function*, Proc. Amer. Math. Soc. **127** (1999), 685–694.

[4] J. Igusa, *An introduction to the theory of local zeta functions*, AMS/IP Studies in Advanced Mathematics, 14, American Mathematical Society and International Press, 1994.

[5] Y. Jang, *An asymptotic expansion of the p-adic Green function*, Tôhoku Math. J. **50** (1998), 229–242.

[6] A. N. Kochubei, *On p-adic Green's functions*, Theoret. and Math. Phys. **96** (1993), 854–865.

[7] ———, *On the asymptotics of p-adic Green functions*, Proc. Steklov Inst. Math. **203** (1994), 105–113.

[8] ———, *Pseudo-differential equations and stochastics over non-Archi-medean fields*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 244, Marcel Dekker, Inc., New York, 2001.

[9] P. Sally and M. Taibleson, *Special functions on locally compact fields*, Acta Math. **116** (1966), 279–309.

[10] F. Sato, *p-adic Green functions and zeta functions*, Comment. Math. Univ. St. Pauli **51** (2002), 79–97.

[11] M. H. Taibleson, *Fourier analysis on local fields*, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1975.

[12] V. S. Vladimirov, I. V. Volovich, and E. I. Zelenov, *p-Adic Analysis and Mathematical Physics*, Series on Soviet & East European Mathematics, Vol. I, World Scientific, Singapore, 1994.

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