Commun. Korean Math. Soc. **20** (2005), No. 4, pp. 649–655

SKEW ENVELOPING ALGEBRAS AND POISSON ENVELOPING ALGEBRAS

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ABSTRACT. The universal mapping property and the Gelfand-Kirillov dimension of a skew enveloping algebra are studied and it is proved that every Poisson enveloping algebra is a homomorphic image of a skew enveloping algebra.

1. Introduction

Poisson algebras play an important role in many mathematical branches related to quantized algebras and have been studied by many mathematicians. Moreover the second author constructed Poisson enveloping algebras in [6] and [7] for studying Poisson modules. The main aim of this note is to explain Poisson enveloping algebras using skew enveloping algebra.

In this note, we study the universal mapping property and the Gelfand Kirillov dimension for a skew enveloping algebra and we prove that every Poisson enveloping algebra is a homomorphic image of a skew enveloping algebra as a corollary.

Throughout the paper, k will be a field of characteristic zero, all vector spaces will be over k and, for an algebra A, A_L will denotes the Lie algebra A with Lie bracket $[a, b] = ab - ba$ for all $a, b \in A$.

Received February 11, 2005.
2000 Mathematics Subject Classification: Primary 16W30; Secondary 17B63.
Key words and phrases: skew enveloping algebra, Poisson enveloping algebra.

2. Skew enveloping algebras

Let $H = (H, m, \mu, \Delta, \epsilon)$ be a bialgebra. We use Sweedler's notation throughout the paper. That is,

$$\Delta(x) = \sum x' \otimes x'',$$
$$(\mathrm{id} \otimes \Delta)\Delta(x) = (\Delta \otimes \mathrm{id})\Delta(x) = \sum x' \otimes x'' \otimes x'''$$

for $x \in H$. Let A be an H-module algebra. That is, A is a left H-module with module structure $H \times A \longrightarrow A$, $(x, a) \mapsto x \cdot a$, satisfying

$$x \cdot (ab) = \sum (x' \cdot a)(x'' \cdot b), \quad x \cdot 1 = \epsilon(x)1.$$

It is well-known that there exists a unique algebra structure on the vector space $A \otimes H$, with multiplicative identity $1 \otimes 1$, such that its product is given by

$$(a \otimes x)(b \otimes y) = \sum a(x' \cdot b) \otimes x''y.$$

This algebra is called a smash product of A and H and denoted by $A\sharp H$. The reader is referred to [2], [4], and [8] for further bialgebra structures and module algebra structures over a bialgebra.

Let $\{a_t \mid t \in I\}$ and $\{x_u \mid u \in J\}$ be k-bases of A and H respectively. Then the set $\{a_t \otimes x_u \mid t \in I, u \in J\}$ forms a k-basis for $A\sharp H$. It follows that $A\sharp H$ is a free left A-module with basis $\{1 \otimes x_u \mid u \in J\}$ and the maps

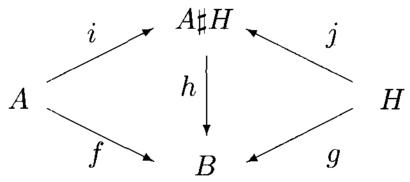
$$i : A \longrightarrow A\sharp H, \quad i(a) = a \otimes 1,$$
$$j : H \longrightarrow A\sharp H, \quad j(x) = 1 \otimes x$$

are monomorphisms. Hence it makes sense that, for $a \in A$ and $x \in H$, the elements $a \otimes 1$ and $1 \otimes x$ in $A\sharp H$ are denoted by a and x respectively. Henceforce we omit the notation $\otimes$ in $A\sharp H$ and each element $a \otimes x \in A\sharp H$ is denoted by ax. Thus the smash product $A\sharp H$ is an algebra generated by A and H subject to the relation $xa = \sum (x' \cdot a)x''$ for all $a \in A$ and $x \in H$.

LEMMA 1. *The triple $(A\sharp H, i, j)$ satisfies the following property: Given an algebra B and algebra homomorphisms $f : A \longrightarrow B$ and $g : H \longrightarrow B$ satisfying*

$$g(x)f(a) = \sum f(x' \cdot a)g(x'') \tag{1}$$

for all $a \in A$ and $x \in H$, there exists a unique algebra homomorphism $h : A\sharp H \longrightarrow B$ such that $f = hi$ and $g = hj$.



Proof. Fix k-bases $\{a_t \mid t \in I\}$ and $\{x_u \mid u \in J\}$ of A and H respectively. Then the set $\{a_t x_u \mid t \in I, u \in J\}$ forms a k-basis of $A\sharp H$, and thus there exists a unique k-linear map $h : A\sharp H \longrightarrow B$ such that $h(a_t x_u) = f(a_t)g(x_u)$ for all $t \in I$ and $u \in J$. Since, by (1),

$$\begin{aligned} h((a_t x_u)(a_r x_s)) &= h(\sum a_t(x'_u \cdot a_r)(x_u{}'' x_s)) \\ &= \sum f(a_t(x'_u \cdot a_r))g(x''_u x_s) \\ &= \sum f(a_t)f(x'_u \cdot a_r)g(x''_u)g(x_s) \\ &= f(a_t)g(x_u)f(a_r)g(x_s) \\ &= h(a_t x_u)h(a_r x_s) \end{aligned}$$

for all $t, r \in I$ and $u, s \in J$, h is an algebra homomorphism such that $f = hi$ and $g = hj$. If h' is an algebra homomorphism from $A\sharp H$ into B such that $f = h'i$ and $g = h'j$ then

$$h'(ax) = h'i(a)h'j(x) = f(a)g(x) = h(ax)$$

for all $a \in A, x \in H$ and thus $h' = h$. It completes the proof. □

Let A be an algebra and let $\mathfrak{g}$ be a Lie algebra such that there is a Lie homomorphism $\phi : \mathfrak{g} \longrightarrow \mathrm{Der}_k(A)$, where $\mathrm{Der}_k(A)$ is the set of all derivations in A. Denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. Then A is a $U(\mathfrak{g})$-module algebra with module structure $x \cdot a = \phi(x)(a)$ for $x \in \mathfrak{g}$ and $a \in A$ since $U(\mathfrak{g})$ is a bialgebra such that the comultiplication Δ and the counit ϵ are given by

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \epsilon(x) = 0$$

for all $x \in \mathfrak{g}$. Hence the smash product $A\sharp U(\mathfrak{g})$ is the algebra generated by A and $U(\mathfrak{g})$ satisfying the relation $xa = x \cdot a + ax$ for all $a \in A$ and $x \in \mathfrak{g}$. In this case, $A\sharp U(\mathfrak{g})$ is called a skew enveloping algebra of A and $\mathfrak{g}$. (See [5, 1.7.10])

Let $\{y_j | j \in I\}$ be a totally ordered basis of $\mathfrak{g}$. Then $U(\mathfrak{g})$ has the basis $\mathfrak{B}$ consisting of

$$y_{j_1}^{r_1} y_{j_2}^{r_2} \cdots y_{j_n}^{r_n}, \quad j_1 \leq j_2 \leq \cdots \leq j_n, \quad r_i \geq 0,$$

called a standard monomial, by the Poincaré-Birkhoff-Witt theorem. Let us give a total order $\prec$ in $\mathfrak{B}$ by

$$y_{j_1}^{r_1} y_{j_2}^{r_2} \cdots y_{j_n}^{r_n} \prec y_{j_1}^{s_1} y_{j_2}^{s_2} \cdots y_{j_n}^{s_n}$$
$$\Longleftrightarrow$$
(i) $\sum_i r_i < \sum_i s_i$ or
(ii) $\sum_i r_i = \sum_i s_i$, $r_1 = s_1, \cdots, r_i = s_i$ and $r_{i+1} < s_{i+1}$.

COROLLARY 2. *$A\sharp U(\mathfrak{g})$ is a free left and right A-module with the standard monomials as a basis.*

PROOF. Since the standard monomials form a k-basis of $U(\mathfrak{g})$ by the Poincaré-Birkhoff-Witt theorem, $A\sharp U(\mathfrak{g})$ is a free left A-module with the standard monomials as a basis. Suppose $\sum_{i\in I} x_i a_i = 0$, where all $x_i, i \in I$ are standard monomials and all $a_i, i \in I$ are nonzero elements of A. Assume that x_1 is the maximal element among all $x_i, i \in I$. Since $\Delta(x) = x \otimes 1 + 1 \otimes x$ for all $x \in \mathfrak{g}$, we have $x_i a_i = a_i x_i + (*)$ for all i, where $(*)$ is an A-combination of standard monomials less than x_i, thus

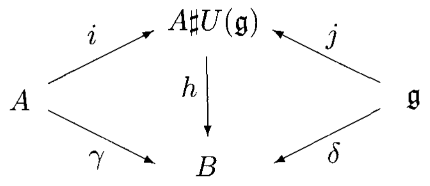
$$0 = \sum_i x_i a_i = a_1 x_1 + (**),$$

where $(**)$ is an A-combination of standard monomials less than x_1. Since $A\sharp U(\mathfrak{g})$ is a free left A-module with standard monomials as a basis, $a_1 = 0$, a contradiction. It completes the proof. □

LEMMA 3. *Let $i : A \longrightarrow A\sharp U(\mathfrak{g})$ be the algebra homomorphism defined by $i(a) = a$ and let $j : \mathfrak{g} \longrightarrow (A\sharp U(\mathfrak{g}))_L$ be the Lie homomorphism defined by $j(x) = x$. Then the triple $(A\sharp U(\mathfrak{g}), i, j)$ satisfies the following condition: Given an algebra B, $\gamma : A \longrightarrow B$ an algebra homomorphism and $\delta : \mathfrak{g} \longrightarrow B_L$ a Lie homomorphism, let the pair (γ, δ) satisfy*

$$\gamma(x \cdot a) = [\delta(x), \gamma(a)] \tag{2}$$

for all $x \in \mathfrak{g}$ and $a \in A$. Then there exists a unique algebra homomorphism h from $A\sharp U(\mathfrak{g})$ into B such that $hi = \gamma$ and $hj = \delta$.



PROOF. Since δ is a Lie homomorphism there exists a unique algebra homomorphism g from $U(\mathfrak{g})$ into B such that $g(x) = \delta(x)$ for all $x \in \mathfrak{g}$. By Lemma 1, it is enough to show that the pair (γ, g) satisfies (1). For any $a \in A$ and $x \in \mathfrak{g}$, since $\Delta(x) = x \otimes 1 + 1 \otimes x$,

$$\begin{aligned}\sum \gamma(x' \cdot a)g(x'') &= \gamma(x \cdot a)g(1) + \gamma(1 \cdot a)g(x) \\ &= [\delta(x), \gamma(a)] + \gamma(a)g(x) = g(x)\gamma(a),\end{aligned}$$

as claimed. □

Let A be an algebra with a filtration $\{F_j | j = 0, 1, \cdots\}$. The filtration of A is said to be standard if $F_1^n = F_n$ for each n.

THEOREM 4. *Let A be a finitely generated commutative algebra and let $\mathfrak{g}$ be a finite dimensional Lie algebra such that there exists a Lie homomorphism from $\mathfrak{g}$ into $Der_k(A)$. Then the Gelfand-Kirillov dimension of $A\natural U(\mathfrak{g})$ is equal to $GKdim(A) + dim(\mathfrak{g})$, where $GKdim(A)$ is the Gelfand-Kirillov dimension of A.*

PROOF. Let $x_1, x_2, \cdots, x_n$ form a k-basis of the Lie algebra $\mathfrak{g}$. We set

$$\deg(a) = 0, \quad \deg(x_i) = 1$$

for all $a \in A$ and $i = 1, 2, \cdots, n$. Note, by Corollary 2, that $A\natural U(\mathfrak{g})$ is a free left and right A-module with the standard monomials as a basis. Let F_j be the left A-combinations of monomials with degree less than or equal to j, where $j = 0, 1, \cdots$. Then $\{F_j | j = 0, 1, \cdots\}$ is a standard filtration of $A\natural U(\mathfrak{g})$ and its associated graded algebra $\mathrm{Gr}(A\natural U(\mathfrak{g})) = \bigoplus_j (F_j/F_{j-1})$ is isomorphic to the polynomial ring $A[x_1, \cdots, x_n]$ since, for all standard monomials X and $a \in A$, $Xa = aX + (*)$, where $(*)$ is an A-combination of standard monomials less than X. Hence the Gelfand-Kirillov dimension of $\mathrm{Gr}(A\natural U(\mathfrak{g}))$ is equal to $\mathrm{GKdim}(A) + \dim(\mathfrak{g})$ by [5, 8.2.7(iii)], and thus the conclusion follows from [5, 8.1.14] immediately. □

3. Poisson enveloping algebras

Recall that a commutative k-algebra A with bilinear map $\{\cdot,\cdot\}$ is said to be a Poisson algebra if $(A,\{\cdot,\cdot\})$ is a Lie algebra such that

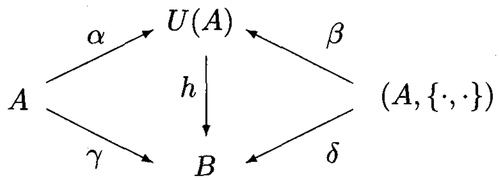
$$\{a,bc\}=b\{a,c\}+\{a,b\}c$$

for all $a,b,c\in A$.

Let us recall the definition for Poisson enveloping algebra in [6, Definition 3]. Let A be a Poisson algebra. A triple $(U(A),\alpha,\beta)$, where $U(A)$ is an algebra, α is an algebra homomorphism from A into $U(A)$ and β is a Lie homomorphism from $(A,\{\cdot,\cdot\})$ into $U(A)_L$ such that (α,β) satisfies

$$\alpha(\{a,b\})=[\beta(a),\alpha(b)],\qquad \beta(ab)=\alpha(a)\beta(b)+\alpha(b)\beta(a) \tag{3}$$

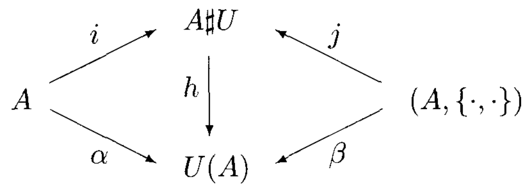
for all $a,b\in A$, is called a Poisson enveloping algebra of A if the following holds: If B is an algebra, γ is an algebra homomorphism from A into B and δ is a Lie homomorphism from $(A,\{\cdot,\cdot\})$ into B_L such that (γ,δ) satisfies (3), there exists a unique algebra homomorphism h from $U(A)$ into B such that $h\alpha=\gamma$ and $h\beta=\delta$.



Note that α is a monomorphism by [7, Proposition 2.2].

THEOREM 5. *Every Poisson enveloping algebra is a homomorphic image of a skew enveloping algebra.*

PROOF. Let a triple $(U(A),\alpha,\beta)$ be the Poisson enveloping algebra of a Poisson algebra A and let U be the universal enveloping algebra of the Lie algebra $(A,\{\cdot,\cdot\})$. We will verify that there exists a unique epimorphism h from $A\sharp U$ onto $U(A)$ such that $hi=\alpha$ and $hj=\beta$, where $i:A\longrightarrow A\sharp U, i(a)=a$ and $j:(A,\{\cdot,\cdot\})\longrightarrow U, j(x)=x$.



Since $x \cdot a = \{x, a\}$ and $\alpha(\{x, a\}) = [\beta(x), \alpha(a)]$ for $x, a \in A$ by (3), we have that the pair (α, β) satisfies (2). Hence there exists a unique homomorphism h from $A\natural U$ into $U(A)$ such that $hi = \alpha$ and $hj = \beta$ by Lemma 3. Moreover h is surjective since $U(A)$ is generated by $\alpha(A)$ and $\beta(A)$ by the proof of [6, Theorem 5], as claimed. □

ACKNOWLEDGEMENT. The second author is supported by the Chungnam National University Research Grant.

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