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Fuzzy equivalence relations and transformations

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Abstract

We investigate the properties of A-transformations, P-transformations and L-transformations in metric spaces, t-norms and T-fuzzy equivalence relations.

Key Words : A-transformations, P-transformations, L-transformations, (quasi-, psudo-) metric spaces, t-norms, T-fuzzy (quai-) equivalence relations.

1. Introduction and Preliminaries

The concept of fuzzy equality [1-4], which is also called the equality relation, the fuzzy equivalence relation, the similarity relation[5,12], the indistinguishability operator[7,11], has a significant concern in various fields[6,10]. It is a graded equality being generalization of the classical equality. We understand two objects to be approximately equal if they are similar. The degree 0 means that objects are completely different and the degree 1 means that objects are indistinguishable.

In this paper, we investigate the properties of A-transformations, P-transformations and L-transformations in metric spaces, t-norms and T-fuzzy equivalence relations. We construct metrics by A-transformations. From A-(resp. P-, L-transformations, we compare t-norms generated by A-(resp. P-, L-) generators. In particular, we investigate the relationship between T-fuzzy quasi-equivalence relations and A-(resp. P-, L-) transformations.

A binary operation $T:[0,1]\times[0,1]\rightarrow[0,1]$ is called a t-norm if it satisfies the following conditions:
for each $x,y,z\in[0,1]$,
(T1) $T(x,y)=T(y,z)$,
(T2) $T(x,T(y,z))=T(T(x,y),z)$
(T3) $T(x,1)=x$,
(T4) if $y\leq z$, then $T(x,y)\leq T(x,z)$.
We denote $T(x,y)=x\odot y$. A t-norm T_1 is called weaker than a t-norm T_2(T_2 is called stronger than a t-norm T_1, denoted by $T_1\leq T_2$, if $T_1(x,y)\leq T_2(x,y)$.
An increasing function $f:[0,1]\rightarrow[0,1]$ is called a P-generator if $T_f(x,y)=f^{-1}(f(x)f(y)\vee f(0))$ is a t-norm.
A strictly decreasing $g:[0,1]\rightarrow[0,\infty]$ is called an A-generator if $T_g(x,y)=g^{-1}((g(x)+g(y))\wedge g(0))$ is a t-norm.
A bijective increasing $h:[0,1]\rightarrow[0,1]$ is called an L-generator if $T_h(x,y)=h^{-1}((h(x)+h(y)-1)\wedge 0)$ is a t-norm.

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Theorem 1.1 [9] If T is an Archimedean t-norm, then there is an increasing continuous $f:[0,1]\rightarrow[f(0),1]$ such that $x\odot y=f^{-1}(f(x)f(y)\vee f(0))$ for all $x,y\in[0,1]$. If $g:[0,1]\rightarrow[g(0),1]$ is an order isomorphism, then $x\odot y=g^{-1}(g(x)g(y)\vee g(0))$ for all $x,y\in[0,1]$ iff $f(x)=g(x)^{r}$ for some $r>0$.

Theorem 1.2 [8] If T is an Archimedean t-norm, then there is a strictly decreasing continuous function $f:[0,1]\rightarrow[0,\infty]$ such that $x\odot y=f^{-1}((f(x)+f(y))\wedge f(0))$ for all $x,y\in[0,1]$. If $g:[0,1]\rightarrow[0,\infty]$ is an order reversing continuous function, then $x\odot y=g^{-1}((g(x)+g(y))\wedge g(0))$ for all $x,y\in[0,1]$ iff $f(x)=ag(x)$ for some $a>0$.

Theorem 1.3 [9] If T is an Archimedean nilpotent t-norm, then there is a bijective continuous function $h:[0,1]\rightarrow[0,1]$ with $x\odot y=h^{-1}((h(x)+h(y)-1)\vee 0)$ for all $x,y\in[0,1]$.

Definition 1.4 A mapping $d:X\times X\rightarrow[0,\infty]$ is called a quasi-metric if it satisfies the following conditions: for each $x,y,z\in X$,
(M1) $d(x,x)=0$,
(M2) $d(x,z)\leq d(x,y)+d(y,z)$.
A quasi-metric d on X is called a pseudo-metric on X if it satisfies:
(M3) $d(x,y)=d(y,x)$, for each $x,y\in X$.
A pseudo-metric d on X is called a metric on X if it satisfies:
(M) if $d(x,y)=0$ for each $x,y\in X$, then $x=y$.

2. Metrics, t-norms and transformations

Definition 2.1 An increasing map $s:[0,b]\rightarrow[0,\infty]$ is called an $A-$transformation if it is sub-additive ,i.e. $s(x)+s(y)\geq s(x+y)$ and $s(0)=0$.

Lemma 2.2 A map $s:[0,b]\to[0,\infty]$ with $s(0)=0$ is an $A-$transformation iff for all $x,y,z\in[0,b]$ with $x+y\geq z$, we have $s(x)+s(y)\geq s(z)$

Proof. Let s be an $A-$transformation. Then for all $x,y,z\in[0,b]$ with $x+y\geq z$, $s(x)+s(y)\geq s(x+y)\geq s(z)$
Put $z=x+y$. Then $s(x)+s(y)\geq s(z)=s(x+y)$. If $x+0\geq y$, then $s(x)=s(x)+s(0)\geq s(y)$ Thus, s is an increasing map.

Theorem 2.3 (1) If $d:X\times X\to[0,\infty]$ is a quasi-metric (resp. pseudo-metric) and $s:[0,b]\to[0,\infty]$ is $A-$transformation, then $e=s\circ d$ is a quasi-metric (resp. pseudo-metric).
(2) If $d:X\times X\to[0,\infty]$ is a metric and $s:[0,b]\to[0,\infty]$ is $A-$transformation such that $s(x)>0$ for all $0<x\leq 1$, then $e=s\circ d$ is a metric.

Proof. (1) Since $d(x,z)\leq d(x,y)+d(y,z)$,
$e(x,z)=s(d(x,z))\leq s(d(x,y))+s(d(y,z))=e(x,y)+e(y,z)$.
(2) $e(x,y)=0$ iff $s(d(x,y))=0$ iff $d(x,y)=0$ iff $x=y$.

Example 2.4 Let $s:[0,\infty)\to[0,\infty]$ a map defined as $s(x)=a\wedge x$ for $a>0$. s is $A-$transformation because $s(x+y)=a\wedge(x+y)\leq(a\wedge x)+(a\wedge y)=s(x)+s(y)$ Let d be a metric. Then $e(x,y)=s(d(x,y))=a\wedge d(x,y)$ is a metric.

Example 2.5 Let $s:[0,\infty)\to[0,\infty]$ a map defined as $s(x)=x^{p}$, $0<p\leq 1$. Put $h(t)=(t+1)^{p}-t^{p}-1$. Since $h'(t)=p(t+1)^{p-1}-pt^{p-1}\leq 0$ for all $t\geq 0$ and $h(0)=0$, then $h(t)=(t+1)^{p}-t^{p}-1\leq 0$. Put $t=\frac{y}{x}$. It follows $(x+y)^{p}\leq x^{p}+y^{p}$ Thus s is $A-$transformation. Let d be a metric. Then $e(x,y)=s(d(x,y))=d^{p}(x,y)$ for all $0<p\leq 1$ is a metric.

Example 2.6 Let $s:[0,\infty)\to[0,\infty]$ a map defined as $s(x)=\frac{px}{x+1}$, $p>0$. Since $\frac{p(x+y)}{x+y+1}\leq\frac{px}{x+1}+\frac{py}{y+1}$
Thus s is $A-$transformation. Let d is a metric.
Then $e(x,y)=s(d(x,y))=\frac{pd(x,y)}{1+d(x,y)}$ for all $p>0$ is a metric.

Example 2.7 Let $s:[0,\infty)\to[0,\infty]$ a map defined as $s(x)=\ln(x+1)$. Since $\ln(x+y+1)\leq\ln(x+1)+\ln(y+1)$ then s is $A-$transformation. Let d is a metric. Then $e(x,y)=s(d(x,y))=\ln(1+d(x,y))$ is a metric.

Theorem 2.8 Let $f,g:[0,1]\to[0,\infty]$ be an $A-$generators. Then gf^{-1} is $A-$transformation iff $T_f\geq T_g$.

Proof. Let $gf^{-1}(a+b)\leq gf^{-1}(a)+gf^{-1}(b)$ be given. Put $f^{-1}(a)=x$ and $f^{-1}(b)=y$.
$gf^{-1}(f(x)+f(y))\leq g(x)+g(y)$.
Since g^{-1} is decreasing and $f(x)+f(y)>f(0)$ implies
$T_f(x,y)=f^{-1}((f(x)+f(y))\wedge f(0))$
$\geq g^{-1}((g(x)+g(y))\wedge g(0))=T_g(x,y)$.

Conversely, it is similarly proved.

Example 2.9 (1) Let $s(x)=x^{p}$, $0<p\leq 1$ be $A-$transformation. Put $gf^{-1}(x)=s(x)=x^{p}$ and $f(x)=1-x$. Then $T_f(x,y)=(x+y-1)\vee 0$ and $g(x)=(1-x)^{p}$. We obtain

$$T_g(x,y)=\{1-((1-x)^{p}+(1-y)^{p})^{\frac{1}{p}}\}\vee 0.$$

(2) Let $s(x)=\frac{px}{x+1}$, $p>0$ be $A-$transformation. Put $gf^{-1}(x)=\frac{px}{x+1}$ and $f(x)=1-x$. Then $T_f=(x+y-1)\vee 0$ and $g(x)=\frac{p(1-x)}{2-x}$. We obtain $T_g(x,y)=\frac{xy}{2-x-y+xy}$.
(3) Let $f,g:[0,1]\to[0,\infty]$ be mappings defined by $f(x)=\frac{1}{x}-1$, $g(x)=-\ln x$. Then we obtain $A-$transformation $g(f^{-1}(x))=\ln(x+1)$. Then $T_f(x,y)=\frac{xy}{x+y-xy}\geq xy=T_g(x,y)$.

Definition 2.10 A non-decreasing map $h:[a,1]\to[0,1]$ for some $a\in[0,1)$ is called $P-$transformation if it satisfies $h(xy)\geq h(x)h(y)$ and $h(1)=1$.

Lemma 2.11 A map $h:[a,1]\to[0,1]$ for some $a\in[0,1)$ with $h(1)=1$ is $P-$transformation iff for all $x,y,z\in[a,1]$ with $xy\leq z$, we have $h(x)h(y)\leq h(z)$.

Proof. Let h be $P-$transformation. Then for all $x,y,z\in[a,1]$ with $xy\leq z$, $h(x)h(y)\leq h(xy)\leq h(z)$
Put $z=xy$. Then $h(x)h(y)\leq h(z)=h(xy)$. If $x1\leq y$, then $h(x)=h(x)h(1)\leq h(y)$. Thus, h is a non-decreasing map.

Theorem 2.12 Let T_f and T_g be a t-norm with $P-$generators f and g. Then gf^{-1} is $P-$transformation iff $T_f\geq T_g$.

Proof. Let $gf^{-1}(ab)\geq gf^{-1}(a)gf^{-1}(b)$ be given. Put $f^{-1}(a)=x$ and $f^{-1}(b)=y$. $gf^{-1}(f(x)f(y))\geq g(x)g(y)$.
Since g is strictly increasing and $f(x)f(y)\leq f(0)$ implies $g(0)\geq gf^{-1}(f(x)f(y))\geq g(x)g(y)$,
$T_f(x,y)=f^{-1}(f(x)f(y)\vee f(0))$
$\geq g^{-1}((g(x)g(y)\vee g(0))=T_g(x,y)$.

Example 2.13 Let $gf^{-1}:[0,1]\to[0,1]$ be a map defined as $gf^{-1}=a^{x-1}$, $a>1$. Then $gf^{-1}(x)gf^{-1}(y)\leq gf^{-1}(xy)$. Put $f(x)=x$. Then $g(x)=a^{x-1}$ and $g^{-1}(x)=(\log_a x)-1$. We obtain
$T_g(x,y)=g^{-1}(g(x)g(y)\vee g(0))=(x+y-1)\vee 0$.
$T_f(x,y)=xy\geq T_g(x,y)$.

Definition 2.14 A nondecreasing map $h:[0,1]\to[0,1]$ is called an $L-$transformation if it satisfies $h(x+y-1)\geq h(x)+h(y)-1$ with $h(0)=1$ and $h(0)=0$.
Lemma 2.15 A map $h:[0,1]\to[0,1]$ with $h(0)=1$ and

$h(0)=0$ is an $L-$transformation iff for all $x, y, z\in[0,1]$ with $x+y-1\leq z$, we have $h(x)+h(y)-1\leq h(z)$.

Proof. Let h be an $L-$transformation. Then for all $x, y, z\in[0,1]$ with $x+y-1\leq z$,

$$h(x)+h(y)-1\leq h(x+y-1)\leq h(z)$$

Put $z=x+y-1$. Then

$$h(x)+h(y)-1\leq h(z)=h(x+y-1).$$

If $x+1-1\leq y$, then $h(x)=h(x)+h(1)-1\leq h(y)$. Thus, h is a non-decreasing map.

Theorem 2.15. Let T_f and T_g be a nilpotent t-norm with $L-$generators f and g.
Then gf^{-1} is $L-$transformation iff $T_f\geq T_g$.

Proof. It is similarly proved as in Theorem 2.12.

Example 2.16 Let $gf^{-1}:[0,1]\to[0,1]$ be a map defined as $gf^{-1}(x)=x^n$, $n\in N$.
Put $h(x,y)=(x+y-1)^n-x^n-y^n+1$. Then $(1,1)$ is a saddle point because

$$h_x(1,1)=h_y(1,1)=0,\ h_{xx}=h_{yy}=0,\ h_{xy}=n(n-1)$$

If $n=2m$, since $h(x,0)$ and $h(0,y)$ are decreasing with $h(1,0)=h(0,1)=0$ and $h(1,x)=h(y,1)=0$. Hence $h(x,y)\geq 0$.

If $n=2m+1$, since $h'(\frac{1}{2},0)=h'(0,\frac{1}{2})=0$ with $h(0,0)=h(1,0)=h(0,1)=0$ and

$$h(\frac{1}{2},0)=h(0,\frac{1}{2})=1-2(\frac{1}{2})^{2m+1}\geq 0.$$

Hence $h(x,y)\geq 0$.
So, $gf^{-1}(f(x)+f(y)-1)\leq g(x)+g(y)-1$.
Put $f(x)=x$. Then $g(x)=x^n$. We obtain

$T_g(x,y)=(x^n+y^n-1)\vee 0)^{\frac{1}{n}}$. Furthermore,
$T_f(x,y)=0\vee(x+y-1)\geq T_g(x,y)$.

3. T-fuzzy equivalence relations

Definition 3.1[1-4] A map $E:X\times X\to[0,1]$ is called a $T-$fuzzy quasi-equivalence relation on X if the following properties hold:
(E1) $E(x,x)=1$, for each $x\in X$.
(E2) $T(E(x,y),E(y,z))\leq E(x,z)$, for each $x,y,z\in X$
A $T-$fuzzy quasi-equivalence relation is called a $T-$fuzzy equivalence relation on X if it satisfies:
(E3) $E(x,y)=E(y,x)$, for each $x,y\in X$.
A $T-$fuzzy equivalence relation is called a $T-$ fuzzy equality on X if it satisfies:
(E) if $E(x,y)=1$ for each $x,y\in X$, then $x=y$.
Let E_1, E_2 be $T-$fuzzy quasi-equivalence relations on X. Then E_1 is called coarser than E_1
if $E_1(x,y)\leq E_2(x,y)$ for each $x,y\in X$.

Remark 3.2 (1) If a t-norm T_1 is weaker than a t-norm T_2, then a T_2-fuzzy (quasi-)equivalence E on X is a T_1-fuzzy (quasi-)equivalence E on X. Thus, $\wedge$-fuzzy (quasi-)equivalence E on X is a $T-$fuzzy (quasi-)equivalence E on X because $T(x,y)\leq x\wedge y$ for every t-norm T.
(2) The condition (E2) is equivalent to the following condition: for each distinct $x,y,z\in X$,
(E2-1) $T(E(x,y),E(y,z))\leq E(x,z)$.

Theorem 3.3 Let E be a T-fuzzy quasi-equivalence relation on X. Then:
(1) if $a\in(0,1)$, then $G(x,y)=E(x,y)\vee a$ for all $x,y\in X$, then G is a T-fuzzy quasi-equivalence relation on X.
(2) If $E^{-1}(x,y)=E(y,x)$ for all $x,y\in X$, then $F(x,y)=T(E(x,y),E^{-1}(x,y))$ is G-fuzzy equivalence relation on X, for any t-norm G such that $G\leq T$.
(3) $F(x,y)=E(x,y)\wedge E^{-1}(x,y)$ is the finest $T-$fuzzy equivalence relation on X which is coarser than E and E^{-1}.

Proof. (1)
$T(G(x,y),G(y,z))$
$=T(E(x,y)\vee a,E(y,z)\vee a)$
$=T(E(x,y),E(y,z))\vee T(E(x,y),a)\vee T(a,E(y,z))\vee T(a,a)$
$\leq T(E(x,y),E(y,z))\vee a$
$\leq E(x,z)\vee a=G(x,z)$.
(2)
$G(F(x,y),F(y,z))$
$\leq G(T(E(x,y),E(y,x)),T(E(y,z),E(z,y)))$
$\leq T(T(E(x,y),E(y,x)),T(E(y,z),E(z,y))$
$=T(T(E(x,y),E(y,z)),T(E(z,y),E(y,x)))$
$=T(T(E(x,y),E(y,z)),T(E(z,y),E(y,x)))$
$=T(E(x,z),E(z,x))=F(x,z)$.
(3) For $T=\wedge$ in (2), it is easy. If $H\leq E$ and $H\leq E^{-1}$, then $H\leq F=E\wedge E^{-1}$.

In above theorem, $E_1\vee E_2$ cannot be $T-$fuzzy quasi-equivalence relation on X, in general.

Example 3.4 Let $X=\{x,y,z\}$ be a set. and Define E_1, E_2 as

$E_1(x,x)=E_1(y,y)=E_1(z,z)=1$
$E_1(x,y)=E_1(y,x)=0.4,\ E_1(x,z)=E_1(z,x)=0.8$
$E_1(y,z)=E_1(z,y)=0.4$
$E_2(x,x)=E_2(y,y)=E_2(z,z)=1$
$E_2(x,y)=E_2(y,x)=0.6,\ E_2(x,z)=E_2(z,x)=0.3$
$E_2(y,z)=E_2(z,y)=0.3$.

Then E_1 and E_2 are $\wedge$-fuzzy quasi-equivalence relations on X. We can obtain $E_1\vee E_2$ on X as follows

$E_1\vee E_2(x,x)=E_1\vee E_2(y,y)=E_1\vee E_2(z,z)=1$
$E_1\vee E_2(x,y)=E_1\vee E_2(y,x)=0.4$,
$E_1\vee E_2(x,z)=E_1\vee E_2(z,x)=0.8$
$E_1\vee E_2(y,z)=E_1\vee E_2(z,y)=0.4$.

But $E_1 \vee E_2$ is not a $\wedge$-fuzzy quasi-equivalence relations on X because

$0.4 = (E_1 \vee E_2)(y, z) \ngeq (E_1 \vee E_2)(y, x) \wedge (E_1 \vee E_2)(x, z) = 0.6.$

Theorem 3.5 Let E be a T_f-fuzzy equivalence relation on X where T_f is a Archimedean t-norm with $A-$generator f and $\phi: [0,1] \to [0,1]$ an increasing function with $\phi(1) = 1$. Then the following statements are equivalent:

(1) $\phi \circ E$ is a T_f-fuzzy equivalence relation on X.

(2) $T_f(\phi(a), \phi(b)) \leq \phi(c)$ for $T_f(a, b) \leq c$.

(3) $f(\phi(a)) + f(\phi(b)) \geq f(\phi(c))$ for each $f(a) + f(b) \geq f(c)$

(4) There exists an $A-$transformation s such that $s = f \circ \phi \circ f^{-1}$.

Proof. (1) $\Rightarrow$(2). Let $E(x, y) = a, E(y, z) = b, E(x, z) = c$. Since E is a T_f-fuzzy equivalence relation, $T_f(a, b) \leq c$. We have $T_f(\phi(a), \phi(b)) \leq \phi(c)$.

(2 $\Rightarrow$(3). Let $f(a) + f(b) \geq f(c)$ be given.

If $f(a) + f(b) > f(0) \geq f(c)$, then $T_f(a, b) \leq c$.

If $f(0) \geq f(a) + f(b) \geq f(c)$, then $T_f(a, b) \leq c$

Thus $T_f(a, b) \leq c$. By (2),

$T_f(\phi(a), \phi(b)) = f^{-1}(f(\phi(a)) + f(\phi(b)) \wedge f(0)) \leq \phi(c)$.

If $f(\phi(a)) + f(\phi(b)) > f(0)$, since $f(0) \geq f(\phi(c))$, $f(\phi(a)) + f(\phi(b)) \geq f(\phi(c))$

If $f(\phi(a)) + f(\phi(b)) \leq f(0)$, $f(\phi(a)) + f(\phi(b)) \geq f(\phi(c))$.

(3) $\Rightarrow$(4). By (3), put $f(a) = x$, $f(b) = y, f(c) = z$, and $s = f \circ \phi \circ f^{-1}$. For each $x + y \geq z$, we have $s(x) + s(y) \geq s(z), s(0) = 0$ and s is increasing. Hence s is an $A-$transformation.

(4) $\Rightarrow$(1). Let $\phi = f^{-1} \circ s \circ f$ with an $A-$transformation s, then ϕ is an increasing map with $\phi(1) = 1$. We will show $\phi \circ E$ is T_f-fuzzy equivalence relation on X. So $\phi(E(x, x)) = \phi(1) = 1$. Since $T_f(a, b) = f^{-1}(f(a) + f(b) \wedge f(0)) \leq c$, we can prove two cases (A) $f(a) + f(b) \leq f(0)$ (B) $f(a) + f(b) > f(0)$.

(Case A) If $f(a) + f(b) \leq f(0)$, then $f(a) + f(b) \geq f(c)$

(I) If $f(\phi(a)) + f(\phi(b)) \leq f(0)$, then

$$\begin{aligned}
&T_f(\phi(a), \phi(b)) \\
&= f^{-1}(f(\phi(a)) + f(\phi(b)) \wedge f(0)) \\
&= f^{-1}(f(f^{-1} \circ s \circ f(a)) + f(f^{-1} \circ s \circ f(b))) \\
&= f^{-1}(s \circ f(a) + s \circ f(b)) \\
&\leq f^{-1}(s(f(a) + f(b))) \\
&\leq f^{-1}(s \circ f(c)) = \phi(c).
\end{aligned}$$

(II) If $f(\phi(a)) + f(\phi(b)) > f(0)$, then $T(\phi(a), \phi(b)) = 0$

(Case B) If $f(a) + f(b) > f(0)$, then

(I) If $s(f(a)) > f(0)$ or $s(f(b)) > f(0)$, then $f^{-1} \circ s \circ f(a) = 0$ or $f^{-1} \circ s \circ f(b) = 0$. Thus,

$$0 = T_f(\phi(a), \phi(b)) \leq \phi(c)$$

(II) If $s(f(a)) \leq f(0)$ and $s(f(b)) \leq f(0)$, we prove similarly as (I) in (Case A).

Theorem 3.6 Let E be a T_f-fuzzy equivalence relation on X where T_f is a Archimedean t-norm with $A-$generator f and $\phi: [0,1] \to [0,1]$ an increasing function with $\phi(1) = 1$. For all $x, y \in [0,1]$,

$\phi(T_f(x, y)) \geq T_f(\phi(x), \phi(y))$ iff $\phi \circ E$ is a T_f-fuzzy equivalence relation on X.

Proof. ($\Rightarrow$). For each $T_f(a, b) \leq c$,

$$\phi(c) \geq \phi(T_f(a, b)) \geq T_f(\phi(a), \phi(b)).$$

By Theorem 3.5(2), $\phi \circ E$ is a T_f-fuzzy equivalence relation.

($\Leftarrow$) Suppose there exists $x, y \in [0,1]$ such that

$$\phi(T_f(x, y)) \ngeq T_f(\phi(x), \phi(y))$$

There exists $c \in [0,1]$ such that

$$\phi(T_f(x, y)) \leq c < T_f(\phi(x), \phi(y))$$

It follows $T_f(x, y) \leq \phi^{-1}(c)$. So,

$T_f(\phi(x), \phi(y)) \leq \phi(\phi^{-1}(c)) = c$. It is a contradiction.

Example 3.7 Let $\phi: [0,1] \to [0,1]$ a map defined as $\phi(x) = \frac{1}{2 - x}$. Let E be a T_f-fuzzy equivalence relation on X such that $T_f(x, y) = (x + y - 1) \vee 0$ and $f(x) = 1 - x$. Then $s(x) = f^{-1}(\phi f(x))) = \frac{x}{1 + x}$.

So, s is an $A-$transformation from Example 2.9(2). Hence, $\phi(E(x, y)) = \frac{1}{2 - E(x, y)}$ is T_f-fuzzy equivalence relation on X.

Example 3.8 Let $s(x) = x^p, 0 < p \leq 1$ be an $A-$transformation. Let E be a T_f-fuzzy equivalence relation on X.

(1) $T_f(x, y) = (x + y - 1) \vee 0$ with $f(x) = k - kx$ for $k > 0$

We obtain

$$\phi E(x, y) = (f^{-1} \circ s \circ f)(E(x, y)) = 1 - \frac{1}{k}(k - kE(x, y))^p.$$

(2) $T_f(x, y) = xy$ with $f(x) = -\ln x$. We obtain

$$\phi E(x, y) = (f^{-1} \circ s \circ f)(E(x, y)) = e^{-(-\ln E(x, y))^p}.$$

Example 3.9 Let $s(x) = \frac{px}{x + 1}$ be an $A-$transformation. Let E be a T_f-fuzzy equivalence relation on X.

(1) $T_f(x, y) = (x + y - 1) \vee 0$ with $f(x) = 1 - x$. Then

$$\phi(E(x, y)) = (f^{-1} \circ s \circ f)(E(x, y)) = 1 - \frac{p(1 - E(x, y))}{2 - E(x, y)}.$$

(2) $T_f(x, y) = xy$ with $f(x) = -\ln x$. We obtain

$$\phi E(x, y) = (f^{-1} \circ s \circ f)(E(x, y)) = e^{p \ln E(x, y) \; 1 - \ln E(x, y)}.$$

Example 3.10 Let $s(x) = \ln(x + 1)$ be an $A-$transformation Let E be a T_f-fuzzy equivalence relation on X.

(1) $T_f(x, y) = (x + y - 1) \vee 0$ with $f(x) = 1 - x$. Then

$$\phi(E(x, y)) = (f^{-1} \circ s \circ f)(E(x, y)) = 1 - \ln(2 - E(x, y)).$$

(2) $T_f(x, y) = xy$ with $f(x) = -\ln x$.

We obtain $\phi E(x, y) = e^{-\ln(1 - \ln E(x, y))}$.

Theorem 3.11 Let E be a T_g-fuzzy equivalence relation on X where T_g is a Archimedean t-norm with $P-$generator g and $\phi: [0,1] \to [0,1]$ an increasing function with

$\phi(1)=1$. Then the following statements are equivalent:

(1) $\phi\circ E$ is a T_g-fuzzy equivalence relation on X.

(2) $T_g(\phi(a),\phi(b))\le\phi(c)$ for $T_g(a,b)\le c$.

(3) $g(\phi(a))g(\phi(b))\le g(\phi(c))$ for each $g(a)g(b)\le g(c)$

(4) There exists a $P-$transformation h such that $h=f\circ\phi\circ f^{-1}$.

Proof. By Lemma 2.11, it is similarly proved as in Theorem 3.5

Example 3.12 Let $h(x)=a^x-1(a>1)$ be a $P-$-transformation. Let E be a T_g-fuzzy equivalence relation on X with a $P-$generator $g(x)=x$.

We obtain $\phi\circ E$ a T_g-fuzzy equivalence relation $\phi\circ E$ as follows:

$\phi(E(x,y))=(g^{-1}\circ h\circ g)(E(x,y))=a^{E(x,y)}-1.$

Theorem 3.13 Let E be a T_g-fuzzy equivalence relation on X where T_g is a Archimedean t-norm with $L-$generator g and $\phi:[0,1]\to[0,1]$ an increasing function with $\phi(1)=1$. Then the following statements are equivalent:

(1) $\phi\circ E$ is a T_g-fuzzy equivalence relation on X.

(2) $T_g(\phi(a),\phi(b))\le\phi(c)$ for $T_g(a,b)\le c$.

(3) $g(\phi(a))+g(\phi(b))-1\le g(\phi(c))$ for each $g(a)+g(b)-1\le g(c)$

(4) There exists an $L-$transformation h such that $h=f\circ\phi\circ f^{-1}$.

Proof. By Lemma 2.15, it is similarly proved as in Theorem 3.5

Example 3.14 Let $h(x)=x^n(n\in N)$ be an $L-$transformation. Let E be a T_g-fuzzy equivalence relation on X with an $L-$generator $g(x)=x^2$.

We obtain $\phi\circ E$ a T_g-fuzzy equivalence relation $\phi\circ E$ as follows:

$\phi(E(x,y))=(g^{-1}\circ h\circ g)(E(x,y))=E(x,y)^n.$

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