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Invited Paper

Merging Two Regional Geoid Estimates by Using Optimal Variance Components of Type repro-BIQUUE: An Algorithmic Approach

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Abstract

When merging various datasets the perennial problem of relative weighting arises. In case of two datasets an iterative algorithm has been developed recently that allows the rigorous determination of optimal variance components of type repro-BIQUUE even for large amounts of data, along with the estimation of the joint parameters. Here we shall present this new algorithm, and show its versatility in an example that will entail the merging of two regional geoid estimates (derived from EGM 96 and CHAMP) in terms of certain series expansions which have been proven previously to belong to the most efficient ones (e.g., wavelets, Hardy's multi-quadrics, etc.). Future attempts will be devoted to the sequential merging of altimeter and tide gauge data.

Keywords : *Variance components, repro-BIQUUE, multi-quadrics, geoid estimates*

1. Introduction

In geodetic science, it is a very common task to find a combined solution for something that had been measured twice (or even more frequently) by different methods, exhibiting their very own observational error characteristics. Consequently, the respective *variance components* for each dataset, not to mention the covariance component in case of stochastically dependent data, ought to be determined along with the parameter estimation. For this purpose, Schaffrin (1983) had proposed the non-iterative *repro-BIQUUE principle* that leads to *nonlinear normal equations* for the estimated variance components and, as a result, to *nonlinear estimators* of the parameters; these, however, may still turn out to be unbiased for *symmetrically* distributed observational errors.

Related to the repro-BIQUUE approach, we face essentially *two problems* that must be solved before an efficient algorithm can be created:

(i) For the set-up of the nonlinear normal equations from which the estimated variance components are to be derived, *numerous traces* need to be calculated of matrices with total-data size, (e.g., three traces for two components, six traces for three components, etc); and:

(ii) A more *suitable iteration scheme* than the "iterated MINQUE" - which latter does indeed provide the repro-BIQUUE in case of convergence, according to Schaffrin (1983), - must be formed in order to widen the "convergence interval" for the initial values and reduce the number of iterations while reducing the costs per iteration step.

In order to circumvent the *first problem (i)*, we succeeded in *rephrasing* the normal equations for the estimated variance components *equivalently* in such a way that, in the case of two components, *only one trace* must be calculated of a *matrix the size of the parameters*. Moreover, if applied *sequentially*, for each additional dataset and its component *one more trace* of a matrix with parameter size needs to be evaluated, - if necessary by one of the randomized methods as proposed by Hutchinson (1990), or Koch and Kusche (2002).

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To solve the *second problem (ii)*, we developed a number of new algorithms for the "*rephrased normal equations*", but will concentrate here on just one of them that appeared to be the most promising to us at this stage where we try to merge CHAMP data with the well established EGM 96 gravity potential field and to determine an "optimal geoid" from both sources.

Obviously, at a later stage, comparisons ought to be done with other existing algorithms, including those by Crocetto et al. (2000) and by Kusche (2003), with emphasis on the dependence of the variance ratio on the chosen resolution. But this is beyond the scope of the present contribution.

2. A Review of the repro-BIQUUE Equations

Let us assume that two datasets have been collected independently to explain a joint set of unknown parameters. Then, a suitable model may read:

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} = \begin{bmatrix} \underset{n_1 \times m}{\mathbf{A}_1} \\ \underset{n_2 \times m}{\mathbf{A}_2} \end{bmatrix} \xi + \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix} = \mathbf{A}\xi + \mathbf{e}, \quad \mathbf{e} : \left(0,\ \Sigma = \sigma_1^2 \mathbf{V}_1 + \sigma_2^2 \mathbf{V}_2\right), \tag{1}$$

with rk $\mathbf{A} = m < n := n_1 + n_2$,

$$\underset{n\times n}{\mathbf{V}_1} := \begin{bmatrix} \mathbf{Q}_1 & 0 \\ 0 & 0 \end{bmatrix}, \text{ and } \underset{n\times n}{\mathbf{V}_2} := \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{Q}_2 \end{bmatrix}. \tag{2}$$

Apparently, this turns out to be a standard *Variance Component Model (VCM)* which has been treated extensively before, e.g. by Rao and Kleffe (1988) or Searle et al. (1992). When following the approach by Schaffrin (1983) that is based on the non-iterative repro-BIQUUE principle, the following *nonlinear normal equations* result formally:

$$\begin{bmatrix} \mathrm{tr}\left(\hat{\mathbf{W}}\mathbf{V}_1\hat{\mathbf{W}}\mathbf{V}_1\right) & \mathrm{tr}\left(\hat{\mathbf{W}}\mathbf{V}_1\hat{\mathbf{W}}\mathbf{V}_2\right) \\ \mathrm{tr}\left(\hat{\mathbf{W}}\mathbf{V}_1\hat{\mathbf{W}}\mathbf{V}_2\right) & \mathrm{tr}\left(\hat{\mathbf{W}}\mathbf{V}_2\hat{\mathbf{W}}\mathbf{V}_2\right) \end{bmatrix} \begin{bmatrix} \hat{\sigma}_1^2 \\ \hat{\sigma}_2^2 \end{bmatrix} = \begin{bmatrix} \mathbf{y}^T\hat{\mathbf{W}}\mathbf{V}_1\hat{\mathbf{W}}\mathbf{y} \\ \mathbf{y}^T\hat{\mathbf{W}}\mathbf{V}_2\hat{\mathbf{W}}\mathbf{y} \end{bmatrix}, \tag{3}$$

where,

$$\hat{\mathbf{W}} = \hat{\Sigma}^{-1} - \hat{\Sigma}^{-1}\mathbf{A}\left(\mathbf{A}^T\hat{\Sigma}^{-1}\mathbf{A}\right)^{-1}\mathbf{A}^T\hat{\Sigma}^{-1} \tag{4}$$

is the "*reduced weight matrix*", but derived from the *estimated* dispersion matrix

$$\hat{\Sigma} = \hat{\sigma}_1^2\mathbf{V}_1 + \hat{\sigma}_2^2\mathbf{V}_2 = \hat{\sigma}_2^2\left(\mathbf{V}_2 + \hat{\lambda}_{12}\mathbf{V}_1\right) \tag{5}$$

for $\hat{\lambda}_{12} = \hat{\sigma}_1^2 / \hat{\sigma}_2^2$, respectively *its inverse*

$$\hat{\Sigma}^{-1} = \hat{\sigma}_2^{-2}\left(\mathbf{V}_2 + \hat{\lambda}_{12}\mathbf{V}_1\right)^{-1} \tag{6}$$

which may be further modified if necessary. We note that $(\hat{\Sigma}\cdot\hat{\mathbf{W}})$ would project the combined data vector $\mathbf{y}$ into the *residual vector*

$$\tilde{\mathbf{e}} := \left(\hat{\Sigma}\hat{\mathbf{W}}\right)\mathbf{y} = \hat{\sigma}_1^2\left(\mathbf{V}_1\hat{\mathbf{W}}\right)\mathbf{y} + \hat{\sigma}_2^2\left(\mathbf{V}_2\hat{\mathbf{W}}\right)\mathbf{y} = \begin{bmatrix} \tilde{\mathbf{e}}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \tilde{\mathbf{e}}_2 \end{bmatrix} \tag{7}$$

so that we obtain

$$\mathbf{y}^T\hat{\mathbf{W}}\mathbf{V}_j\hat{\mathbf{W}}\mathbf{y} = \left(\tilde{\mathbf{e}}_j^T\mathbf{Q}_j^{-1}\tilde{\mathbf{e}}_j\right)\cdot\hat{\sigma}_j^{-4} \text{ for } j \in \{1,2\} \tag{8}$$

on the RHS of (3). On the other hand, the matrix $\hat{\mathbf{W}}$ from (4) would more explicitly read:

$$\hat{\mathbf{W}} = \hat{\sigma}_1^{-2}\left(\begin{bmatrix} \mathbf{Q}_1^{-1} & 0 \\ 0 & \hat{\lambda}_{12}\mathbf{Q}_2^{-1} \end{bmatrix} - \left[\begin{array}{c|c} \mathbf{Q}_1^{-1}\mathbf{A}_1\left(\mathbf{N}_{11} + \hat{\lambda}_{12}\mathbf{N}_{22}\right)^{-1}\mathbf{A}_1^T\mathbf{Q}_1^{-1} & \hat{\lambda}_{12}\mathbf{Q}_1^{-1}\mathbf{A}_1\left(\mathbf{N}_{11} + \hat{\lambda}_{12}\mathbf{N}_{22}\right)^{-1}\mathbf{A}_2^T\mathbf{Q}_2^{-1} \\ \hat{\lambda}_{12}\mathbf{Q}_2^{-1}\mathbf{A}_2\left(\mathbf{N}_{11} + \hat{\lambda}_{12}\mathbf{N}_{22}\right)^{-1}\mathbf{A}_1^T\mathbf{Q}_1^{-1} & \hat{\lambda}_{12}^2\mathbf{Q}_2^{-1}\mathbf{A}_2\left(\mathbf{N}_{11} + \hat{\lambda}_{12}\mathbf{N}_{22}\right)^{-1}\mathbf{A}_2^T\mathbf{Q}_2^{-1} \end{array}\right]\right) \tag{9}$$

with $\mathbf{N}_{11} = \mathbf{A}_1^T\mathbf{Q}_1^{-1}\mathbf{A}_1$ and $\mathbf{N}_{22} = \mathbf{A}_2^T\mathbf{Q}_2^{-1}\mathbf{A}_2$; this leads to the identities:

$$\hat{\mathbf{W}}\mathbf{V}_1 = \hat{\sigma}_1^{-2}\left[\begin{array}{c|c} \left[\mathbf{I}_{n_1} - \mathbf{Q}_1^{-1}\mathbf{A}_1\left(\mathbf{N}_{11} + \hat{\lambda}_{12}\mathbf{N}_{22}\right)^{-1}\mathbf{A}_1^T\right] & 0 \\ -\hat{\lambda}_{12}\mathbf{Q}_2^{-1}\mathbf{A}_2\left(\mathbf{N}_{11} + \hat{\lambda}_{12}\mathbf{N}_{22}\right)^{-1}\mathbf{A}_1^T & 0 \end{array}\right], \tag{10}$$

$$\hat{\mathbf{W}}\mathbf{V}_2 = \hat{\sigma}_1^{-2}\left[\begin{array}{c|c} 0 & -\hat{\lambda}_{12}\mathbf{Q}_1^{-1}\mathbf{A}_1\left(\mathbf{N}_{11} + \hat{\lambda}_{12}\mathbf{N}_{22}\right)^{-1}\mathbf{A}_2^T \\ 0 & \hat{\lambda}_{12}\cdot\left[\mathbf{I}_{n_2} - \hat{\lambda}_{12}\mathbf{Q}_2^{-1}\mathbf{A}_2\left(\mathbf{N}_{11} + \hat{\lambda}_{12}\mathbf{N}_{22}\right)^{-1}\mathbf{A}_2^T\right] \end{array}\right], \tag{11}$$

as well as to the following expressions for the traces on the LHS of (3), namely:

$$\begin{aligned} \hat{m}_{11} &= \hat{\sigma}_1^4 \mathrm{tr}\left(\hat{\mathbf{W}}\mathbf{V}_1\hat{\mathbf{W}}\mathbf{V}_1\right) = \mathrm{tr}\,\mathbf{I}_{n_2} - 2\,\mathrm{tr}\,\mathbf{N}_{11}\left(\mathbf{N}_{11} + \hat{\lambda}_{12}\mathbf{N}_{22}\right)^{-1} \\ &\quad + \mathrm{tr}\,\mathbf{N}_{11}\left(\mathbf{N}_{11} + \hat{\lambda}_{12}\mathbf{N}_{22}\right)^{-1}\mathbf{N}_{11}\left(\mathbf{N}_{11} + \hat{\lambda}_{12}\mathbf{N}_{22}\right)^{-1} \\ &= n_1 - \mathrm{tr}\,\mathbf{N}_{11}\left(\mathbf{N}_{11} + \hat{\lambda}_{12}\mathbf{N}_{22}\right)^{-1} \\ &\quad - \hat{\lambda}_{12}\mathrm{tr}\,\mathbf{N}_{11}\left(\mathbf{N}_{11} + \hat{\lambda}_{12}\mathbf{N}_{22}\right)^{-1}\mathbf{N}_{22}\left(\mathbf{N}_{11} + \hat{\lambda}_{12}\mathbf{N}_{22}\right)^{-1}, \end{aligned} \tag{12}$$

$$\begin{aligned} \hat{m}_{12} &= \hat{\sigma}_1^4 \mathrm{tr}\left(\hat{\mathbf{W}}\mathbf{V}_1\hat{\mathbf{W}}\mathbf{V}_2\right) \\ &= \hat{\lambda}_{12}^2\cdot\mathrm{tr}\,\mathbf{N}_{11}\left(\mathbf{N}_{11} + \hat{\lambda}_{12}\mathbf{N}_{22}\right)^{-1}\mathbf{N}_{22}\left(\mathbf{N}_{11} + \hat{\lambda}_{12}\mathbf{N}_{22}\right)^{-1}, \end{aligned} \tag{13}$$

$$\hat{m}_{22} = \hat{\sigma}_1^4 \mathrm{tr}\left(\hat{\mathbf{W}}\mathbf{V}_2\hat{\mathbf{W}}\mathbf{V}_2\right)$$
$$= \hat{\lambda}_{12}^2 \cdot \left[n_2 - \hat{\lambda}_{12}\mathrm{tr}\,\mathbf{N}_{22}\left(\mathbf{N}_{11} + \hat{\lambda}_{12}\mathbf{N}_{22}\right)^{-1} - \hat{\lambda}_{12}\mathrm{tr}\,\mathbf{N}_{11}\left(\mathbf{N}_{11} + \hat{\lambda}_{12}\mathbf{N}_{22}\right)^{-1}\mathbf{N}_{22}\left(\mathbf{N}_{11} + \hat{\lambda}_{12}\mathbf{N}_{22}\right)^{-1}\right]. \quad (14)$$

Most importantly, the "*redundancy identity*" holds true:

$$\boxed{\hat{m}_{11} + 2\hat{\lambda}_{12}^{-1}\hat{m}_{12} + \hat{\lambda}_{12}^{-2}\hat{m}_{22} = n - m} \quad (15)$$

as can be easily checked. By applying the formulas (12-14) along with (8), the *nonlinear normal equations* (3) for the estimated variance components become:

$$\tilde{\mathbf{e}}_1^{\mathrm{T}}\mathbf{Q}_1^{-1}\tilde{\mathbf{e}}_1 = \left(\hat{m}_{11} + \hat{\lambda}_{12}^{-1}\hat{m}_{12}\right)\cdot\hat{\sigma}_1^2$$
$$= \left[n_1 - \mathrm{tr}\,\mathbf{N}_{11}\left(\mathbf{N}_{11} + \hat{\lambda}_{12}\mathbf{N}_{22}\right)^{-1}\right]\cdot\hat{\sigma}_1^2, \quad (16)$$
$$\hat{\lambda}_{12}^2\cdot\left(\tilde{\mathbf{e}}_2^{\mathrm{T}}\mathbf{Q}_2^{-1}\tilde{\mathbf{e}}_2\right) = \left(\hat{m}_{12} + \hat{\lambda}_{12}^{-1}\hat{m}_{22}\right)\cdot\hat{\sigma}_1^2$$
$$= \hat{\lambda}_{12}\left[n_2 - \hat{\lambda}_{12}\mathrm{tr}\,\mathbf{N}_{22}\left(\mathbf{N}_{11} + \hat{\lambda}_{12}\mathbf{N}_{22}\right)^{-1}\right]\cdot\hat{\sigma}_1^2 \quad (17)$$

By combining (16-17), we arrive at

$$\boxed{\left(\tilde{\mathbf{e}}_1^{\mathrm{T}}\mathbf{Q}_1^{-1}\tilde{\mathbf{e}}_1\right) + \hat{\lambda}_{12}\cdot\left(\tilde{\mathbf{e}}_2^{\mathrm{T}}\mathbf{Q}_2^{-1}\tilde{\mathbf{e}}_2\right) = (n-m)\cdot\hat{\sigma}_1^2} \quad (18)$$

where the two quadratic forms in the residuals may be computed from the estimated parameters in the previous step via:

$$\tilde{\mathbf{e}}_1^{\mathrm{T}}\mathbf{Q}_1^{-1}\tilde{\mathbf{e}}_1 = \mathbf{y}_1^{\mathrm{T}}\mathbf{Q}_1^{-1}\mathbf{y}_1 - 2\mathbf{c}_1^{\mathrm{T}}\hat{\xi} + \hat{\xi}^{\mathrm{T}}\mathbf{N}_{11}\hat{\xi}, \quad (19)$$

$$\tilde{\mathbf{e}}_2^{\mathrm{T}}\mathbf{Q}_2^{-1}\tilde{\mathbf{e}}_2 = \mathbf{y}_2^{\mathrm{T}}\mathbf{Q}_2^{-1}\mathbf{y}_2 - 2\mathbf{c}_2^{\mathrm{T}}\hat{\xi} + \hat{\xi}^{\mathrm{T}}\mathbf{N}_{22}\hat{\xi}, \quad (20)$$

with $\mathbf{c}_j := \mathbf{A}_j^{\mathrm{T}}\mathbf{Q}_j^{-1}\mathbf{y}_j$ for $j \in \{1, 2\}$ and

$$\boxed{\hat{\xi} = \left(\mathbf{N}_{11} + \hat{\lambda}_{12}\mathbf{N}_{22}\right)^{-1}\left(\mathbf{c}_1 + \hat{\lambda}_{12}\mathbf{c}_2\right)}\ . \quad (21)$$

3. A New Algorithm for repro-BIQUUE

Now all ingredients are available to construct a variety of algorithms to find the variance component estimates in the case of two datasets. By taking the ratio of (17) over (16), namely, we first get:

$$\hat{\lambda}_{12}\cdot\frac{\left(\tilde{\mathbf{e}}_2^{\mathrm{T}}\mathbf{Q}_2^{-1}\tilde{\mathbf{e}}_2\right)}{\left(\tilde{\mathbf{e}}_1^{\mathrm{T}}\mathbf{Q}_1^{-1}\tilde{\mathbf{e}}_1\right)} = \frac{n_2 - \hat{\lambda}_{12}\mathrm{tr}\,\mathbf{N}_{22}\left(\mathbf{N}_{11} + \hat{\lambda}_{12}\mathbf{N}_{22}\right)^{-1}}{n_1 - \mathrm{tr}\,\mathbf{N}_{11}\left(\mathbf{N}_{11} + \hat{\lambda}_{12}\mathbf{N}_{22}\right)^{-1}}$$
$$= \frac{(n_2 - m) + \hat{t}}{n_1 - \hat{t}} \quad (22)$$

if $\hat{t} := \mathrm{tr}\,\mathbf{N}_{11}\left(\mathbf{N}_{11} + \hat{\lambda}_{12}\mathbf{N}_{22}\right)^{-1}$ and finally:

$$\boxed{\hat{\lambda}_{12} = \left(\frac{n-m}{n_1 - \hat{t}} - 1\right)\cdot\frac{\left(\tilde{\mathbf{e}}_1^{\mathrm{T}}\mathbf{Q}_1^{-1}\tilde{\mathbf{e}}_1\right)}{\left(\tilde{\mathbf{e}}_2^{\mathrm{T}}\mathbf{Q}_2^{-1}\tilde{\mathbf{e}}_2\right)},\ \hat{t} = \mathrm{tr}\,\mathbf{N}_{11}\left(\mathbf{N}_{11} + \hat{\lambda}_{12}\mathbf{N}_{22}\right)^{-1}}\ , \quad (23)$$

which needs iteration since $\hat{t}$ depends on $\hat{\lambda}_{12}$ again. After convergence, using 1 as initial value, the variance component estimates are obtained from (18) using $\hat{\lambda}_{12}$, or through

$$\boxed{\hat{\sigma}_1^2 = \frac{\left(\tilde{\mathbf{e}}_1^{\mathrm{T}}\mathbf{Q}_1^{-1}\tilde{\mathbf{e}}_1\right)}{n_1 - \hat{t}} \text{ and } \hat{\sigma}_2^2 = \frac{\hat{\sigma}_1^2}{\hat{\lambda}_{12}}} \quad (24)$$

using $\hat{t}$ before recomputing the parameter estimates through (21) and the corresponding residuals through

$$\tilde{\mathbf{e}}_j = \mathbf{y}_j - \mathbf{A}_j\hat{\xi} \text{ for } j \in \{1,2\}. \quad (25)$$

The use of (23-24) has been most promising among all algorithms that we tried for the solution of the general system (3) so far. Here only one trace must be computed, and that is from a $m \times m$ (and thus comparatively small) matrix. Once the variance components $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ have been determined, the parameter estimates $\hat{\xi}$ follow from (21) and their dispersion matrix approximately from:

$$\boxed{\mathbf{D}\{\hat{\xi}\} \approx \hat{\sigma}_1^2\left(\mathbf{N}_{11} + \hat{\lambda}_{12}\mathbf{N}_{22}\right)^{-1}}\ . \quad (26)$$

It would be of great interest to find a way to compute the trace $\hat{t}$ in (23) without having to invert the matrix $\mathbf{N}_{11} + \hat{\lambda}_{12}\mathbf{N}_{22}$ itself, although it will be used in (26) again.

4. An Example: A Geoid Patch Derived from CHAMP and EGM-96

As an example, we shall consider geoid data on the patch with $\varphi \in [-11.25°; +11.25°]$ latitude, and with $\lambda \in [100°; 122.5°]$ longitude. The first dataset has been derived from the EGM 96 field with a resolution of 65 by 65, and the second one from CHAMP data with a resolution of 33 by 33. Since, at this stage, all data are gridded, suitable basis functions for the continuous geoid representation may include biharmonic spline functions, Hardy's multiquadrics (at fixed positions), and further alternatives listed by Mautz et al. (2003). Here we present the results for the multiquadric functions.

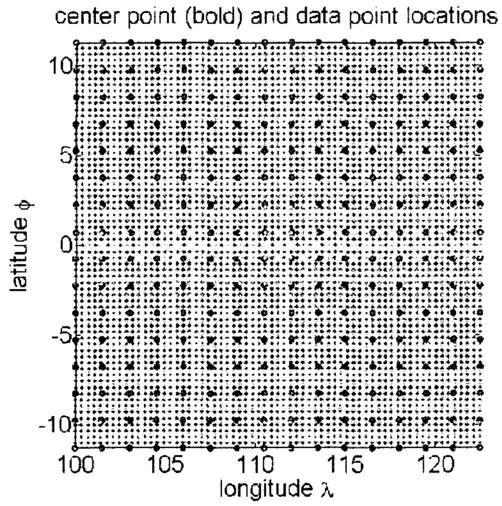

Fig. 1. Location of 256 center points of the multiquadric functions (bold) and the distribution of the data points.

Figure 1 illustrates the principal data point locations relative to the center points of the multiquadric basis functions while Figures 2 and 3 show the original datasets of *geoid undulations* from EGM 96 and CHAMP, respectively. An *equally weighted* combination solution is presented in Figure 4 along with the two residual fields in Figures 5 and 6. In contrast, Figure 7 shows the *properly weighted* combination solution, and Figures 8 and 9 the respective residual fields for both EGM 96 and CHAMP. For the estimation of the variance components, essentially *six iterations* were required as can be taken from the Table 1 where the individual values are presented for $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$, as well as their ratio $\hat{\lambda}_{12}$ and the respective sums of squared residuals ($\tilde{\mathbf{e}}_j^T \mathbf{Q}_j^{-1} \tilde{\mathbf{e}}_j$).

Finally, we investigated the relative increase in the CPU time for evaluating the trace $\hat{t}$ in (2) when increasing the number of parameters in the series

Table 1. Values of variance components, their ratios and the sums of squared residuals while iterating the formulas given in chapter 2.

Iteration Number	$\hat{\sigma}_1^2$ [m^2]	$\hat{\sigma}_2^2$ [m^2]	$\hat{\lambda}_{12}$	$\tilde{\mathbf{e}}_1^T \mathbf{Q}_1^{-1} \tilde{\mathbf{e}}_1$ [m^2]	$\tilde{\mathbf{e}}_2^T \mathbf{Q}_2^{-1} \tilde{\mathbf{e}}_2$ [m^2]
0	1.00000	1.00000	1.00000	4837.58	3993.32
1	1.20161	3.86916	0.31056	3892.44	5503.09
2	0.97549	5.15391	0.18927	3802.66	5863.84
3	0.95486	5.45193	0.17514	3794.55	5908.38
4	0.95305	5.48851	0.17364	3793.72	5913.14
5	0.95286	5.49241	0.17349	3793.63	5913.63
6	0.95284	5.49282	0.17347	3793.62	5913.69
7	0.95284	5.49286	0.17347	3793.62	5913.69
8	0.95284	5.49287	0.17347	3793.62	5913.69

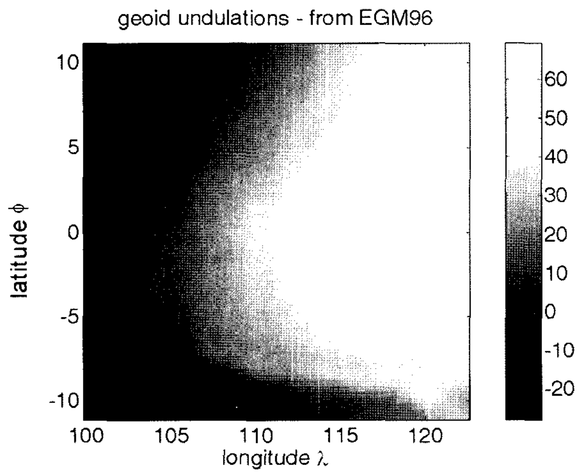

Fig. 2. Geoid undulations from EGM 96 on a 65 by 65 grid. The unit is [m].

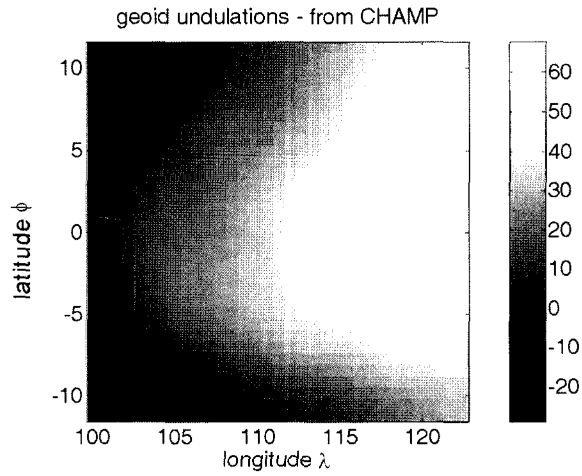

Fig. 3. Geoid undulations from CHAMP on a 33 by 33 grid. The unit is [m].

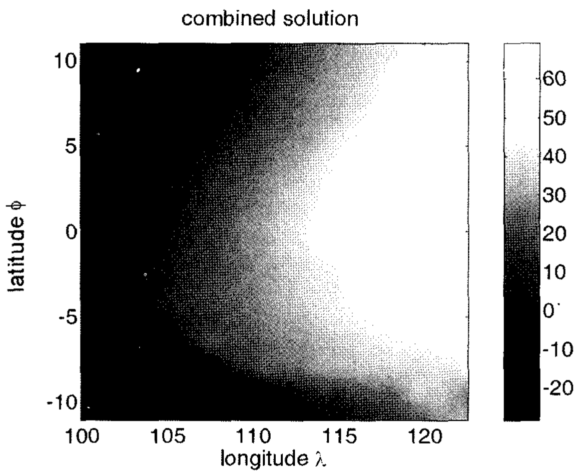

Fig. 4. Combined solution of EGM 96 and CHAMP with equal weights (λ_{12} = 1). The unit is [m].

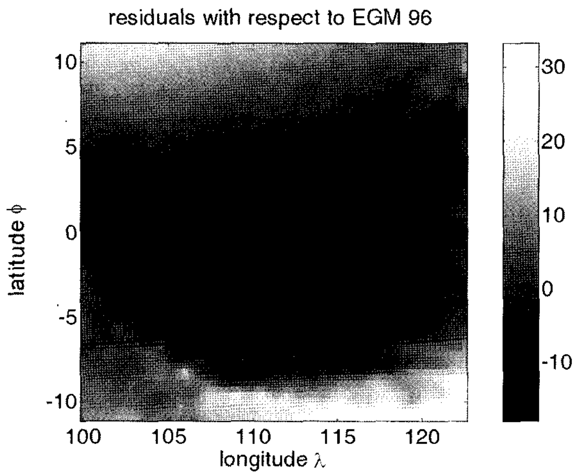

Fig. 5. Residuals of the equally weighted combined solution with respect to EGM 96. The unit is [m].

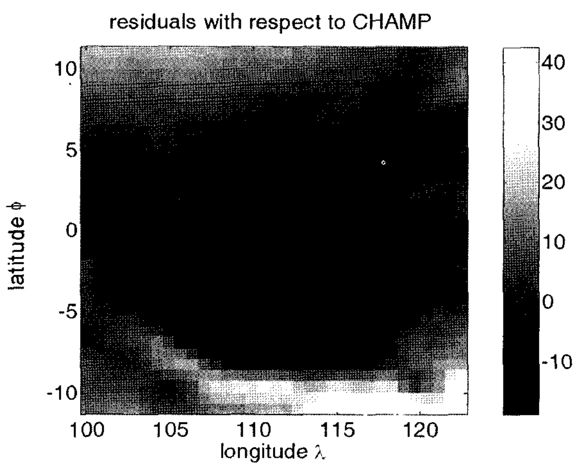

Fig. 6. Residuals of the equally weighted combined solution with respect to CHAMP data. The unit is [m].

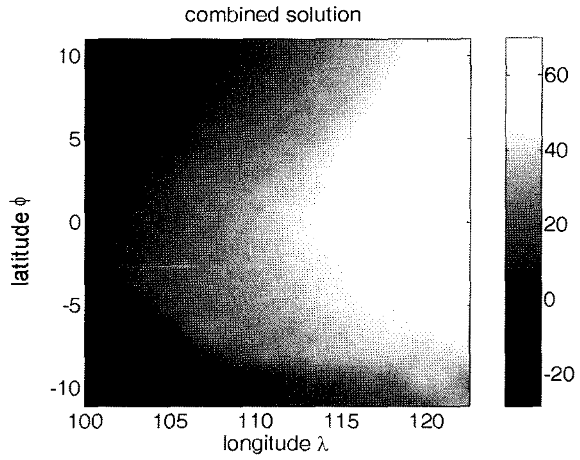

Fig. 7. Combined solution of EGM 96 and CHAMP with proper weights (λ_{12} = 0.17). The unit is [m].

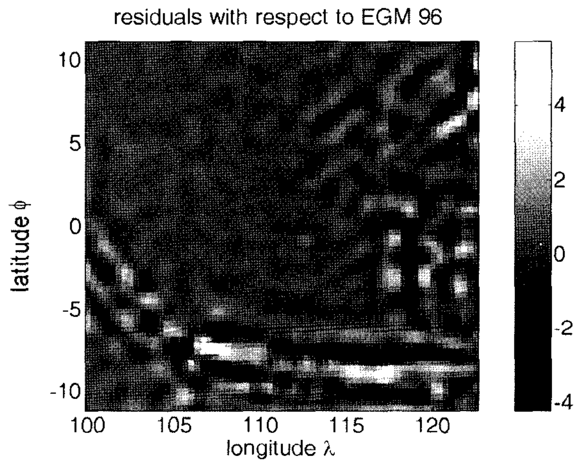

Fig. 8. Residuals of the properly weighted combined solution with respect to EGM 96. The unit is [m].

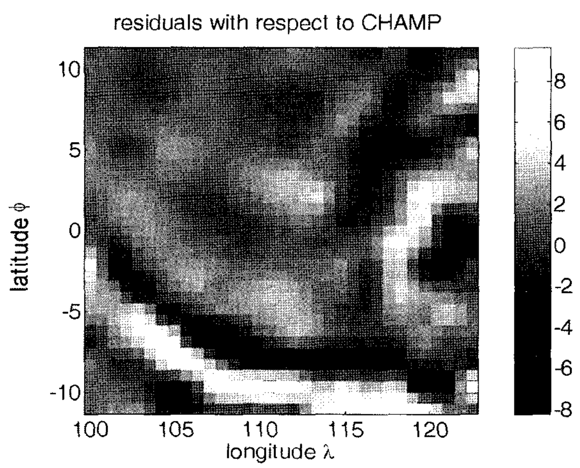

Fig. 9. Residuals of the properly weighted combined solution with respect to CHAMP data. The unit is [m].

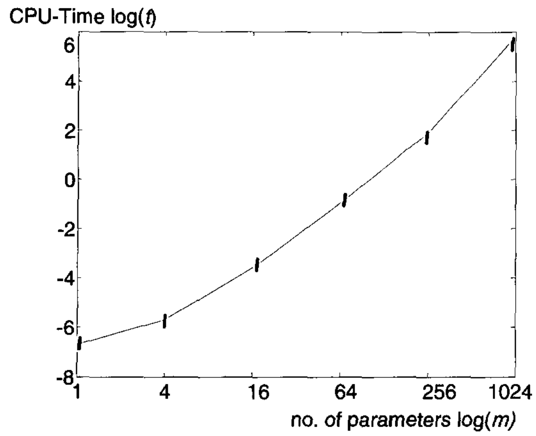

Fig. 10. CPU-time in [log(t)] (time t in [s]) versus number of parameters m in the model. Both axes are in logarithmic scale.

expansion for the representation of the geoid undulations. The results are provided in Figure 10.

5. Conclusions and Outlook

We have presented a new efficient algorithm that allows us to merge two datasets with *proper weighting* while only one trace needs to be computed, plus another one for the next dataset, and so forth. The example that *merges geoid data* from EGM 96 and CHAMP, has shown successfully that *weighting matters* by giving EGM 96 six times the influence over CHAMP in this area.

Further studies are necessary to enhance the efficiency of the algorithm even more (if possible) so that truly global problems can be handled in the near future. Also, the dependence of the estimated variance ratio on the chosen resolution needs to be investigated. In our example, the resolution was about 150 km which explains the relative preference of EGM 96 over CHAMP in this case.

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