350 *Journal of Mechanical Science and Technology, Vol. 19, No. 1, ⟨Special Edition⟩ pp. 350~356, 2005*

Dynamic Analysis of Multi-body Systems Considering Probabilistic Properties

Dong Hwan Choi
School of Mechanical Engineering, Hanyang University, Haengdang-dong, Seongdong-gu, Seoul, Korea, 133791
Se Jeong Lee
Department of Mechanical and Information Engineering, University of Seoul, Siripdae-gil 13, Dongdaemun-gu, Seoul, Korea, 130743
Hong Hee Yoo*
School of Mechanical Engineering, Hanyang University, Haengdang-dong, Seongdong-gu, Seoul, Korea, 133791

A method of dynamic analysis of mechanical systems considering probabilistic properties is proposed in this paper. Probabilistic properties that result from manufacturing tolerances can be represented by means and standard deviations (or variances). The probabilistic characteristics of dynamic responses of constrained multi-body systems are obtained by two ways: the proposed analytical approach and the Monte Carlo simulation. The former necessitates sensitivity information to calculate the standard deviations. In this paper, a direct differentiation method is employed to find the sensitivities of constrained multi-body systems. To verify the accuracy of the proposed method, numerical examples are solved and the results obtained by using the proposed method are compared to those obtained by Monte Carlo simulation.

Key Words: Manufacturing Tolerance, Multibody System, Probabilistic Property, Monte Carlo Method, Direct Differentiation Method (DDM)

1. Introduction

Designers want to minimize mechanical response errors that result from manufacturing tolerances. Particularly in spatial systems, flexibility of manipulators could affect the system performance including the accuracy of the dynamic response. In the past, many researchers studied the flexibility effect due to the joint clearance. It is well known that the clearance often degrades the performance of a mechanical system and causes wear, noise, and vibration.

Hartenberg and Denavit (1964) first addressed the issue of mechanical errors in linkages. They estimated the mechanical errors based on the maximum allowable tolerances of the link lengths in four-bar linkages. Their approach employed a deterministic method and offered a "worst case" analysis of tolerance. Garrett and Hall (1969) developed a statistical approach to determine mechanical errors due to tolerances and clearances and represented the errors as mobility bands. They carried out Monte Carlo simulations for a four-bar linkage. Dubowsky and Freudenstein (1971) developed an impact pair and Dubowsky (1974) presented a prediction model for the dynamic effects of clearances in planar mechanisms through the use of impact pair. Recently, Dubowsky et al. (1987) proposed a dynamic modeling of spatial mechanism with multiple clea-

* Corresponding Author,
E-mail: hhyoo@hanyang.ac.kr
TEL: +82-2-2290-0446; **FAX**: +82-2-2293-5070
School of Mechanical Engineering, Hanyang University, Haengdang-dong, Seongdong-gu, Seoul, Korea, 133791. (Manuscript **Received** November 29, 2004; **Revised** December 15, 2004)

rances and flexible links. Lee (1989) proposed an effective link model and later Choi et al. (1998) solved the same problem with the clearance vector model. They considered various uncertainties associated with tolerances and locations. Gilmore and Cipra (1991) discussed a simulation method for planar mechanical systems with changing topologies. Farahanchi and Shaw (1994) considered the model of planar, rigid-link mechanism with clearance at the slider joint. They observed that system response appears to be chaotic, although periodic motion occurs more commonly as dissipation effects are increased. Ravn (1998) suggested a continuous analysis approach for a planar mechanism with a contact force model to describe joint clearance in rotational joints. Equations of motion for a multi-body system of rigid bodies and Hertz contact law are employed. More recently, Stoenescu and Marghitu (2003) investigated the dynamic behavior of a planar, rigid-link mechanism with a sliding joint clearance. They considered the clearance of link length. When the topology change occurs, the equations of motion are reformulated to reflect the changes of the system topology.

Most of the studies mentioned so far are limited to the application of planar mechanism and mostly focus on the joint clearance modeling issues. Little work has been done to consider uncertainties associated with manufacturing tolerances.(for example, material and geometric deviations) This paper presents a method of dynamic analysis of multi-body system considering various probabilistic properties based on a general multi-body formulation. Especially when manufacturing tolerances have specific probability distributions, the purpose of this paper is to propose a numerical method to predict the dynamic response of mechanical systems with manufacturing tolerances. These tolerances can be represented by variances or standard deviations in the probabilistic manner. The standard deviation is usually taken as one third of the bilateral tolerance. The mean and variances can be calculated on a certain confidence interval. The probabilistic characteristics of dynamic responses of mechanical systems with tolerances are obtained by two ways: the approximate approach and the Monte Carlo simulation (Rubinstein, 1981). Since the approximate method estimates the means and variances of dynamic responses of a multi-body system with tolerances by using Taylor series expansion, it necessitates sensitivity information. In this paper, a direct differentiation method proposed by Serban and Freeman (1996) is used to find the sensitivities of multi-body systems. The first order variance is considered in tolerance analysis. To verify the accuracy of the proposed method, numerical examples are solved and the results obtained by using the proposed method are compared to those obtained by Monte Carlo simulation.

2. Approximate Method

The statistical method has been used to determine mechanical errors in terms of standard deviations (or means and variations). The governing equations for mechanical systems are usually implicit in the form of dynamics. In general, the equations of motion for a constrained multi-body system are a set of index-3 differential-algebraic equations (Brenan et al., 1989). Therefore, the dynamic response can be obtained from integrating the differential-algebraic equations. For the purpose of illustration, consider a typical response variable Y that can be represented by a relation g of a set of random variables X_i as

$$Y=g(X_1, X_2, \cdots, X_n) \tag{1}$$

If the mean and variance of each X_i are known but the distribution is unknown, the approximate mean and variance of Y can be estimated by using Taylor series expansion.

Expanding the function $g(X_1, X_2, \cdots, X_n)$ in a Taylor series about the mean values $\mu_{X_1}, \mu_{X_2}, \cdots, \mu_{X_n}$, one obtains

$$\begin{aligned} Y=&g(\mu_{X_1}, \mu_{X_2}, \cdots, \mu_{X_n})+\sum_{i=1}^{n}(X_i-\mu_{Xi})\frac{\partial g}{\partial X_i} \\ &+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(X_i-\mu_{X_i})(X_j-\mu_{X_j})\frac{\partial^2 g}{\partial X_i \partial X_j}+\cdots \end{aligned} \tag{2}$$

Truncating the series at the linear terms, the first order approximate mean of Y, denoted as $E(Y)$, can be obtained as

$$E(Y)=g(\mu_{X_1},\mu_{X_2},\cdots,\mu_{X_n}) \tag{3}$$

The above equation indicates that the first order mean of Y is approximated by the value of the function evaluated at the mean values of the X_i's.

The first order variance of Y, denoted as $Var(Y)$, can be obtained as

$$\begin{aligned}Var(Y)&=\sum_{i=1}^{n}\left(\frac{\partial g}{\partial X_i}\right)^2 Var(X_i)\\&\quad+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial g}{\partial X_i}\frac{\partial g}{\partial X_j}Cov(X_i,X_j)\,(i\neq j)\\&=\sum_{i=1}^{n}S_i^2 Var(X_i)+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}S_iS_jCov(X_i,X_j)\,(i\neq j)\end{aligned} \tag{4}$$

where $Cov(X_i, X_j)$ denotes covariance of two random variables. S_i is constant and it represents the value of the partial derivative $\partial g/\partial X_i$ evaluated at the mean values of the X_i's, in general, which is defined as the sensitivity in multi-body dynamics. In engineering practice, since the random variables are assumed to be independent of each other, the variance of Y reduces to

$$Var(Y)\sum_{i=1}^{n}S_i^2 Var(X_i) \tag{5}$$

Including the higher order terms in the Taylor series expansion, the second order mean of Y can be obtained as

$$\begin{aligned}E(Y)&=g(\mu_{X_1},\mu_{X_2},\cdots,\mu_{X_n})+\frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial^2 g}{\partial X_i^2}\right)Var(X_i)\\&=g(\mu_{X_1},\mu_{X_2},\cdots,\mu_{X_n})+\frac{1}{2}\sum_{i=1}^{n}H_i Var(X_i)\end{aligned} \tag{6}$$

Again, the partial derivatives H_i's are evaluated at the mean values of all X_i's. However, to estimate the second order variance, the information on the third and fourth moments of the X_i's must be available. Unfortunately, in most cases this information is rarely available. In general, the first order variance is considered adequate for most practical engineering applications.

3. Sensitivity Analysis of CMBS

The Euler-Lagrange equations of motion for a constrained multi-body system can be expressed in the form

$$M\ddot{q}+\Phi_q^T\lambda=Q \tag{7}$$

along with the algebraic constraint equations

$$\Phi(q)=0 \tag{8}$$

where q is a generalized coordinate vector, M is a mass matrix, Q is a generalized force vector, and λ is a Lagrange multiplier vector.

Let b be a random variable, which is assumed a scalar for derivational simplicity. Taking the derivative of Eqs. (7) and (8) with respect to the design variable b results in the following system of equations

$$\begin{aligned}M\ddot{q}_b+\Phi_q^T\lambda_b=&-M_b\ddot{q}-(M\ddot{q})_q q_b-(\Phi_q^T\lambda)_b\\&-(\Phi_q^T\lambda)_q q_b+Q_b+Q_q q_b+Q_{\dot{q}}\dot{q}_b\end{aligned} \tag{9}$$

and

$$\Phi_q q_b=-\Phi_b \tag{10}$$

where subscripts mean partial derivatives.

The fact (the system sensitivity equations expressed by Eqs. (9) and (10) is not a set of DAE's) leads to inconsistent solutions when applying DAE solution techniques. Considering Eq. (7) through Eq. (10) together, Serban and Freeman (1996) proposed a sensitivity equation as the form of DAE's which can be solved using standard technique. Sensitivity equations for a constrained multi-body system can be obtained as

$$\hat{M}\ddot{r}+\Pi_r^T\mu=\hat{Q} \tag{11}$$

$$\Pi=0 \tag{12}$$

where

$$\hat{M}=\begin{bmatrix}(M_b+P) & M\\ M & 0\end{bmatrix} \tag{13}$$

$$r=\begin{Bmatrix}q\\ q^b\end{Bmatrix} \tag{14}$$

$$\Pi_r=\begin{bmatrix}\Phi_q & 0\\ \{(\Phi_q q_b)_q+(\Phi_b)_q\} & \Phi_q\end{bmatrix} \tag{15}$$

$$\mu=\begin{Bmatrix}\lambda_b\\ \lambda\end{Bmatrix} \tag{16}$$

$$\hat{Q}=\begin{bmatrix}Q_b+Q_q q_b+Q_{\dot{q}}\dot{q}_b\\ Q\end{bmatrix} \tag{17}$$

The composite constraints Π are defined as

$$\boldsymbol{\Pi}=\left\{\begin{matrix}\boldsymbol{\Phi}\\ \boldsymbol{\Phi}_q\boldsymbol{q}_b+\boldsymbol{\Phi}_b\end{matrix}\right\}=0 \tag{18}$$

A new matrix $\boldsymbol{P}$ is defined as

$$(\boldsymbol{M}\ddot{\boldsymbol{q}})_q\boldsymbol{q}_b=\boldsymbol{P}\ddot{\boldsymbol{q}} \tag{19}$$

where the components, p_{ij}, of the matrix $\boldsymbol{P}$ are given as

$$p_{ij}=\sum_{k=1}^{n}\frac{\partial m_{ij}}{\partial q_k}(q_k)_b \tag{20}$$

In order to solve the differential-algebraic sensitivity equations given by Eqs. (11) and (12), a set of $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$ must be defined. It can be shown (Haug, 1987) that this procedure yields unique initial conditions for $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$.

Therefore, after obtaining sensitivity values, if the distributions of random variables are defined, then the means and variances of response variables can be obtained from Eqs. (3) and (5).

4. Numerical Examples

The simple pendulum example is shown in Fig. 1. The rigid bar, mass $m=3$ kg and length $L=1$ m, is connected by a pin joint. At initial state, the angle θ_1 is 0.5236 radian and it oscillates under the influence of gravity. In this system, since only the length of pendulum affects the dynamic response, so it is taken as a random variable. Figure 2 shows the mean values of the dynamic response of the angle θ_1, when the length of the pendulum has the normal distribution with 99.73% confidence interval and its tolerance is 10.0%. Two analytic methods and a Monte Carlo method are used. Since the second order approximate method necessitates the partial derivative of sensitivity with respect to random variable, the finite difference method is used in this paper. As shown in the plot, the mean value obtained by using the second order approximate method is almost identical with the result of the Monte Carlo method. Especially, the amplitude of mean value is decaying as time progresses. The reason is that, in this system, the variation of the length directly causes the variation of the natural frequency of the system.

In the Fig. 2, the mean value obtained by using the first order approximate method still oscillates with the same amplitude. The reason is that there are no sensitivity values concerning the tolerances in calculating the first order mean value.

Figure 3 shows the variation of mean value with respect to the variation of the magnitude of

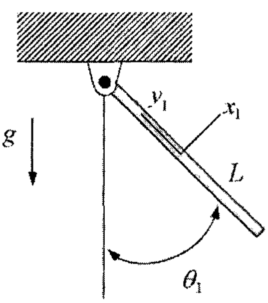

Fig. 1 Simple pendulum

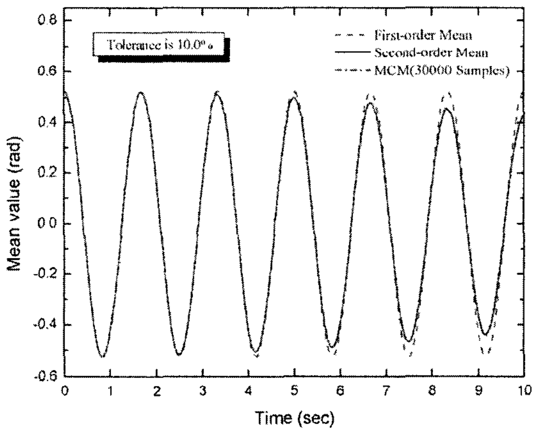

Fig. 2 Mean values of angle of pendulum

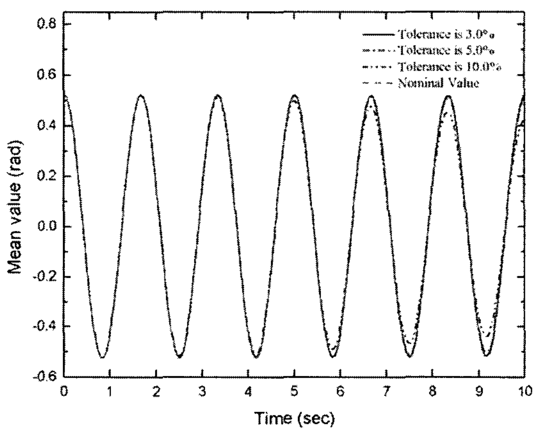

Fig. 3 Mean values w.r.t. tolerances

tolerance. The second order approximate method is used. Naturally, it shows that the response has the more error as the tolerance increases and time progresses.

Figure 4 shows the variance of the angle θ_1, when the length of pendulum has 10.0% tolerance. The first order approximate method is used. Since the sensitivity value with respect to the length, θ_{1L}, is directly proportional to time, the variance from the approximate method diverges as time progresses. Since the actual tolerance is, in general, less than 1 percent of nominal value, the first order variance is considered adequate for most engineering applications. However, to obtain the reliable responses from the approximate method, the simulation should be done under one or two period when the dynamic responses of the mechanical system represent the periodic motions.

As a second example for the tolerance analysis of multibody systems, consider the slider-crank mechanism shown in Fig. 5. Body 1 of the system is the crank and its mass $m_1=10$ kg and its length is $L_1=1.414$ m. Body 2 is the connecting rod and its mass is $m_2=2$ kg and its length is $L_2=2$ m. Body 3 is the slider and its mass is $m_3=1$ kg. The crank and the ground ; the crank and the connecting rod ; and the connecting rod and the slider are all connected by revolute joints. The slider and the ground are connected by a translational joint. At initial state, the orientation of the crank is 0.7854 radian. In this example, the constraint force acting on the slider from the ground (with respect to the tolerance of the mass of crank) is considered.

Figure 6 shows the mean value of reaction force when the mass of the crank has the normal distribution with 99.73% confidence interval and its tolerance is 10.0%. Figure 7 shows the standard

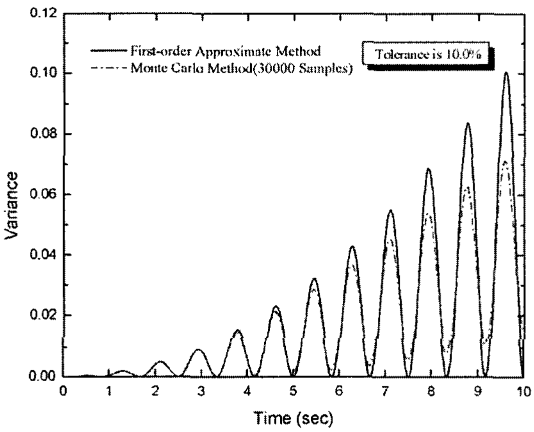

Fig. 4 Variances of angle of pendulum

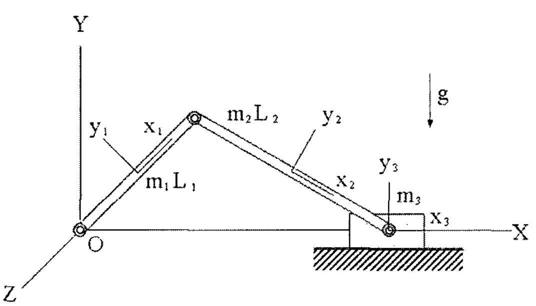

Fig. 5 Slider-crank mechanism

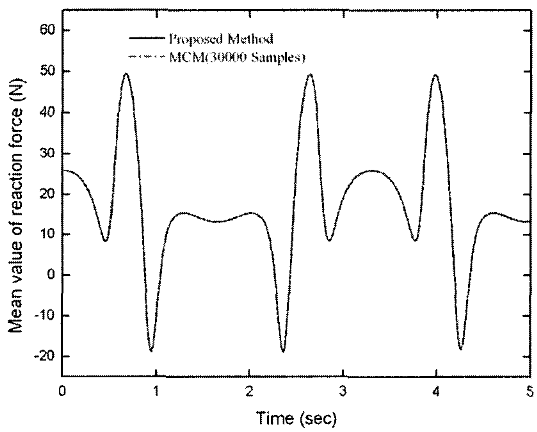

Fig. 6 Mean values of reaction force

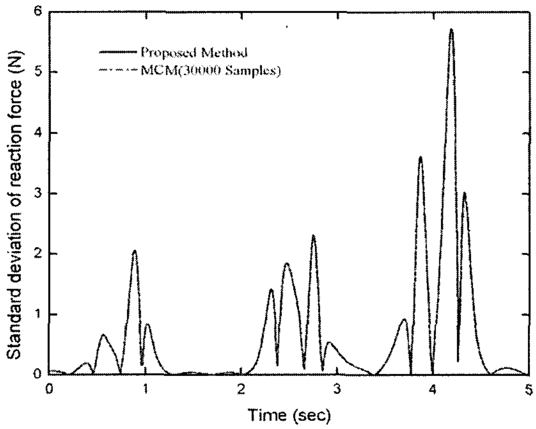

Fig. 7 Standard deviations of reaction force

deviation of reaction force. As shown in the results, the results of the proposed method are almost identical with the results of Monte Carlo method.

5. Conclusions

In this paper, a computational algorithm is proposed to predict the various dynamic responses of mechanical systems with probabilistic properties, which result from various manufacturing tolerances. These tolerances can be represented by standard deviations or means and variances in the probabilistic manner. The means, variances, and standard deviations of dynamic responses of mechanical system with tolerances are obtained by two ways : the analytical method and Monte Carlo method. Monte Carlo method is very expensive although it gives accurate results for means and variances. In this paper, the approximate method using Taylor series expansion is used as the analytical method. Since the approximate method necessitates the sensitivity information, the direct differentiation method is used to calculate sensitivities of a constrained multi-body system. To verify the accuracy of the proposed method, two numerical examples are solved and the results obtained by using the proposed method are compared to those obtained by Monte Carlo method. It is proved that the proposed method provides accurate means, variances, and standard deviations.

Acknowledgment

This research was supported by Center of Innovative Design Optimization Technology (iDOT), Korea Science and Engineering Foundation.

References

Brenan, K. B., Campbell, S. L. and Petzold, L. R., 1989, *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*, North Holland Publishing Co., New York, N.Y.

Choi, J. H., Lee, S. J. and Choi, D. H., 1998, "Stochastic Linkage Modeling for Mechanical Error Analysis of Planar Mechanisms," *Mechanics of Structures and Machines*, Vol. 26, No. 3, pp. 257~276.

Dobowsky, S. and Freudenstein, F., 1971, "Dynamic Analysis of Mechanical Systems with Clearances, Part 1 : Formulation of Dynamic Model," *Journal of Engineering for Industry, ASME*, Vol. 90, No. 1, pp. 305-316.

Dobowsky, S., 1974, "On Predicting the Dynamic Effects of Clearances in Planar Mechanisms," *Journal of Engineering for Industry, ASME*, Vol. 96, No. 1, pp. 317~323.

Dobowsky, S., Deck, J. F. and Costello, H., 1987, "The Dynamic Modeling of Flexible Spatial Machine Systems with Clearance Connections," *Journal of Mechanisms, Transmissions and Automation in Design, ASME*, Vol. 109, pp. 87~94.

Farahanchi, F. and Shaw, S. W., 1994, "Chaotic and Periodic Dynamics of A Slider-Crank Mechanism with Slider Clearance," *Journal of Sound and Vibration*, Vol. 177, No. 3, pp. 307~324.

Garret, R. E. and Hall, A. S., 1969, "Effects of Tolerance and Clearance in Linkage Design," *Journal of Engineering for Industry, ASME*, Vol. 91, pp. 198~202.

Gilmore, B. J. and Cipra, R. J., 1991, "Simulation of Planar Dynamic Mechanical Systems with Changing Topologies : Part I — Characterization and Prediction of the Kinematic Constraint Changes, Part II — Implementation Strategy and Simulation Results for Example Dynamic System," *Journal of Mechanical Design, ASME*, Vol. 113, pp. 70~83.

Hartenberg, R. S., and Denavit, J., 1964, *Kinematic Synthesis of Linkages*, McGraw-Hill, New York.

Haug, E. J., 1987, *Design Sensitivity Analysis of Dynamic Systems, Computer-Aided Design : Structural and Mechanical Systems*, C. A. Mota-Soares, Ed., Springer-Verlag, Berlin.

Lee, S. J., 1989, *Performance Reliability and Tolerance Allocation of Stochastically Defined Mechanical Systems*, Ph.D. Dissertation, The Pennsylvania State University.

Ravn P. A., 1998, "Continuous Analysis

Method for Planar Multibody Systems with Clearance," *Multibody System Dynamics*, Vol. 2, pp. 1~24.

Rubinstein, R. Y., 1981, *Simulation and Monte Carlo Method*, John Wiley and Sons.

Serban, R. and Freeman, J. S., 1996, "Direct Differentation Methods for the Design Sensitivity of Multibody Dynamic Systems," *The 1996 ASME Design Engineering Technical Conferences and Computers in Engineering Conference* August 18-22, DETC/DAC-1087.

Stoenescu, E. D. and Marghitu, D. B., 2003 "Dynamic Analysis of a Planar Rigid-link Mechanism with Rotating Slider Joint and Clearance," *Journal of Sound and Vibration*, Vol. 266 pp. 394~404.