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A SKEWED GENERALIZED t DISTRIBUTION

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ABSTRACT

Skewed t distributions have attracted significant attention in the last few years. In this paper, a generalization – referred to as the *skewed generalized t distribution* – with the pdf $f(x) = 2g(x)G(\lambda x)$ is introduced, where $g(\cdot)$ and $G(\cdot)$ are taken, respectively, to be the pdf and the cdf of the generalized t distribution due to McDonald and Newey (1984, 1988). Several particular cases of this distribution are identified and various representations for its moments derived. An application is provided to rainfall data from Orlando, Florida.

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1. INTRODUCTION

Skewed t distributions – both univariate and multivariate – have attracted significant attention in the last few years. Most notable are the work by Branco and Dey (2001), Gupta *et al* (2002), and Gupta (2003). In this paper, we introduce a generalization of skewed t distributions as follows. Consider the generalized t distribution with the probability density function (pdf) specified by

$$g(x) = \frac{k\Gamma(h)}{2\lambda\Gamma(1/k)\Gamma(h-1/k)} \left\{1 + \left(\frac{|x|}{\lambda}\right)^k\right\}^{-h} \tag{1.1}$$

for $-\infty < x < \infty$, $\lambda > 0$, $k > 0$, $h > 0$ and $h > 1/k$. This distribution was used by for McDonald and Newey for partially adaptive estimation of regression models. It has been followed up more recently by Theodossiou (1998), Arslan

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and Genc (2003) and Kim (2005). The corresponding cumulative distribution function (cdf) can be expressed as:

$$G(x) = \begin{cases} \frac{1}{2}\left[1 + I_{1-\left\{1+\left(\frac{x}{\lambda}\right)^k\right\}^{-1}}\left(\frac{1}{k}, h - \frac{1}{k}\right)\right], & \text{if } x \geq 0, \\ \frac{1}{2}\left[1 - I_{1-\left\{1+\left(-\frac{x}{\lambda}\right)^k\right\}^{-1}}\left(\frac{1}{k}, h - \frac{1}{k}\right)\right], & \text{if } x \leq 0, \end{cases} \tag{1.2}$$

where $I_x(a, b)$ denotes the incomplete beta function ratio defined by

$$I_x(a, b) = \frac{1}{B(a, b)} \int_0^x w^{a-1}(1-w)^{b-1} dw. \tag{1.3}$$

Following the usual definition of skew symmetric distributions (see, for example, Gupta *et al.* (2002)), we define a random variable X to have the skewed generalized t distribution if its pdf is given by

$$f(x) = 2g(x)G(\gamma x), \tag{1.4}$$

where $-\infty < x < \infty$. We assume without loss of generality that $\gamma \geq 0$ in (1.4) since the corresponding properties for $\gamma < 0$ can be obtained using the fact $G(\gamma x) = 1 - G(-\gamma x)$. It follows from (1.1), (1.2) and (1.4) that the pdf of X is

$$f(x) = \begin{cases} \frac{k\Gamma(h)}{2\lambda\Gamma(1/k)\Gamma(h-1/k)}\left\{1 + \left(\frac{x}{\lambda}\right)^k\right\}^{-h} \\ \quad \times \left[1 + I_{1-\left\{1+\left(\frac{\gamma x}{\lambda}\right)^k\right\}^{-1}}\left(\frac{1}{k}, h - \frac{1}{k}\right)\right], \text{if } x \geq 0, \\ \frac{k\Gamma(h)}{2\lambda\Gamma(1/k)\Gamma(h-1/k)}\left\{1 + \left(-\frac{x}{\lambda}\right)^k\right\}^{-h} \\ \quad \times \left[1 - I_{1-\left\{1+\left(-\frac{\gamma x}{\lambda}\right)^k\right\}^{-1}}\left(\frac{1}{k}, h - \frac{1}{k}\right)\right], \text{if } x \leq 0. \end{cases} \tag{1.5}$$

When $\gamma = 0$, (1.5) reduces to the generalized t pdf (1.1). Figures 1.1 and 1.2 below illustrate the shape of the pdf (1.5) for a range of values of γ, h and k. The effect of the parameters γ, h and k is evident.

The generalized t distribution given by (1.1) has major applications in the sciences. The main feature of the skewed generalized t distribution in (1.5) is that a new parameter λ is introduced to control skewness and kurtosis. Thus, (1.5) allows for a greater degree of flexibility and one can expect this to be useful in many more practical situations.

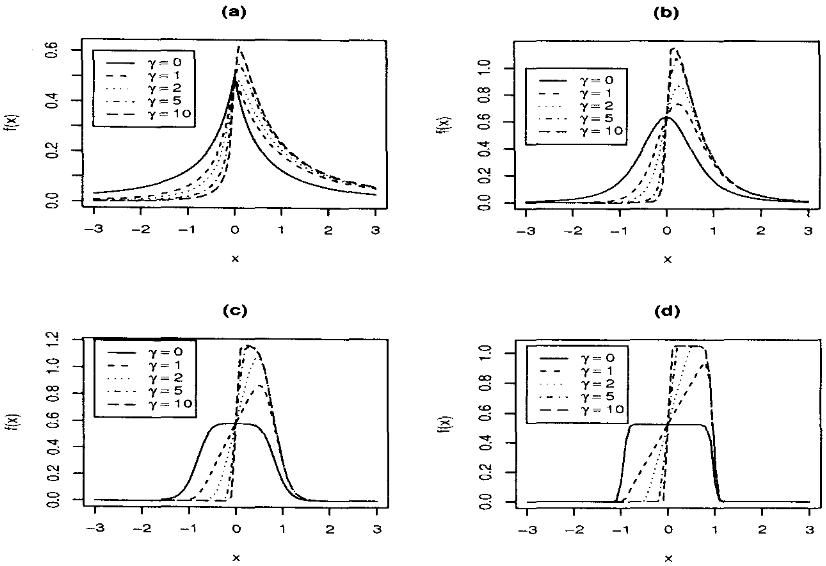


FIGURE 1.1 *The skewed generalized t pdf (1.5) for* $\gamma = 0, 1, 2, 5, 10$, $\lambda = 1$ *and (a):* $(h, k) = (2, 1)$; *(b):* $(h, k) = (2, 2)$; *(c):* $(h, k) = (2, 5)$; *and, (d):* $(h, k) = (2, 20)$.

Various skewed distributions can be obtained from (1.4) by taking $g(\cdot)$ and $G(\cdot)$ to belong to standard parametric families. The skewed normal distribution with $g(\cdot)$ in (1.4) taken to be standard normal pdf was introduced in the seminal paper by Azzalini (1985). This distribution has been studied extensively by several authors. Henze (1986), Liseo and Loperfido (2003) and Gupta *et al* (2004) provided various characterizations and representations of this distribution. Gupta and Chen (2001) and Monti (2003) considered goodness–of–fit and estimation issues. Arellano-Valle *et al* (2004) and Gupta and Gupta (2004) developed certain generalizations of the skewed normal distribution. Pewsey (2000) developed the wrapped skewed normal distribution for circular data. Azzalini and Chiogna (2004) considered stress–strength modeling using the skewed normal distribution.

Among other skewed distributions arising from (1.4), see Arnold and Beaver (2000) for skewed Cauchy, Kozubowski and Panorska (2004) and Aryal and Nadarajah (2005) for skewed Laplace, and Wahed and Ali (2001) for skewed logistic. See also Gupta *et al* (2002).

It should be noted that original skewed generalized t distribution due to Theo-

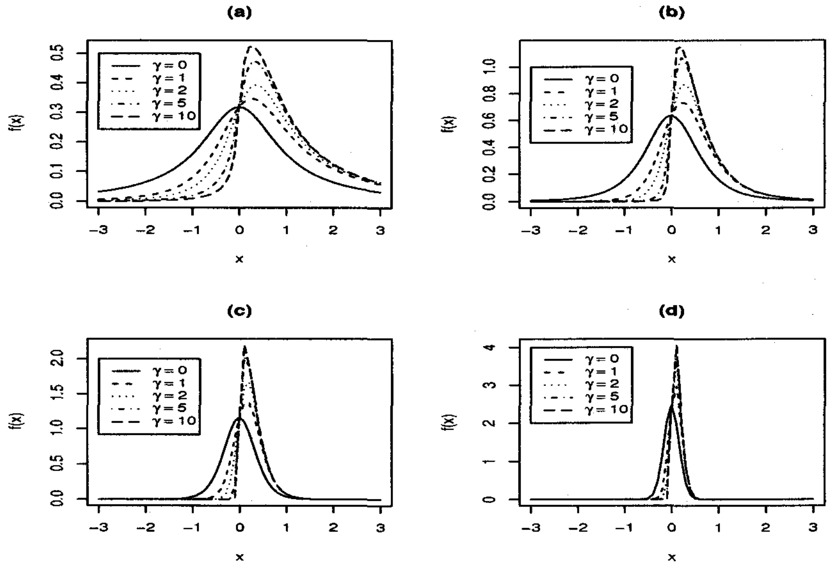


FIGURE 1.2 *The skewed generalized t pdf (1.5) for* $\gamma = 0, 1, 2, 5, 10$, $\lambda = 1$ *and (a):* $(h, k) = (1, 2)$; *(b):* $(h, k) = (2, 2)$; *(c):* $(h, k) = (5, 2)$; *and, (d):* $(h, k) = (20, 2)$.

dossiou (1998) has no relation to (1.5). Its pdf is given by

$$f(x) = \frac{p}{2B(1/p, q) q^{1/p} \sigma} \left[1 + \frac{\mid x - \mu \mid^p}{[1 + \operatorname{sign}(x - \mu)\lambda]^p q \sigma^p} \right]^{q+1/p} \tag{1.6}$$

for $-\infty < x < \infty$, $-\infty < \mu < \infty$ (location parameter), $\sigma > 0$ (scale parameter), $\mid \lambda \mid < 1$ (skewness parameter), and $p > 0$ and $q > 0$ (shape parameters). This distribution has been studied extensively by Grottke (1999), Hueng and Brashier (2003), Ioannides *et al* (2004), Hueng *et al* (2003), Adcock and Meade (2003) and Arslan and Genc (2006). The distribution family (1.6) includes many of the well-known distributions, including the skewed t distribution defined by Hansen (1994), the t distribution, the normal distribution, and the skewed normal distribution. But, there is no known relationships between (1.5) and (1.6).

The rest of this paper is organized as follows. In Section 2, we derive several particular forms of (1.5). In Section 3, various representations for the moments of the distribution are derived. An application to rainfall data is discussed in Section 4 to show that the generalization given by (1.5) can be useful in practice and that it outperforms the traditional gamma model for rainfall data. We also provide an appendix which notes some technical results.

2. PARTICULAR CASES

In this section, we derive six particular forms of (1.5). These derivations are based on the properties of the incomplete beta function ratio noted in Appendix. Further properties of this function ratio can be read from Prudnikov *et al.* (1986) and Gradshteyn and Ryzhik (2000).

PROPOSITION 2.1. *If $1/k \geq 1$ is an integer then, on using (A.1), (1.5) reduces to*

$$f(x) = \begin{cases} \dfrac{k\Gamma(h)}{2\lambda\Gamma(1/k)\Gamma(h-1/k)} \left\{1 + \left(\dfrac{x}{\lambda}\right)^k\right\}^{-h} \\ \quad \times \left[2 - \displaystyle\sum_{i=1}^{1/k} \dfrac{\Gamma(h+i-1-1/k)}{\Gamma(h-1/k)\Gamma(i)} \left(\dfrac{\gamma x}{\lambda}\right)^{k(i-1)} \right. \\ \qquad \left. \times \left\{1 + \left(\dfrac{\gamma x}{\lambda}\right)^k\right\}^{1-i-h+1/k}\right], \textit{if } x \geq 0, \\ \dfrac{k\Gamma(h)}{2\lambda\Gamma(1/k)\Gamma(h-1/k)} \left\{1 + \left(-\dfrac{x}{\lambda}\right)^k\right\}^{-h} \\ \quad \times \left[\displaystyle\sum_{i=1}^{1/k} \dfrac{\Gamma(h+i-1-1/k)}{\Gamma(h-1/k)\Gamma(i)} \left(-\dfrac{\gamma x}{\lambda}\right)^{k(i-1)} \right. \\ \qquad \left. \times \left\{1 + \left(-\dfrac{\gamma x}{\lambda}\right)^k\right\}^{1-i-h+1/k}\right], \textit{if } x \leq 0. \end{cases} \tag{2.1}$$

PROPOSITION 2.2. *If $h - 1/k \geq 1$ is an integer then, on using (A.2), (1.5) reduces to*

$$f(x) = \begin{cases} \dfrac{k\Gamma(h)}{2\lambda\Gamma(1/k)\Gamma(h-1/k)} \left\{1 + \left(\dfrac{x}{\lambda}\right)^k\right\}^{-h} \\ \quad \times \left[1 + \displaystyle\sum_{i=1}^{h-1/k} \dfrac{\Gamma(i-1+1/k)}{\Gamma(1/k)\Gamma(i)} \right. \\ \qquad \left. \times \left(\dfrac{\gamma x}{\lambda}\right) \left\{1 + \left(\dfrac{\gamma x}{\lambda}\right)^k\right\}^{1-i-1/k}\right], \textit{if } x \geq 0, \\ \dfrac{k\Gamma(h)}{2\lambda\Gamma(1/k)\Gamma(h-1/k)} \left\{1 + \left(-\dfrac{x}{\lambda}\right)^k\right\}^{-h} \\ \quad \times \left[1 + \displaystyle\sum_{i=1}^{h-1/k} \dfrac{\Gamma(i-1+1/k)}{\Gamma(1/k)\Gamma(i)} \right. \\ \qquad \left. \times \left(\dfrac{\gamma x}{\lambda}\right) \left\{1 + \left(-\dfrac{\gamma x}{\lambda}\right)^k\right\}^{1-i-1/k}\right], \textit{if } x \leq 0. \end{cases} \tag{2.2}$$

If $k = 2$ and $h - 1/k = \nu/2 \geq 1$ is an integer then (2.2) reduces to the well known skewed t distribution with degrees of freedom ν.

PROPOSITION 2.3. *If $k = 2$ and $h = 1$ then, on using (A.3), (1.5) reduces to the skewed Cauchy distribution given by*

$$f(x) = \frac{1}{\lambda\pi} \left\{ 1 + \left(\frac{x}{\lambda} \right)^2 \right\}^{-1} \left\{ 1 + \frac{2}{\pi} \arctan \left(\frac{\gamma x}{\lambda} \right) \right\}. \tag{2.3}$$

PROPOSITION 2.4. *If $h - 1/k = 1/2$ and $h \geq 2$ is an integer then, on using (A.4), (1.5) reduces to*

$$f(x) = \begin{cases} \dfrac{k\Gamma(h)}{2\lambda\sqrt{\pi}\Gamma(1/k)} \left\{ 1 + \left(\dfrac{\gamma x}{\lambda} \right)^k \right\}^{-h} \\ \quad \times \left[1 + \dfrac{2}{\pi} \arctan \left(\dfrac{\gamma x}{\lambda} \right)^{k/2} - \dfrac{1}{\sqrt{\pi}} \displaystyle\sum_{i=1}^{h-1} \dfrac{\Gamma(i)}{\Gamma(i + 1/2)} \left(\dfrac{\gamma x}{\lambda} \right)^{k(i-1/2)} \right. \\ \qquad \left. \times \left\{ 1 + \left(\dfrac{\gamma x}{\lambda} \right)^k \right\}^{-i} \right], \text{ if } x \geq 0, \\ \dfrac{k\Gamma(h)}{2\lambda\sqrt{\pi}\Gamma(1/k)} \left\{ 1 + \left(-\dfrac{\gamma x}{\lambda} \right)^k \right\}^{-h} \\ \quad \times \left[1 - \dfrac{2}{\pi} \arctan \left(-\dfrac{\gamma x}{\lambda} \right)^{k/2} + \dfrac{1}{\sqrt{\pi}} \displaystyle\sum_{i=1}^{h-1} \dfrac{\Gamma(i)}{\Gamma(i + 1/2)} \left(-\dfrac{\gamma x}{\lambda} \right)^{k(i-1/2)} \right. \\ \qquad \left. \times \left\{ 1 + \left(-\dfrac{\gamma x}{\lambda} \right)^k \right\}^{-i} \right], \text{ if } x \leq 0. \end{cases} \tag{2.4}$$

PROPOSITION 2.5. *If $1/k = 1/2$ and $h \geq 2$ is an integer then, on using (A.5), (1.5) reduces to*

$$\begin{aligned} f(x) = \frac{\Gamma(h)}{\lambda\sqrt{\pi}\Gamma(h - 1/2)} \left\{ 1 + \left(\frac{x}{\lambda} \right)^2 \right\}^{-h} & \left[1 + \frac{2}{\pi} \arctan \left(\frac{\gamma x}{\lambda} \right) \right. \\ & \left. + \frac{1}{\sqrt{\pi}} \sum_{i=1}^{h-1} \frac{\Gamma(i)}{\Gamma(i + 1/2)} \left(\frac{\gamma x}{\lambda} \right) \left\{ 1 + \left(\frac{\gamma x}{\lambda} \right)^2 \right\}^{-i} \right]. \end{aligned} \tag{2.5}$$

PROPOSITION 2.6. *If $1/k = p - 1/2$ and $h - 1/k = q - 1/2$, where $p \geq 2$ and*

$q \geq 2$ are integers, then, on using (A.6), (1.5) reduces to

$$f(x) = \begin{cases} \dfrac{k\Gamma(p+q-1)}{2\lambda\Gamma(p-1/2)\Gamma(q-1/2)}\left\{1+\left(\dfrac{x}{\lambda}\right)^k\right\}^{1-p-q} \\ \quad \times\left[1+\dfrac{2}{\pi}\arctan\left(\dfrac{\gamma x}{\lambda}\right)^{k/2}\right. \\ \quad -\dfrac{1}{\sqrt{\pi}}\displaystyle\sum_{i=1}^{p-1}\dfrac{\Gamma(i)}{\Gamma(i+1/2)}\left(\dfrac{\gamma x}{\lambda}\right)^{k(i-1/2)}\left\{1+\left(\dfrac{\gamma x}{\lambda}\right)^k\right\}^{-i} \\ \quad +\displaystyle\sum_{i=1}^{q-1}\dfrac{\Gamma(p+i-1)}{\Gamma(p-1/2)\Gamma(i+1/2)}\left(\dfrac{\gamma x}{\lambda}\right)^{k(p-1/2)} \\ \quad \left.\times\left\{1+\left(\dfrac{\gamma x}{\lambda}\right)^k\right\}^{1-i-p}\right], \text{ if } x \geq 0, \\ \dfrac{k\Gamma(p+q-1)}{2\lambda\Gamma(p-1/2)\Gamma(q-1/2)}\left\{1-\left(-\dfrac{x}{\lambda}\right)^k\right\}^{1-p-q} \\ \quad \times\left[1+\dfrac{2}{\pi}\arctan\left(-\dfrac{\gamma x}{\lambda}\right)^{k/2}\right. \\ \quad +\dfrac{1}{\sqrt{\pi}}\displaystyle\sum_{i=1}^{p-1}\dfrac{\Gamma(i)}{\Gamma(i+1/2)}\left(-\dfrac{\gamma x}{\lambda}\right)^{k(i-1/2)}\left\{1+\left(-\dfrac{\gamma x}{\lambda}\right)^k\right\}^{-i} \\ \quad -\displaystyle\sum_{i=1}^{q-1}\dfrac{\Gamma(p+i-1)}{\Gamma(p-1/2)\Gamma(i+1/2)}\left(-\dfrac{\gamma x}{\lambda}\right)^{k(p-1/2)} \\ \quad \left.\times\left\{1+\left(-\dfrac{\gamma x}{\lambda}\right)^k\right\}^{1-i-p}\right], \text{ if } x \leq 0. \end{cases} \tag{2.6}$$

3. MOMENTS

By Lemma A.2 in Gupta *et al.* (2002), the even order moments of X (having the pdf (1.5)) are the same as those of the generalized t distribution given by (1.1). The moments of the generalized t distribution are known in the literature (see, for example, McDonald and Newey (1984, 1988)) to take the form

$$E\left(X^n\right) = \frac{\lambda^n\Gamma\left(h-\dfrac{n+1}{k}\right)\Gamma\left(\dfrac{n+1}{k}\right)}{\Gamma\left(\dfrac{1}{k}\right)\Gamma\left(h-\dfrac{1}{k}\right)} \tag{3.1}$$

for $n < hk - 1$. In the following we derive expressions for the odd order moments of X. Theorems 3.1 to 3.6 provide closed form expressions for $E(X^n)$ when X has the pdfs (2.1)–(2.6).

THEOREM 3.1. *If X is a random variable having the pdf (2.1) then*

$$E\left(X^{n}\right) = \frac{\lambda^{n}\Gamma\left(h-(n+1)/k\right)\Gamma\left((n+1)/k\right)}{\Gamma\left(1/k\right)\Gamma\left(h-1/k\right)} - \frac{\lambda^{n}\gamma^{-bk}\Gamma\left(h\right)}{\Gamma\left(1/k\right)\Gamma^{2}\left(h-1/k\right)}\sum_{i=1}^{1/k}\frac{\Gamma\left(h+i-1-1/k\right)}{\Gamma(i)}M(i) \tag{3.2}$$

for odd integers $n < 2(hk-1)$, *where*

$$M(i) = B\left(2h-\frac{n+2}{k},\frac{n+1}{k}+i-1\right) \times {}_2F_1\left(i+h-1-\frac{1}{k},2h-\frac{n+2}{k};2h+i-1-\frac{1}{k};1-\gamma^{-k}\right). \tag{3.3}$$

PROOF. Using (2.1), one can write

$$E\left(X^{n}\right) = \frac{\lambda^{n}\Gamma\left(h-(n+1)/k\right)\Gamma\left((n+1)/k\right)}{\Gamma\left(1/k\right)\Gamma\left(h-1/k\right)} - \frac{k\Gamma\left(h\right)}{\lambda\Gamma\left(1/k\right)\Gamma^{2}\left(h-1/k\right)}\sum_{i=1}^{1/k}\frac{\Gamma\left(h+i-1-1/k\right)}{\Gamma(i)}N(i), \tag{3.4}$$

where

$$N(i) = \int_0^\infty x^n\left\{1+\left(\frac{x}{\lambda}\right)^k\right\}^{-h}\left(\frac{\gamma x}{\lambda}\right)^{k(i-1)}\left\{1+\left(\frac{\gamma x}{\lambda}\right)^k\right\}^{1-i-h+1/k} dx.$$

Setting $y=(x/\lambda)^k$, the integral $N(i)$ can be reduced to

$$N(i) = \frac{\lambda^{n+1}\gamma^{-kb}}{k}\int_0^\infty y^{i-2+(n+1)/k}(1+y)^{-h}\left(\gamma^{-k}+y\right)^{1-i-h+1/k} dy. \tag{3.5}$$

By application of Lemma A.1 (in the appendix), the integral in (3.5) can be reduced to $M(i)$ in (3.3). The result in (3.2) follows by combining (3.4) and (3.5). □

THEOREM 3.2. *If X is a random variable having the pdf (2.2) then*

$$E\left(X^{n}\right) = \frac{\lambda^{n}\Gamma\left(h\right)}{\Gamma^{2}\left(1/k\right)\Gamma\left(h-1/k\right)}\sum_{i=1}^{h-1/k}\frac{\gamma^{k(1-i)}\Gamma\left(1/k+i-1\right)}{\Gamma(i)}M(i) \tag{3.6}$$

for odd integers $n < hk - 1$, where

$$M(i) = B\left(h+i-1-\frac{n+1}{k}, \frac{n+2}{k}\right) \times {}_2F_1\left(i-1+\frac{1}{k}, h+i-1-\frac{n+1}{k}; h+i-1+\frac{1}{k}; 1-\gamma^{-k}\right). \quad (3.7)$$

PROOF. Using (2.2), one can write

$$E(X^n) = \frac{k\Gamma(h)}{\lambda\Gamma^2(1/k)\Gamma(h-1/k)} \sum_{i=1}^{h-1/k} \frac{\Gamma(i-1+1/k)}{\Gamma(i)} N(i), \quad (3.8)$$

where

$$N(i) = \int_0^\infty x^n \left\{1+\left(\frac{x}{\lambda}\right)^k\right\}^{-h} \left(\frac{\gamma x}{\lambda}\right) \left\{1+\left(\frac{\gamma x}{\lambda}\right)^k\right\}^{1-i-1/k} dx.$$

Setting $y = (x/\lambda)^k$, the integral $N(i)$ can be reduced to

$$N(i) = \frac{\lambda^{n+1}\gamma}{k} \int_0^\infty y^{(n+2)/k-1}(1+y)^{-h}\left(\gamma^{-k}+y\right)^{1-i-1/k} dy. \quad (3.9)$$

By application of Lemma A.1 (in the appendix), the integral in (3.9) can be reduced to $M(i)$ in (3.7). The result in (3.6) follows by combining (3.8) and (3.9). □

THEOREM 3.3. *If X is a random variable having the pdf (2.3) then*

$$E(X^n) = \frac{4\lambda^n}{\pi^2} \int_0^\infty \frac{y^n \arctan(\gamma y)}{1+y^2} dy \quad (3.10)$$

for $n < 1$.

PROOF. Using (2.3), one can write

$$E(X^n) = \frac{4}{\pi^2\lambda} \int_0^\infty x^n \left\{1+\left(\frac{x}{\lambda}\right)^2\right\}^{-1} \arctan\left(\frac{\gamma x}{\lambda}\right) dx.$$

The result in (3.10) follows by setting $y = x/\lambda$. □

THEOREM 3.4. *If X is a random variable having the pdf (2.4) then*

$$E(X^n) = \frac{\lambda^n\Gamma(h)}{\pi\Gamma(h-1/2)} \left\{\frac{4}{\sqrt{\pi}} \int_0^\infty \frac{y^n \arctan(\gamma y)}{(1+y^2)^h} dy + \sum_{i=1}^{h-1} \frac{\gamma^{1-2i}\Gamma(i)}{\Gamma(i+1/2)} M(i)\right\} \quad (3.11)$$

for odd integers $n < 2h - 1$, where

$$M(i) = B\left(h+i-1-\frac{n}{2}, \frac{n}{2}+1\right) \times {}_2F_1\left(i, h+i-1-\frac{n}{2}; h+i; 1-\gamma^{-2}\right). \tag{3.12}$$

PROOF. Using (2.4), one can write

$$E\left(X^n\right) = \frac{2\Gamma\left(h\right)}{\pi\lambda\Gamma\left(h-1/2\right)}\left\{\frac{2}{\sqrt{\pi}}N_1 + \sum_{i=1}^{h-1}\frac{\Gamma(i)}{\Gamma\left(i+1/2\right)}N_2(i)\right\}, \tag{3.13}$$

where

$$N_1 = \int_0^\infty x^n \left\{1+\left(\frac{x}{\lambda}\right)^2\right\}^{-h} \arctan\left(\frac{\gamma x}{\lambda}\right) dx$$

and

$$N_2(i) = \int_0^\infty x^n \left\{1+\left(\frac{x}{\lambda}\right)^2\right\}^{-h} \left(\frac{\gamma x}{\lambda}\right)\left\{1+\left(\frac{\gamma x}{\lambda}\right)^2\right\}^{-i} dx.$$

Setting $y = (x/\lambda)^2$, the integrals N_1 and $N_2(i)$ can be reduced to

$$N_1 = \lambda^{n+1}\int_0^\infty \frac{y^n \arctan(\gamma y)}{\left(1+y^2\right)^h} dy \tag{3.14}$$

and

$$N_2(i) = \frac{\lambda^{n+1}\gamma^{1-2i}}{2}\int_0^\infty y^{n/2}\left(1+y\right)^{-h}\left(\gamma^{-2}+y\right)^{-i} dy. \tag{3.15}$$

By application of Lemma A.1 (in the appendix), the integral in (3.15) can be reduced to $M(i)$ in (3.12). The result in (3.11) follows by combining (3.13), (3.14) and (3.15). □

THEOREM 3.5. *If X is a random variable having the pdf (2.5) then*

$$E\left(X^n\right) = \frac{\lambda^n\Gamma\left(h\right)}{\pi\Gamma\left(1/k\right)}\left\{\frac{2k}{\sqrt{\pi}}\int_0^\infty \frac{y^n \arctan(\gamma y)^{k/2}}{\left(1+y^k\right)^h} dy - \gamma^{-k/2}\sum_{i=1}^{h-1}\frac{\Gamma(i)}{\Gamma\left(i+1/2\right)}M(i)\right\} \tag{3.16}$$

for odd integers $n < \min(k/2, hk-1)$, *where*

$$M(i) = B\left(h+\frac{1}{2}-\frac{n+1}{k}, \frac{n+1}{k}+i-\frac{1}{2}\right) \times {}_2F_1\left(i, h-\frac{n+1}{k}+\frac{1}{2}; h+i; 1-\gamma^{-k}\right). \quad (3.17)$$

PROOF. Using (2.5), one can write

$$E\left(X^n\right) = \frac{k\Gamma(h)}{\pi\lambda\Gamma(1/k)}\left\{\frac{2}{\sqrt{\pi}}N_1 - \sum_{i=1}^{h-1}\frac{\Gamma(i)}{\Gamma(i+1/2)}N_2(i)\right\}, \quad (3.18)$$

where

$$N_1 = \int_0^\infty x^n \left\{1+\left(\frac{x}{\lambda}\right)^k\right\}^{-h} \arctan\left(\frac{\gamma x}{\lambda}\right)^{k/2} dx$$

and

$$N_2(i) = \int_0^\infty x^n \left\{1+\left(\frac{x}{\lambda}\right)^k\right\}^{-h} \left(\frac{\gamma x}{\lambda}\right)^{k(i-1/2)} \left\{1+\left(\frac{\gamma x}{\lambda}\right)^k\right\}^{-i} dx.$$

Setting $y = (x/\lambda)^k$, the integrals N_1 and $N_2(i)$ can be reduced to

$$N_1 = \lambda^{n+1}\int_0^\infty \frac{y^n \arctan(\gamma y)^{k/2}}{\left(1+y^k\right)^h} dy \quad (3.19)$$

and

$$N_2(i) = \frac{\lambda^{n+1}\gamma^{-k/2}}{k}\int_0^\infty y^{i-3/2+(n+1)/k}\left(1+y\right)^{-h}\left(\gamma^{-k}+y\right)^{-i} dy. \quad (3.20)$$

By application of Lemma A.1 (in the appendix), the integral in (3.20) can be reduced to $M(i)$ in (3.17). The result in (3.16) follows by combining (3.18), (3.19) and (3.20). □

THEOREM 3.6. *If X is a random variable having the pdf (2.6) then*

$$E\left(X^n\right) = \frac{\lambda^n\Gamma(h)}{\Gamma(1/k)\Gamma(h-1/k)}\left\{\frac{2k}{\pi}\int_0^\infty \frac{y^n \arctan(\gamma y)^{k/2}}{\left(1+y^2\right)^h} dy - \frac{\gamma^{-k/2}}{\sqrt{\pi}}\sum_{i=1}^{p-1}\frac{\Gamma(i)}{\Gamma(i+1/2)}M_1(i) + \frac{1}{\Gamma(p-1/2)}\sum_{i=1}^{q-1}\frac{\Gamma(p+i-1)\gamma^{k(1/2-i)}}{\Gamma(i+1/2)}M_2(i)\right\} \quad (3.21)$$

for odd integers $n < \min(k/2, 2h-1)$, *where*

$$M_1(i) = B\left(q + \frac{1}{k} - \frac{n+1}{k}, \frac{n+1}{k} + i - \frac{1}{2}\right) \times {}_2F_1\left(i, q - \frac{n}{k}; h+i; 1-\gamma^{-k}\right) \tag{3.22}$$

and

$$M_2(i) = B\left(q + i - 1 - \frac{n}{k}, p - \frac{1}{2} + \frac{n+1}{k}\right) \times {}_2F_1\left(p+i-1, q+i-1-\frac{n}{k}; h+i+p-1; 1-\gamma^{-k}\right). \tag{3.23}$$

PROOF. Using (2.6), one can write

$$E\left(X^n\right) = \frac{k\Gamma(h)}{\lambda\Gamma(1/k)\Gamma(h-1/k)}\left\{\frac{2}{\pi}N_1 - \frac{1}{\sqrt{\pi}}\sum_{i=1}^{p-1}\frac{\Gamma(i)}{\Gamma(i+1/2)}N_2(i) + \sum_{i=1}^{q-1}\frac{\Gamma(p+i-1)}{\Gamma(p-1/2)\Gamma(i+1/2)}N_3(i)\right\}, \tag{3.24}$$

where

$$N_1 = \int_0^\infty x^n \left\{1 + \left(\frac{x}{\lambda}\right)^k\right\}^{-h} \arctan\left(\frac{\gamma x}{\lambda}\right)^{k/2} dx,$$

$$N_2(i) = \int_0^\infty x^n \left\{1 + \left(\frac{x}{\lambda}\right)^k\right\}^{-h} \left(\frac{\gamma x}{\lambda}\right)^{k(i-1/2)} \left\{1 + \left(\frac{\gamma x}{\lambda}\right)^k\right\}^{-i} dx$$

and

$$N_3(i) = \int_0^\infty x^n \left\{1 + \left(\frac{x}{\lambda}\right)^k\right\}^{-h} \left(\frac{\gamma x}{\lambda}\right)^{k(p-1/2)} \left\{1 + \left(\frac{\gamma x}{\lambda}\right)^k\right\}^{1-i-p} dx.$$

Setting $y = (x/\lambda)^k$, the integrals N_1, $N_2(i)$ and $N_3(i)$ can be reduced to

$$N_1 = \lambda^{n+1}\int_0^\infty \frac{y^n \arctan(\gamma y)^{k/2}}{\left(1+y^k\right)^h} dy, \tag{3.25}$$

$$N_2(i) = \frac{\lambda^{n+1}\gamma^{-k/2}}{k}\int_0^\infty y^{i-3/2+(n+1)/k}(1+y)^{-h}\left(\gamma^{-k}+y\right)^{-i} dy \tag{3.26}$$

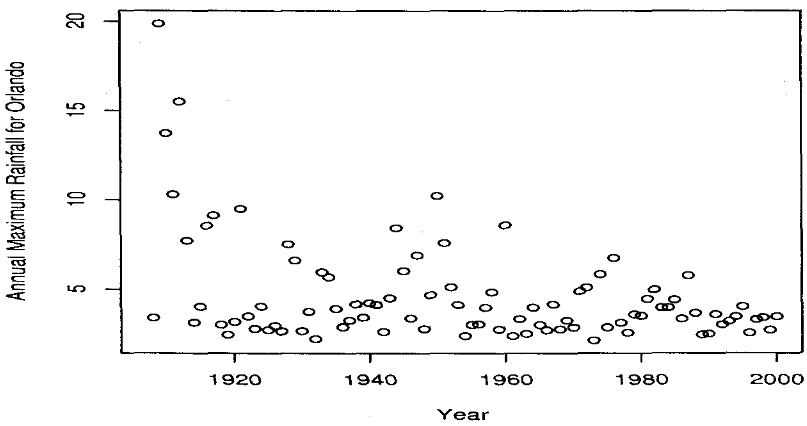


FIGURE 4.1 *Annual maxima daily rainfall for Orlando (1907–2000).*

and

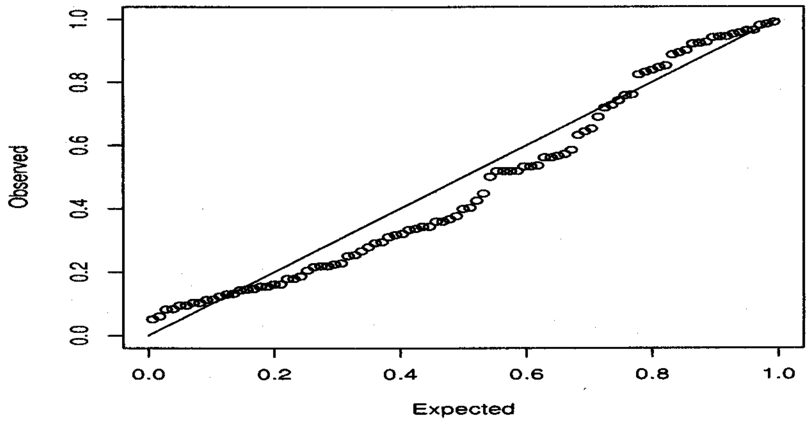
$$N_3(i) = \frac{\lambda^{n+1}\gamma^{k(1/2-i)}}{k} \int_0^\infty y^{p-3/2+(n+1)/k} (1+y)^{-h} \left(\gamma^{-k} + y\right)^{1-i-p} dy. \quad (3.27)$$

By application of Lemma A.1 (in the appendix), the integrals in (3.26) and (3.27) can be reduced to $M_1(i)$ and $M_2(i)$ in (3.22) and (3.23), respectively. The result in (3.21) follows by combining (3.24), (3.25), (3.26) and (3.27). □

4. APPLICATION

In this section we show that the skewed generalized t distribution given by (1.5) can be a better model than one just based on the generalized t distribution given by (1.1). We illustrate this for rainfall data from Orlando, Florida. The data consists of annual maximum daily rainfall for the years from 1907 to 2000; see Figure 4.1 below. The independence of the data values was verified by plotting the auto-correlation function. The data were obtained from the Department of Meteorology in Tallahassee, Florida.

Since rainfall is always non-negative, we first transformed the data to log (rainfall/median(rainfall)). We then fitted both (1.1) and (1.5) to the transformed



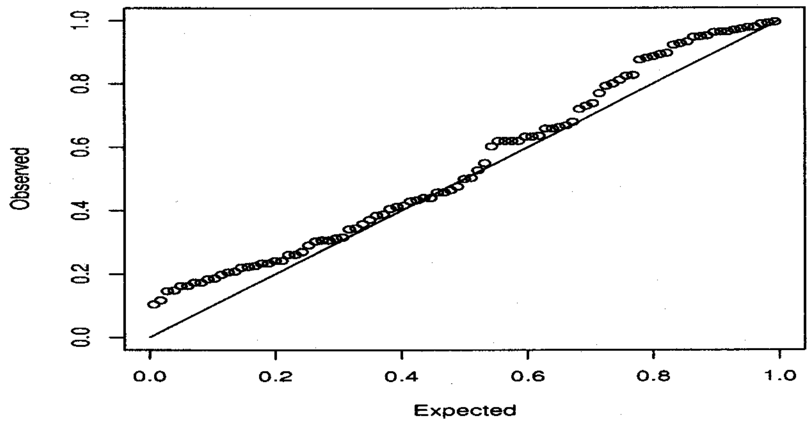


FIGURE 4.2 *Probability plots for the models based on the skewed generalized t distribution (top) and the generalized t distribution (bottom).*

data by the method of maximum likelihood. The quasi-Newton algorithm nlm in the R software package (Dennis and Schnabel, 1983; Schnabel *et al*, 1985; Ihaka and Gentleman, 1996) was used to solve the likelihood equations. The following estimates were obtained:

$$\hat{\lambda} = 0.845, \hat{k} = 1.557, \hat{h} = 3.570 \text{ with } -\log L = 55.4$$

and

$$\hat{\lambda} = 0.574, \hat{k} = 1.942, \hat{h} = 2.186, \hat{\gamma} = 0.280 \text{ with } -\log L = 51.8$$

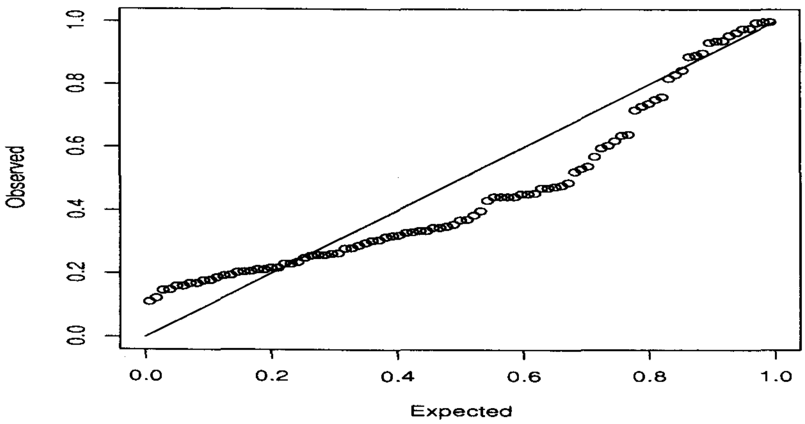


FIGURE 4.3 *Probability plot for the model based on the gamma distribution.*

for the models based on (1.1) and (1.5), respectively. Thus, it follows by the standard likelihood ratio test that (1.5) should be preferred to (1.1). In order words, the skewed generalized t distribution provides a better fit than the generalized t distribution.

The goodness of fit of these models can be examined by probability plots. A probability plot is where the observed probability is plotted against the probability predicted by the fitted model. To check the goodness of fit given by (1.5), one would plot $F(y_{(i)})$ versus $(i - 0.375)/(n + 0.25)$ (as recommended by Blom (1958) and Chambers *et al* (1983)), where $F(\cdot)$ is the cdf corresponding to (1.5) and $y_{(i)}$ are the sorted values, in the ascending order, of the observed annual maximum daily rainfall. Similarly, to check the goodness of fit given by (1.1), one would plot $G(y_{(i)})$ versus $(i-0.375)/(n+0.25)$, where $G(\cdot)$ is given by (1.2). These plots are shown in Figure 4.2. It is evident that the skewed generalized t distribution significantly improves the fit given by the generalized t distribution.

The traditional model for rainfall data is based on the gamma distribution given by the pdf $1/\{s^a\Gamma(a)\}x^{a-1}\exp(-x/s)$ for $x > 0$. Fitting this model to the data in Figure 4.1, one obtains the maximum likelihood estimates $\hat{a} = 4.247$ and $\hat{s} = 1.097$. The corresponding probability plot for this fit is shown in Figure 4.3. It is evident that the fit of the gamma distribution is even worse than that of (1.1). Thus, the traditional model is no match to the model given by the skewed generalized t distribution (1.5).

APPENDIX

The calculations of this paper require the following technical results.

LEMMA A.1. *(equation (3.197.9), Gradshteyn and Ryzhik, 2000) For* $a > b > 0$,

$$\int_0^\infty x^{b-1}(1+x)^{c-a}(x+d)^{-c}dx = B(a-b,b)\ {}_2F_1\left(c,a-b;a;1-d\right),$$

where

$${}_2F_1\left(a,b;c;x\right) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{x^k}{k!}$$

denotes the Gauss hypergeometric function and $(z)_k = z(z+1)\cdots(z+k-1)$ *denotes the ascending factorial.*

LEMMA A.2. *Six important properties of the incomplete beta function ratio given by (1.3) are:*

- *if* a *is an integer then*

$$I_x(a,b) = 1 - \sum_{i=1}^{a} \frac{\Gamma(b+i-1)}{\Gamma(b)\Gamma(i)} x^{i-1}(1-x)^b; \qquad (A.1)$$

- *if* b *is an integer then*

$$I_x(a,b) = \sum_{i=1}^{b} \frac{\Gamma(a+i-1)}{\Gamma(a)\Gamma(i)} x^{a}(1-x)^{i-1}; \qquad (A.2)$$

- *if* $a = 1/2$ *and* $b = 1/2$ *then*

$$I_x(a,b) = \frac{2}{\pi}\arctan\sqrt{\frac{x}{1-x}}; \qquad (A.3)$$

- *if* $a = k - 1/2$ *and* $b = 1/2$ *then*

$$I_x\left(a,b\right) = I_x\left(\frac{1}{2},\frac{1}{2}\right) - \sqrt{\frac{x(1-x)}{\pi}} \sum_{i=1}^{k-1} \frac{\Gamma(i)}{\Gamma(i+1/2)} x^{i-1}; \qquad (A.4)$$

- *if* $a = 1/2$ *and* $b = j - 1/2$ *then*

$$I_x(a,b) = I_x\left(\frac{1}{2},\frac{1}{2}\right) + \sqrt{\frac{x(1-x)}{\pi}} \sum_{i=1}^{j-1} \frac{\Gamma(i)}{\Gamma(i+1/2)} (1-x)^{i-1}; \qquad (A.5)$$

- *if* $a = k - 1/2$ *and* $b = j - 1/2$ *then*

$$I_x(a,b) = I_x\left(k-\frac{1}{2},\frac{1}{2}\right) + \sum_{i=1}^{j-1} \frac{\Gamma(k+i-1)}{\Gamma(k-1/2)\Gamma(i+1/2)} x^{k-1/2}(1-x)^{i-1/2}. \qquad (A.6)$$

Further properties of the above special functions can be found in Prudnikov *et al.* (1986) and Gradshteyn and Ryzhik (2000).

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