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ESTIMATION OF THE SECOND ORDER PARAMETER CHARACTERIZING THE TAIL BEHAVIOR OF PROBABILITY DISTRIBUTIONS: CONSISTENCY †

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ABSTRACT

In this paper we introduce an estimator of the second order parameter characterizing the tail behavior of probability distributions and prove its consistency.

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1. INTRODUCTION

Let $X_1, ..., X_n$ be an independent and identically distributed (*i.i.d.*) sample from an unknown distribution function (*d.f.*) F. Suppose F belongs to the domain of attraction of an extreme value *d.f.* G_β for some $\beta \in \mathbb{R}$ $[F \in \mathcal{D}(G_\beta)]$, where

$$G_\beta(x) := \exp\{-(1+\beta x)^{-1/\beta}\},\ 1+\beta x > 0.$$

That is, suppose there exist some constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$\lim_{n\to\infty} P\{(\max\{X_1, ..., X_n\} - b_n)/a_n \le x\} = \lim_{n\to\infty} F^n(a_n x + b_n) = G_\beta(x) \quad (1.1)$$

for all x with $1+\beta x > 0$. Throughout the paper the case $\beta = 0$ is interpreted as the limit when $\beta \to 0$, so that $G_0(x) = \exp(-e^{-x})$, $x \in \mathbb{R}$.

The parameter β is called the extreme value index (or tail index) of F, which represents how heavy the right tail of F is. The right tail of F becomes heavier as β gets larger. F is said to have a short, medium or heavy tail, respectively, if

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$\beta < 0$, $\beta = 0$ or $\beta > 0$. There is a rich literature on the estimation of β using the sample $X_1, ..., X_n$. For example, see Hill (1975), Pickands (1975), Dekkers and de Haan (1989), Dekkers, Einmahl and de Haan (1989), Drees (1995) and Yun (2002). To prove asymptotic normality of the estimators of β introduced in these papers, one needs to consider a second order condition which controls the speed of convergence in (1.1) and contains another parameter $\rho \leq 0$ called the second order parameter.

Those estimators of β are typically defined using m, say, upper order statistics from $X_1, ..., X_n$. The optimal value of m minimizing the asymptotic mean squared error of the estimator of β depends on ρ, and therefore one needs to estimate ρ using the sample $X_1, ..., X_n$. For the estimation of ρ, Gomes, de Haan and Peng (2002) dealt with the case of $\beta > 0$, *i.e.* heavy tail.

In this paper we deal with the general case of $\beta \in \mathbb{R}$. Under $\beta \in \mathbb{R}$, we introduce an estimator of ρ and prove its consistency.

2. MAIN RESULTS

Let the function U be defined by $U(x) := F^{-1}(1 - 1/x)$, $x > 1$, where F^{-1} denotes the quantile function of F. Then a necessary and sufficient condition for $F \in \mathcal{D}(G_\beta)$ for some $\beta \in \mathbb{R}$ is the existence of a function $a(t) > 0$ such that, for $x > 0$,

$$\lim_{t\to\infty} \frac{U(tx) - U(t)}{a(t)} = \frac{x^\beta - 1}{\beta} \tag{2.1}$$

(cf. de Haan (1984)). In this case the function $a(t)$ is regularly varying at infinity with index β [$a(t) \in RV_\beta$], *i.e.* it holds that, for $x > 0$, $\lim_{t\to\infty} a(tx)/a(t) = x^\beta$. If (2.1) holds and $x_F := \sup\{x : F(x) < 1\} > 0$, then it also holds that, for $x > 0$,

$$\lim_{t\to\infty} \frac{\log U(tx) - \log U(t)}{a(t)/U(t)} = \frac{x^{\beta_-} - 1}{\beta_-} =: D(x; \beta_-), \tag{2.2}$$

where $\beta_- := \min\{\beta, 0\}$.

For asymptotic normality of estimators of β appeared in the literature, one needs to consider a second order condition that, for $x > 0$,

$$\frac{\log U(tx) - \log U(t)}{a(t)/U(t)} = D(x; \beta_-) + A(t)H(x; \beta_-, \rho) + o(A(t)) \text{ as } t \to \infty, \tag{2.3}$$

where $A(t)$ is a function of constant sign for large values of t, satisfying $A(t) = o(1)$ as $t \to \infty$, and

$$H(x; \beta_-, \rho) := \frac{1}{\rho}\left(\frac{x^{\beta_-+\rho} - 1}{\beta_- + \rho} - \frac{x^{\beta_-} - 1}{\beta_-}\right)$$

for some $\rho \le 0$ (*cf.* de Haan and Stadtmüller (1996)). In this case we must have $|A(t)| \in RV_\rho$. Throughout the case $\rho = 0$ is interpreted as the limit when $\rho \to 0$, so that $H(x; \beta_-, 0) = (x^{\beta_-}(\log x)\beta_- - x^{\beta_-} + 1)/\beta_-^2$. Under (2.3) which is clearly a stronger condition than (2.2), we want to estimate the second order parameter ρ using the sample $X_1, ..., X_n$.

Let $X_1^{(n)} \ge X_2^{(n)} \ge \cdots \ge X_n^{(n)}$ denote the descending order statistics of $X_1, ..., X_n$. For $k = 1, 2, ...,$ we define

$$M_{n,m}^{(k)} := \frac{1}{m} \sum_{j=1}^{m} (\log X_j^{(n)} - \log X_{m+1}^{(n)})^k$$

and

$$N_{n,m}^{(k)} := (M_{n,m}^{(1)})^k / M_{n,m}^{(k)},$$

provided that $X_{m+1}^{(n)} > 0$, where $1 \le m < n$. For $k = 2, 3, ...,$ we also define the function $\phi_k : (-\infty, 1/k) \to (0, 1)$ by

$$\phi_k(x) := \frac{\prod_{j=2}^{k}(1 - jx)}{k!(1 - x)^{k-1}}, \quad x < \frac{1}{k},$$

which is strictly decreasing, and let $\phi_k^{-1} : (0, 1) \to (-\infty, 1/k)$ denote its inverse function. Let the functions c_i, $i = 1, 2, 3, 4$, be defined by

$$\begin{aligned}
c_1(x) &:= 2(1 - 4x)(3 - 17x + 23x^2), \\
c_2(x) &:= -(1 - 3x)(6 - 33x + 46x^2), \\
c_3(x) &:= 2(3 - 17x + 23x^2), \\
c_4(x) &:= -(1 - 3x)(3 - 8x).
\end{aligned}$$

In this paper we introduce an estimator $\hat{\rho}_{n,m}$ of ρ, defined by

$$\hat{\rho}_{n,m} := \frac{c_1(\widehat{\beta_-}) + c_2(\widehat{\beta_-})R_{n,m}}{c_3(\widehat{\beta_-}) + c_4(\widehat{\beta_-})R_{n,m}},$$

where $1 \le m < n$ and

$$\begin{aligned}
\widehat{\beta_-} &:= \phi_2^{-1}(N_{n,m}^{(2)}), \\
R_{n,m} &:= \frac{\phi_2^{-1}(N_{n,m}^{(2)}) - \phi_3^{-1}(N_{n,m}^{(3)})}{\phi_3^{-1}(N_{n,m}^{(3)}) - \phi_4^{-1}(N_{n,m}^{(4)})}.
\end{aligned}$$

For $k = 0, 1, 2, \ldots$ and for $\alpha \le 0$ and $\beta \in \mathbb{R}$, define

$$s(k; \alpha, \beta_-) := \int_1^\infty x^{\alpha-2} D^k(x; \beta_-)\, dx = \frac{k!}{\prod_{j=0}^{k}(1 - \alpha - j\beta_-)}$$

Notice that $s(k; \alpha, \beta_-) = E(Z^\alpha D^k(Z; \beta_-))$ if Z is a random variable with d.f. $1 - 1/x$, $x > 1$. The following lemma, an extension of Lemma 5.1 of Draisma, de Haan, Peng and Pereira (1999), is needed to prove consistency of $\hat{\rho}_{n,m}$. By $\xrightarrow{d}$ and $\xrightarrow{p}$ we denote convergence in distribution and convergence in probability, respectively.

LEMMA 2.1. *Let $Y_1, \ldots, Y_n$ be i.i.d. random variables with d.f. $1 - 1/x$, $x > 1$, and let $Y_1^{(n)} \ge Y_2^{(n)} \ge \cdots \ge Y_n^{(n)}$ be their descending order statistics. Let $m = m(n)$ be any sequence of integers such that $1 \le m < n$ and $m \to \infty$ as $n \to \infty$.*

(i) For $k = 1, 2, \ldots$ and for $\beta \in \mathbb{R}$, define

$$Q_{n,m}^{(k)}(\beta_-) := \frac{1}{m} \sum_{j=1}^{m} D^k(Y_j^{(n)}/Y_{m+1}^{(n)}; \beta_-) - s(k; 0, \beta_-).$$

Then, for $k = 1, 2, \ldots$ and for $\beta \in \mathbb{R}$,

$$\sqrt{m}(Q_{n,m}^{(1)}(\beta_-), \ldots, Q_{n,m}^{(k)}(\beta_-)) \xrightarrow{d} (Q_1(\beta_-), \ldots, Q_k(\beta_-)) \text{ as } n \to \infty,$$

where $(Q_1(\beta_-), \ldots, Q_k(\beta_-))$ has a k-variate normal distribution with mean vector $(0, \ldots, 0)$ and covariance matrix $(v_{ij}(\beta_-))$ given by

$$v_{ij}(\beta_-) := s(i + j; 0, \beta_-) - s(i; 0, \beta_-) s(j; 0, \beta_-), \ i, j = 1, \ldots, k.$$

(ii) For $k = 1, 2, \ldots$ and for $\beta \in \mathbb{R}$ and $\rho \le 0$,

$$\frac{1}{m} \sum_{j=1}^{m} H(Y_j^{(n)}/Y_{m+1}^{(n)}; \beta_-, \rho) D^{k-1}(Y_j^{(n)}/Y_{m+1}^{(n)}; \beta_-) \xrightarrow{p} b(k; \beta_-, \rho) \text{ as } n \to \infty,$$

where

$$\begin{aligned} b(k; \beta_-, \rho) &:= \int_1^\infty x^{-2} H(x; \beta_-, \rho) D^{k-1}(x; \beta_-)\, dx \\ &= \frac{1}{\rho} \left(\frac{s(k-1; \beta_- + \rho, \beta_-) - s(k-1; 0, \beta_-)}{\beta_- + \rho} - s(k; 0, \beta_-) \right). \end{aligned}$$

PROOF. Let $Z_1, ..., Z_m$ be *i.i.d.* random variables with *d.f.* $1 - 1/x$, $x > 1$, and let $Z_1^{(m)} \geq Z_2^{(m)} \geq \cdots \geq Z_m^{(m)}$ be their order statistics.

(i) Since $(Y_j^{(n)}/Y_{m+1}^{(n)})_{j=1}^m \stackrel{d}{=} (Z_j^{(m)})_{j=1}^m$, we have

$$\sqrt{m}(Q_{n,m}^{(1)}(\beta_-), ..., Q_{n,m}^{(k)}(\beta_-))$$
$$\stackrel{d}{=} \sqrt{m}\left(\frac{1}{m}\sum_{j=1}^m D(Z_j^{(m)};\beta_-) - s(1;0,\beta_-), ..., \frac{1}{m}\sum_{j=1}^m D^k(Z_j^{(m)};\beta_-) - s(k;0,\beta_-)\right)$$
$$= \sqrt{m}\left(\frac{1}{m}\sum_{j=1}^m D(Z_j;\beta_-) - s(1;0,\beta_-), ..., \frac{1}{m}\sum_{j=1}^m D^k(Z_j;\beta_-) - s(k;0,\beta_-)\right)$$

Since $D(Z_j;\beta_-)$ has a *d.f.* $1-(1+\beta_- x)^{-1/\beta_-}$, $x > 0$, $1+\beta_- x > 0$, it can be seen that $(D(Z_j;\beta_-), ..., D^k(Z_j;\beta_-))$, $j = 1, ..., m$, are *i.i.d.* random vectors with mean $(s(1;0,\beta_-), ..., s(k;0,\beta_-))$ and covariance matrix $(v_{ij}(\beta_-))$. The assertion now follows from the multivariate central limit theorem (*cf.* Serfling (1980), Theorem B, page 28).

(ii) Similarly as in (i), we have

$$\frac{1}{m}\sum_{j=1}^m H(Y_j^{(n)}/Y_{m+1}^{(n)};\beta_-,\rho)D^{k-1}(Y_j^{(n)}/Y_{m+1}^{(n)};\beta_-)$$
$$\stackrel{d}{=} \frac{1}{m}\sum_{j=1}^m H(Z_j;\beta_-,\rho)D^{k-1}(Z_j;\beta_-).$$

The assertion follows from the law of large numbers. □

In the following theorem we establish consistency of $\hat{\rho}_{n,m}$. A sequence of positive integers $m = m(n)$ is called an intermediate sequence if $m \to \infty$ and $m/n \to 0$ as $n \to \infty$.

THEOREM 2.2. *Suppose $F \in \mathcal{D}(G_\beta)$ for some $\beta \in \mathbb{R}$ and $x_F > 0$ and that (2.3) holds for some $\rho \leq 0$. Then, for any intermediate sequence $m = m(n)$ such that $\sqrt{m}|A(n/m)| \to \infty$ as $n \to \infty$, we have*

$$\hat{\rho}_{n,m} \xrightarrow{p} \rho \text{ as } n \to \infty.$$

PROOF. Let $m = m(n)$ be any intermediate sequence. Let $Y_1, ..., Y_n$ be *i.i.d.* random variables with *d.f.* $1 - 1/x$, $x > 1$. Then $(X_j)_{j=1}^n \stackrel{d}{=} (U(Y_j))_{j=1}^n$ and so $(X_j^{(n)})_{j=1}^n \stackrel{d}{=} (U(Y_j^{(n)}))_{j=1}^n$, where $Y_1^{(n)} \geq Y_2^{(n)} \geq \cdots \geq Y_n^{(n)}$ denote the order statistics of $Y_1, ..., Y_n$. By (2.3), we have, for $k = 1, 2, ...$ and for $x > 0$, as $t \to \infty$,

$$\begin{aligned}\left(\frac{\log U(tx) - \log U(t)}{a(t)/U(t)}\right)^k &= (D(x;\beta_-) + A(t)H(x;\beta_-,\rho) + o(A(t)))^k \\ &= D^k(x;\beta_-) + kA(t)H(x;\beta_-,\rho)D^{k-1}(x;\beta_-) + o(A(t)).\end{aligned}$$

Replacing t by $Y_{m+1}^{(n)}$ and x by $Y_j^{(n)}/Y_{m+1}^{(n)}$, adding the equalities for $j = 1, ..., m$ and dividing by m, we have, for $k = 1, 2, ...$, as $n \to \infty$,

$$\begin{aligned}&\frac{1}{m}\sum_{j=1}^{m}\left(\frac{\log U(Y_j^{(n)}) - \log U(Y_{m+1}^{(n)})}{a(Y_{m+1}^{(n)})/U(Y_{m+1}^{(n)})}\right)^k \\ &= \frac{1}{m}\sum_{j=1}^{m}\{D^k(Y_j^{(n)}/Y_{m+1}^{(n)};\beta_-) \\ &\quad + kA(Y_{m+1}^{(n)})H(Y_j^{(n)}/Y_{m+1}^{(n)};\beta_-,\rho)D^{k-1}(Y_j^{(n)}/Y_{m+1}^{(n)};\beta_-)\} + o_p(A(Y_{m+1}^{(n)})),\end{aligned}$$

which can be written by Lemma 2.1 as

$$s(k;0,\beta_-) + Q_{n,m}^{(k)}(\beta_-) + kb(k;\beta_-,\rho)A(n/m) + o_p(A(n/m))$$

since $Y_{m+1}^{(n)} \stackrel{p}{\sim} n/m$ and so since $A(Y_{m+1}^{(n)}) \stackrel{p}{\sim} A(n/m)$ by applying the uniform convergence theorem to $|A(t)| \in RV_\rho$. Thus we have, for $k = 1, 2, ...$, as $n \to \infty$,

$$\begin{aligned}M_{n,m}^{(k)} &\stackrel{d}{=} (a(Y_{m+1}^{(n)})/U(Y_{m+1}^{(n)}))^k \cdot \frac{1}{m}\sum_{j=1}^{m}\left(\frac{\log U(Y_j^{(n)}) - \log U(Y_{m+1}^{(n)})}{a(Y_{m+1}^{(n)})/U(Y_{m+1}^{(n)})}\right)^k \\ &= (a(Y_{m+1}^{(n)})/U(Y_{m+1}^{(n)}))^k\{s(k;0,\beta_-) + Q_{n,m}^{(k)}(\beta_-) + kb(k;\beta_-,\rho)A(n/m) \\ &\quad + o_p(A(n/m))\}. \qquad (2.4)\end{aligned}$$

Now, for $k = 2, 3, ...$, we have the Taylor expansion that, for any $x_0 \in (0,1)$,

$$\phi_k^{-1}(x) = \phi_k^{-1}(x_0) + \frac{1}{\phi_k'(\phi_k^{-1}(x_0))}(x - x_0) + o(x - x_0) \text{ as } x \to x_0,$$

and so putting $x_0 = \phi_k(\beta_-)$ leads to

$$\phi_k^{-1}(x) = \beta_- + \frac{1}{\phi_k'(\beta_-)}(x - \phi_k(\beta_-)) + o(x - \phi_k(\beta_-)) \text{ as } x \to \phi_k(\beta_-),$$

from which it follows that, as $n \to \infty$,

$$\begin{aligned}
&\phi_k^{-1}(N_{n,m}^{(k)}) - \beta_- \\
&= \frac{1}{\phi_k'(\beta_-)}(N_{n,m}^{(k)} - \phi_k(\beta_-)) + o_p(N_{n,m}^{(k)} - \phi_k(\beta_-)) \\
&= \frac{(a(Y_{m+1}^{(n)})/U(Y_{m+1}^{(n)}))^k}{\phi_k'(\beta_-)M_{n,m}^{(k)}} \left[\left(\frac{M_{n,m}^{(1)}}{a(Y_{m+1}^{(n)})/U(Y_{m+1}^{(n)})} \right)^k - \frac{\phi_k(\beta_-)M_{n,m}^{(k)}}{(a(Y_{m+1}^{(n)})/U(Y_{m+1}^{(n)}))^k} \right] \\
&\quad + o_p(N_{n,m}^{(k)} - \phi_k(\beta_-)), \qquad (2.5)
\end{aligned}$$

where we have from (2.4), as $n \to \infty$,

$$\begin{aligned}
&\left(\frac{M_{n,m}^{(1)}}{a(Y_{m+1}^{(n)})/U(Y_{m+1}^{(n)})} \right)^k - \frac{\phi_k(\beta_-)M_{n,m}^{(k)}}{(a(Y_{m+1}^{(n)})/U(Y_{m+1}^{(n)}))^k} \\
&\overset{d}{=} \{s(1;0,\beta_-) + Q_{n,m}^{(1)}(\beta_-) + b(1;\beta_-,\rho)A(n/m) + o_p(A(n/m))\}^k \\
&\quad - \phi_k(\beta_-)\{s(k;0,\beta_-) + Q_{n,m}^{(k)}(\beta_-) + kb(k;\beta_-,\rho)A(n/m) + o_p(A(n/m))\} \\
&= \{ks^{k-1}(1;0,\beta_-)Q_{n,m}^{(1)}(\beta_-) - \phi_k(\beta_-)Q_{n,m}^{(k)}(\beta_-)\} \\
&\quad + k\{b(1;\beta_-,\rho)s^{k-1}(1;0,\beta_-) - b(k;\beta_-,\rho)\phi_k(\beta_-)\}A(n/m) \\
&\quad + o_p(Q_{n,m}^{(1)}(\beta_-)) + o_p(A(n/m)) \qquad (2.6)
\end{aligned}$$

since $s^k(1;0,\beta_-) = \phi_k(\beta_-)s(k;0,\beta_-)$. Suppose $\sqrt{m}|A(n/m)| \to \infty$ as $n \to \infty$. Then, combining (2.4)–(2.6) with Lemma 2.1, we have, for $k = 2, 3, \ldots$, as $n \to \infty$,

$$\begin{aligned}
\frac{\phi_k^{-1}(N_{n,m}^{(k)}) - \beta_-}{A(n/m)} &\overset{p}{\to} \frac{k\{b(1;\beta_-,\rho)s^{k-1}(1;0,\beta_-) - b(k;\beta_-,\rho)\phi_k(\beta_-)\}}{\phi_k'(\beta_-)s(k;0,\beta_-)} \\
&=: B(k;\beta_-,\rho).
\end{aligned}$$

Thus,

$$\begin{aligned}
R_{n,m} &= \frac{(\phi_2^{-1}(N_{n,m}^{(2)}) - \beta_-)/A(n/m) - (\phi_3^{-1}(N_{n,m}^{(3)}) - \beta_-)/A(n/m)}{(\phi_3^{-1}(N_{n,m}^{(3)}) - \beta_-)/A(n/m) - (\phi_4^{-1}(N_{n,m}^{(4)}) - \beta_-)/A(n/m)} \\
&\overset{p}{\to} \frac{B(2;\beta_-,\rho) - B(3;\beta_-,\rho)}{B(3;\beta_-,\rho) - B(4;\beta_-,\rho)} = \frac{-c_1(\beta_-) + \rho c_3(\beta_-)}{c_2(\beta_-) - \rho c_4(\beta_-)} =: r(\beta_-,\rho)
\end{aligned}$$

as $n \to \infty$. Since $\widehat{\beta_-} = \phi_2^{-1}(N_{n,m}^{(2)}) \overset{p}{\to} \beta_-$ as $n \to \infty$ by (2.4)–(2.6), we finally have

$$\hat{\rho}_{n,m} = \frac{c_1(\widehat{\beta_-}) + c_2(\widehat{\beta_-})R_{n,m}}{c_3(\widehat{\beta_-}) + c_4(\widehat{\beta_-})R_{n,m}} \overset{p}{\to} \frac{c_1(\beta_-) + c_2(\beta_-)r(\beta_-,\rho)}{c_3(\beta_-) + c_4(\beta_-)r(\beta_-,\rho)} = \rho$$

as $n \to \infty$. □

To apply the estimator $\hat{\rho}_{n,m}$ in practice, the following explicit forms of ϕ_k^{-1}, $k = 2, 3, 4$, are useful: for $y \in (0, 1)$,

$$\phi_2^{-1}(y) = \frac{1-2y}{2(1-y)},$$
$$\phi_3^{-1}(y) = \frac{5-12y-\sqrt{48y+1}}{12(1-y)},$$
$$\phi_4^{-1}(y) = \frac{-7-522y+(13-36y)\phi_{41}^{1/3}(y)-\phi_{41}^{2/3}(y)}{36(1-y)\phi_{41}^{1/3}(y)},$$

where

$$\phi_{41}(y) = 2^3 \cdot 3^6 y^2 + 3^3 \cdot 5 \cdot 47y - 10 - 9(1-y)\sqrt{3(2^6 \cdot 3^7 y^2 - 840y - 1)}.$$

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