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Estimation of Bivariate Survival Function for Possibly Censored Data[1)]

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Abstract

We consider to obtain an estimate of bivariate survival function for the right censored data with the assumption that the two components of censoring vector are independent. The estimate is derived from an *ad hoc* approach based on the representation of survival function. Then the resulting estimate can be considered as an extension of the Susarla-Van Ryzin estimate to the bivariate data. Also we show the consistency and weak convergence for the proposed estimate. Finally we compare our estimate with Dabrowska's estimate with an example and discuss some properties of our estimate with brief comment on the extension to the multivariate case.

Keywords : bivariate survival function, consistency, Susarla-Van Ryzin estimate, weak convergence

1. Introduction

The estimation of the bivariate survival function for possibly right censored data has long been a subject for research among statisticians. However the results have not been quite satisfactory for some reasons. For examples, some estimates require the iterative method since they have the implicit forms (Campbell, 1981, Korwar and Dahiya, 1982 and Hanley and Parnes, 1983) or are too complicated to derive its asymptotic variance (cf. Tsai, Leurgans and Crowley, 1986). Or some estimates depend on the choice of the conditioning component since different choice of conditioning component produces different estimate (cf. Campbell, 1982 and Burke, 1988). Dabrowska (1988) and Prentice and Cai (1991) considered the estimates based on the representations of a bivariate survival function in terms of its conditional bivariate hazard function. However we note that the representation is not unique and so each representation

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gives rise to a different estimate (cf. Andersen, Borgan, Gill and Keiding, 1991). Also most of the mentioned estimates are not proper since they contain negative measures. Pruitt (1991) identified all the points which assign the negative measures. Recently, Van Der Laan (1996) proposed an estimate to eliminate the negative measures and improve the efficiency based on a reduced (or transformed) data set. However this also requires an iterative method and induces the bandwidth selection problem to improve the efficiency of the estimate. All the estimates mentioned up to now are based on the general random censoring schemes.

Also Tsai and Crowley (1998) considered to estimate bivariate survival function under somewhat alleviated condition, which confines univariate censoring. In this paper, we consider to propose an estimate of bivariate survival function, which is easy to compute under the assumption that the two components of censoring vector are independent. This situation would not be rare. For example, suppose that there is an electronic system with two components whose functioning are dependent each other and each component may be out of order by absorbing different shocks, which give independent effect on each component. We propose an estimate in the next section and then show the consistency and weak convergence in Section 3. Finally we illustrate our proposed estimate with an example and discuss some properties of our proposed estimate with brief comment on the extension to the multivariate case.

2. Estimation of the Bivariate Survival Function

Let $(X_{11}, X_{12}), \cdots, (X_{n1}, X_{n2})$ be a bivariate random sample of survival times on some probability space with a continuous bivariate survival function S. Also let $(Y_{11}, Y_{12}), \cdots, (Y_{n1}, Y_{n2})$ be an independent bivariate random sample of censoring times with a continuous distribution function $\overline{G}$ on the same probability space. Since the censoring schemes are involved, we may only observe that

$$\{(T_{i1}, T_{i2}), (\delta_{i1}, \delta_{i2}), 1 \le i \le n\}$$

with $T_{ij} = \min\{X_{ij}, Y_{ij}\}$ and $\delta_{ij} = I(X_{ij} \le Y_{ij})$ for each $1 \le i \le n$ and $j = 1, 2$, where $I(\cdot)$ is an indicator function. Based on this sample, we consider to obtain a nonparametric estimate of the bivariate survival function S,

$$S(\cdot, \cdot) = P\{X_{11} > \cdot, X_{12} > \cdot\}$$

with the additional assumption that Y_{11} and Y_{12} are independent. For this purpose, first of all, we note that

$$\begin{aligned} H(t_1, t_2) &= P\{T_{i1} > t_1, T_{i2} > t_2\} \\ &= P\{X_{i1} > t_1, Y_{i1} > t_1, X_{i2} > t_2, Y_{i2} > t_2\} \\ &= S(t_1, t_2) G_1(t_1) G_2(t_2), \end{aligned}$$

where $G_j(t_j) = P\{Y_{1j} > t_j\}$ for each $j = 1, 2$. Then we observe that

$$S(t_1,t_2)=H(t_1,t_2)/\{G_1(t_1)G_2(t_2)\}, \tag{2.1}$$

for $G_j(t_j)>0$ for each $j=1,2$. Therefore we may propose an estimate S_n for S as follows:

$$\begin{aligned}
&S_n(t_1,t_2)\\
&=\frac{1}{n}\sum_{i=1}^{n}I(T_{i1}>t_1,T_{i2}>t_2)\prod_{i:\,T_{(i1)}\le t_1}\left\{1+\frac{I(\delta_{(i1)}=0)}{n-i+1}\right\}\prod_{i:\,T_{(i2)}\le t_2}\left\{1+\frac{I(\delta_{(i2)}=0)}{n-i+1}\right\}\\
&=H_n(t_1,t_2)W_{1n}(t_1)W_{2n}(t_2),
\end{aligned}$$

where $T_{(ij)}$ is the ith ordered observation among $T_{1j},\cdots,T_{nj}$ for each $j=1,2$ and $\delta_{(ij)}$ is the concomitant of $T_{(ij)}$. We note that H_n is just the empirical survival function of H and $W_{jn}(t_j)$, a consistent estimate of $1/G_j(t_j)$ for each j, $j=1,2$, which will be shown in the next section. Therefore the consistency of S_n will follow easily. Also we note that for the univariate case, for each $j=1,2$

$$S_{jn}(t_j)=\frac{1}{n}\sum_{i=1}^{n}I(T_{ij}>t_j)\prod_{i:\,T_{(ij)}\le t_j}\left\{1+\frac{I(\delta_{(ij)}=0)}{n-i+1}\right\}$$

is just the Susarla-Van Ryzin estimate (1980) of jth marginal survival function, $S_j(t_j)$, which is a version of the Bayesian estimates proposed by Susarla and Van Ryzin (1976). Therefore our estimate can be considered as an extension of the Susarla-Van Ryzin estimate to the bivariate data. In the next section, we deal with the consistency and weak convergence.

3. Consistency and Weak Convergence

For discussions of the asymptotic properties for S_n, first of all, we introduce several notations. Let for each j, $j=1,2$

$$K_j(t_j)=P\{T_{1j}\le t_j,\delta_{1j}=0\}.$$

Also let $K_{jn}(t_j)=(1/n)\sum_{i=1}^{n}I(T_{ij}\le t_j,\delta_{ij}=0)$ be the empirical counterpart of $K_j(t_j)$ for each $j=1,2$. Then we note that $K_j(t_j)$ is the jth marginal sub-distribution function. Also we define the jth marginal function $H_j(t_j)=P\{T_{1j}>t_j\}$ of H and its empirical counterpart $H_{jn}(t_j)=(1/n)\sum_{i=1}^{n}I(T_{ij}>t_j)$. Now we consider the following decomposition.

$$\begin{aligned}
S_n(t_1,t_2)-S(t_1,t_2)&=H_n(t_1,t_2)\{W_{1n}(t_1)W_{2n}(t_2)-1/[G_1(t_1)G_2(t_2)]\}\\
&\quad+[H_n(t_1,t_2)-H(t_1,t_2)]/[G_1(t_1)G_2(t_2)]\\
&=H_n(t_1,t_2)[W_{1n}(t_1)-1/G_1(t_1)]W_{2n}(t_2)\\
&\quad+[H_n(t_1,t_2)/G_1(t_1)][W_{2n}(t_2)-1/G_2(t_2)]\\
&\qquad+[H_n(t_1,t_2)-H(t_1,t_2)]/[G_1(t_1)G_2(t_2)].
\end{aligned} \tag{3.1}$$

From the Taylor's expansion, we have for each j, $j=1,2$

$$\begin{aligned}W_{jn}(t_j)-1/G_j(t_j) &= \exp\{\ln[W_{jn}(t_j)]\}-\exp\{\ln[1/G_j(t_j)]\}\\ &= \exp\{\ln[1/G_j(t_j)]\}\{\exp[\ln[W_{jn}(t_j)]-\ln[1/G_j(t_j)]]-1\}\\ &= \{\ln[W_{jn}(t_j)]-\ln[1/G_j(t_j)]\}/G_j(t_j)\\ &\quad+\frac{\{\ln[W_{jn}(t_j)]-\ln[1/G_j(t_j)]\}^2}{2}\exp\{c\ln[1/G_j(t_j)]\},\end{aligned}$$

where c is a constant between 0 and $\ln[W_{jn}(t_j)]-\ln[1/G_j(t_j)]$. The following two results are due to Susarla and Van Ryzin (1978).

Lemma 1. For each j, $j=1,2$ and for any τ_j such that $H_j(\tau_j)>0$

$$n^{\beta}\|\ln[W_{jn}]-\ln[1/G_j]\|_{\tau_j}\to 0 \text{ almost surely for any } 2\beta<1,$$

where $\|\ \|_{\tau_j}$ is the supremum norm over $(0,\tau_j]$.

Lemma 2. For each j, $j=1,2$,

$$\left\|\sqrt{n}\{\ln[W_{jn}(t_j)]-\ln[1/G_j(t_j)]\}-\sqrt{n}\left\{\int_0^{t_j}\frac{dK_{jn}}{H_{jn}}-\int_0^{t_j}\frac{dK_j}{H_j}\right\}\right\|_{\tau_j}\to 0$$

almost surely at a rate $O(n^{-\beta})$ with $2\beta<1$.

Then we can show the strong uniform consistency of S_n in the following sense using Lemma 1.

Theorem 1. For any (τ_1,τ_2) such that $H(\tau_1,\tau_2)>0$, almost surely

$$\|S_n-S\|_{(\tau_1,\tau_2)}\to 0.$$

Proof. From the decomposition (3.1) and the triangle inequality, we have that

$$\begin{aligned}\|S_n-S\|_{(\tau_1,\tau_2)} &\le \|H_n\|_{(\tau_1,\tau_2)}\|W_{1n}-1/G_1\|_{(\tau_1,\tau_2)}\|W_{2n}\|_{(\tau_1,\tau_2)}\\ &\quad+\|H_n/G_1\|_{(\tau_1,\tau_2)}\|W_{2n}-1/G_2\|_{(\tau_1,\tau_2)}+\|H_n-H\|_{(\tau_1,\tau_2)}/[G_1(\tau_1)G_2(\tau_2)].\end{aligned}$$

Then the result follows from Lemma 1 with the fact that the logarithm function is continuous.

For the derivation of the weak convergence, we note that for each j, by denoting

$$U_{jn}=\sqrt{n}(H_{jn}-H_j) \text{ and } V_{jn}=(K_{jn}-K_j),$$

$$\sqrt{n}\left\{\int_0^{t_j}\frac{dK_{jn}}{H_{jn}}-\int_0^{t_j}\frac{dK_j}{H_j}\right\}=\sqrt{n}\int_0^{t_j}\left(\frac{1}{H_{jn}}-\frac{1}{H_j}\right)dK_{jn}+\sqrt{n}\int_0^{t_j}\frac{1}{H_j}d(K_{jn}-K_j)$$

$$=-\int_0^{t_j}\frac{U_{jn}}{H_j^2}dK_j+\frac{V_{jn}(t_j)}{H_j(t_j)}+\int_0^{t_j}\frac{V_{jn}}{H_j^2}dH_j+R_{1jn}+R_{2jn}$$

$$=Z_{jn}(t_j)+R_{1jn}+R_{2jn},\ \text{say} \tag{3.2}$$

where for each j, $R_{1jn}=\frac{1}{\sqrt{n}}\int_0^{t_j}\frac{U_{jn}^2}{H_j^2H_{jn}}dK_j$ and $R_{2jn}=-\frac{1}{\sqrt{n}}\int_0^{t_j}\frac{U_{jn}}{H_jH_{jn}}dV_{jn}$.

Then we note that almost surely,

$$\|R_{1jn}\|=\|R_{2jn}\|=O(n^{-1/2}).$$

Thus with Lemma 2, we may approximate $\sqrt{n}\{S_n(t_1,t_2)-S(t_1,t_2)\}$ as follows: for any β such as $2\beta<1$, with the notation that $U_n(t_1,t_2)=\sqrt{n}\{H_n(t_1,t_2)-H(t_1,t_2)\}$and for each j,

$$Z_{jn}(t_j)=\left[-\int_0^{t_j}\frac{U_{jn}(v)}{H_j^2(v)}dK_j(v)+\frac{V_{jn}(t_j)}{H_2(t_j)}+\int_0^{t_j}\frac{V_{jn}(v)}{H_j^2(v)}dH_j(v)\right]$$

we have almost surely,

$$\begin{aligned}
&\sqrt{n}\{S_n(t_1,t_2)-S(t_1,t_2)\}\\
&=S(t_1,t_2)\sqrt{n}\{[\ln[W_{1n}(t_1)]-\ln[1/G_1(t_1)]]+[\ln[W_{2n}(t_2)]-\ln[1/G_2(t_2)]]\}\\
&\qquad+\sqrt{n}\{H_n(t_1,t_2)-H(t_1,t_2)\}/[G_1(t_1)G_2(t_2)]+O(n^{-\beta})\\
&=S(t_1,t_2)\left[-\int_0^{t_1}\frac{U_{1n}(u)}{H_1^2(u)}dK_1(u)+\frac{V_{1n}(t_1)}{H_1(t_1)}+\int_0^{t_1}\frac{V_{1n}(u)}{H_1^2(u)}dH_1(u)\right]\\
&\quad+S(t_1,t_2)\left[-\int_0^{t_2}\frac{U_{2n}(v)}{H_2^2(v)}dK_2(v)+\frac{V_{2n}(t_2)}{H_2(t_2)}+\int_0^{t_2}\frac{V_{2n}(v)}{H_2^2(v)}dH_2(v)\right]\\
&\quad+S(t_1,t_2)\frac{U_n(t_1,t_2)}{H(t_1,t_2)}+O(n^{-\beta})\\
&=S(t_1,t_2)\{Z_{1n}(t_1)+Z_{2n}(t_2)+U_n(t_1,t_2)/H(t_1,t_2)\}+O(n^{-\beta}).
\end{aligned} \tag{3.3}$$

Now we state our main result of this section. For this, we need more notation for the expression of covariance function of the limiting process. In the sequel, T_j is the maximum observation in the jth component and $a\wedge b=\min\{a,b\}$ and $a\vee b=\max\{a,b\}$. Also let

$K_{10}(t_1,t_2)=P\{T_{11}\le t_1,T_{12}>t_2,\delta_{11}=0\}$, $K_{20}(t_1,t_2)=P\{T_{11}>t_1,T_{12}\le t_2,\delta_{12}=0\}$

and $\qquad K(t_1,t_2)=P\{T_{11}\le t_1,\delta_{11}=0,T_{12}\le t_2,\delta_{12}=0\}$.

Theorem 2. If $H(T_1,T_2)>0$, then $\sqrt{n}\{S_n(t_1,t_2)-S(t_1,t_2)\}$ converges weakly to a normal process $S(t_1,t_2)\{Z_1(t_1)+Z_2(t_2)+U(t_1,t_2)/H(t_1,t_2)\}$ with 0 mean vector and covariance function $\sigma^2(t_1,t_2)$, where $Z_j(t_j)$ is the limiting process of $Z_{jn}(t_j)$ for each j, $j=1,2$ and

$$\sigma^2(t_1,t_2)=S^2(t_1,t_2)\left\{\frac{1-H(t_1,t_2)}{H(t_1,t_2)}+\int_0^{t_1}\frac{dK_1(u)}{H_1^2(u)}+\int_0^{t_2}\frac{dK_2(v)}{H_2^2(v)}\right\}$$
$$+S^2(t_1,t_2)\{A+B+C+D+E+F+G+H+I\}$$

with

$$A=2\int_0^{t_1}\int_0^{t_2}\frac{H(u,v)-H_1(u)H_2(v)}{H_1^2(u)H_2^2(v)}dK_1(u)dK_2(v),$$

$$B=\frac{2}{H_1(t_1)H_2(t_2)}\{K(t_1,t_2)-K_1(t_1)K_2(t_2)\},$$

$$C=-2\int_0^{t_1}\int_0^{t_2}\frac{K(u,v)-K_1(u)K_2(v)}{H_1^2(u)H_2^2(v)}dH_1(u)dH_2(v),$$

$$D=-\frac{2}{H_2(t_2)}\int_0^{t_1}\frac{K_{20}(u,t_2)-H_1(u)K_2(t_2)}{H_1^2(u)}dK_1(u),$$

$$E=-2\int_0^{t_1}\int_0^{t_2}\frac{K_{20}(u,v)-H_1(u)K_2(v)}{H_1^2(u)H_2^2(v)}dK_1(u)dH_2(v),$$

$$F=-\frac{2}{H_1(t_1)}\int_0^{t_2}\frac{K_{10}(t_1,v)-K_1(t_1)H_2(v)}{H_2^2(u)}dK_2(u),$$

$$G=\frac{2}{H_1(t_1)}\int_0^{t_2}\frac{K(t_1,v)-K_1(t_1)K_2(v)}{H_2^2(v)}dH_2(v),$$

$$H=-2\int_0^{t_1}\int_0^{t_2}\frac{K_{10}(u,v)-K_1(u)H_2(v)}{H_1^2(u)H_2^2(v)}dH_1(u)dK_2(v),$$

$$I=\frac{2}{H_2(t_2)}\int_0^{t_1}\frac{K(u,t_2)-K_1(u)K_2(t_2)}{H_1^2(u)}dH_1(u).$$

Proof. First of all, we note that $U_n(t_1,t_2)$ converges weakly to a two-dimensional -time normal process $U(t_1,t_2)$ (cf. Campbell 1982) with mean $(0,0)$ and covariance function

$$Cov\{U(s_1,s_2),\,U(t_1,t_2)\}=H(s_1\vee t_1,s_2\vee t_2)-H(s_1,s_2)H(t_1,t_2).$$

Also for each j, it is obvious that $V_{jn}(t_j)$ converges weakly to a normal process $V_j(t_j)$ with mean 0 and covariance function

$$Cov\{V_j(s_j),\,V_j(t_j)\}=K_j(s_j\wedge t_j)-K_j(s_j)K_j(t_j).$$

Therefore from the decomposition (3.3), we see the result. The derivation of the covariance function will be postponed until the Appendix.

4. An Example and Discussion

In order to illustrate our procedure and compare with other method, we show an example using the following artificial bivariate data in Hougaard (2000).

<Table 1> Artificial data (+ means censored component)

variables	1	2	3	4	5	6	7
X_1	1	1	1	3	6+	7+	7+
X_2	4	5	8	4	5	5	7+

First of all, we consider obtaining the Dabrowska's (1988) estimate. For this purpose, let

$$R(t_1, t_2) = \sum_{i=1}^{n} I(T_{i1} \geq t_1, T_{i2} \geq t_2)$$

be the bivariate risk set. Also we need the following three types of bivariate events:

$$M_{11}(t_1, t_2) = \sum_{i=1}^{n} \delta_{i1}\delta_{i2} I(T_{i1} = t_1, T_{i2} = t_2)$$

$$M_{10}(t_1, t_2) = \sum_{i=1}^{n} \delta_{i1} I(T_{i1} = t_1, T_{i2} \geq t_2)$$

$$M_{01}(t_1, t_2) = \sum_{i=1}^{n} \delta_{i2} I(T_{i1} \geq t_1, T_{i2} = t_2).$$

Finally, we define the following three quantities:

$$L_{11}(t_1, t_2) = M_{11}(t_1, t_2)/R(t_1, t_2)$$

$$L_{10}(t_1, t_2) = M_{10}(t_1, t_2)/R(t_1, t_2)$$

$$L_{01}(t_1, t_2) = M_{01}(t_1, t_2)/R(t_1, t_2).$$

Then one may obtain the Dabrowska's (1988) estimate for the bivariate survival function as follows:

$$\hat{S}_n(t_1, t_2) = \hat{S}_{1n}(t_1)\hat{S}_{2n}(t_2) \prod_{0 < u \leq t_1, 0 < v \leq t_2} [1 - Q(u, v)],$$

where $\hat{S}_{jn}(t_j)$ is the Kaplan-Meier estimate of the jth component and $Q(u, v)$ is defined as

$$Q(u, v) = \frac{L_{10}(u, v)L_{01}(u, v) - L_{11}(u, v)}{\{1 - L_{10}(u, v)\}\{1 - L_{01}(u, v)\}}.$$

For the calculation, we follow the tradition that $0/0 = 0$. For the proposed estimate, all we have to do is that we obtain the bivariate empirical survival function, $H_n(t_1, t_2)$, and two reciprocals of marginal survival functions, $W_{1n}(t_1)$ and $W_{2n}(t_2)$. Then the two estimates for the bivariate survival function are summarized in Table 2.

<Table 2> Two estimates of bivariate survival function

	(0,0)	(1,4)	(1,5)	(1,8)	(3,4)	(6+,5)	(7+,5)	(7+,7+)
Dabrowska estimate	1	0.429	0.343	0	0.429	0.343	0.343	0.343
Proposed estimate	1	0.429	0.143	0	0.429	0.190	0	0

As illustrated in the above example, the proposed estimate is easy to calculate from data as well as easy to understand even though the extra assumption of independence between two components of the censoring vector is required. However because of the intrinsic structure of our estimate, the monotonicity of S_n cannot be guaranteed (cf. Pruitt, 1991). Also we note that when no censoring occurs, S_n becomes the usual empirical survival function. Therefore our estimate may be used as a complementary one until the advent of better estimate of survival function.

The comparison between the proposed estimate with Dabrowska's might have been carried out through simulation study. However even for the bivariate case, only the bivariate normal distribution can generate pseudo-random vectors in any available software. Also the ambiguity of order among pseudo-random vectors and the difficulty of calculation of volumes of pseudo-random vectors for the use of distribution or survival function prohibit us from the simulation study.

Already we noted that W_{jn} is a consistent estimate of $1/G_j$. Also we might have used the Kaplan-Meier estimate for G_j obtained by switching the roles of the life time and censoring random variables. However when the largest observation is censored, the Kapaln-Meier estimate becomes 0 at the largest observation. Then we can not use this value as the denominator because in the expression of (2.1), the reciprocal forms of G_j's are used. Therefore we have followed the approach of the Susarla-Van Ryzin estimate instead of that of Kaplan-Meier estimate.

We have assumed that two bivariate survival functions of life time and censoring random vectors are continuous for the derivation of the weak convergence. However we may generalize our estimate for the discrete case when we deal with the weak convergence by spreading the jumps of the underlying distributions uniformly over small intervals inserted at each jump point. For technical treatments for this problem, you may refer to Shorack and Wellner (1986).

The extension to the multivariate data is obvious and straightforward with the assumption that all the components of the censoring vector are mutually independent. As an example, in case of the p-variate problem, (2.1) becomes

$$S(t_1, \cdots, t_p) = H(t_1, \cdots, t_p)/\{G_1(t_1) \cdots G_p(t_p)\}.$$

Thus all we need to obtain the estimate for $S(t_1, \cdots, t_p)$ are the terms of the empirical survival function with the p number of estimates of the reciprocal forms of survival functions for censoring vector. Then the asymptotic properties for the p-variate S_n can be easily derived such as the consistency and the weak convergence with variance function.

Appendix

We proceed to derive the asymptotic variance by identifying the terms of the univariate case in Susarla and Van Ryzin (1978). For this matter, we need the following lemma.

Lemma 3. For any $u \leq t_1$ and $v \leq t_2$, we have that

$$I(T_{i1} > t_1, T_{i2} > t_2)I(T_{i1} > u) = I(T_{i1} > t_1, T_{i2} > t_2)$$

and

$$I(T_{i1} > t_1, T_{i2} > t_2)I(T_{i2} > v) = I(T_{i1} > t_1, T_{i2} > t_2).$$

First of all, we the following results with Lemma 3.

(1) $$-2Cov\left\{\int_0^{t_1} \frac{U_1(u)}{H_1^2(u)} dK_1(u), \frac{U(t_1,t_2)}{H(t_1,t_2)}\right\}$$

$$= -2Cov\left\{\int_0^{t_1} \frac{U_{1n}(u)}{H_1^2(u)} dK_1(u), \frac{U_n(t_1,t_2)}{H(t_1,t_2)}\right\}$$

$$= -\frac{2}{H(t_1,t_2)} \int_0^{t_1} \frac{H(t_1,t_2)(1-H_1(u))}{H_1^2(u)} dK_1(u)$$

$$= -2\left\{\int_0^{t_1} \frac{dK_1(u)}{H_1^2(u)} + \ln[1-G_1(t_1)]\right\}$$

$$= -2Cov\left\{\int_0^{t_1} \frac{U_1(u)}{H_1^2(u)} dK_1(u), \frac{U_1(t_1)}{H_1(t_1)}\right\}.$$

Also with similar arguments for (1) with integration by parts, we have

(2) $$2Cov\left\{\int_0^{t_1} \frac{V_1(u)}{H_1^2(u)} dH_1(u), \frac{U(t_1,t_2)}{H(t_1,t_2)}\right\} = 2Cov\left\{\int_0^{t_1} \frac{V_1(u)}{H_1^2(u)} dH_1(u), \frac{U_1(t_1)}{H_1(t_1)}\right\}.$$

Also we have

(3) $$2Cov\left\{\frac{V_1(t_1)}{H_1(t_1)}, \frac{U(t_1,t_2)}{H(t_1,t_2)}\right\} = 2Cov\left\{\frac{V_{1n}(t_1)}{H_1(t_1)}, \frac{U_n(t_1,t_2)}{H(t_1,t_2)}\right\} = 2Cov\left\{\frac{V_1(t_1)}{H_1(t_1)}, \frac{U(t_1)}{H(t_1)}\right\}.$$

Also we note that $Var\{Z_1(t_1)\}$ are exactly the same as those from (5) to (10) in Susarla and Van Ryzin (1978). Then by adding them, we obtain that

$$Var\{Z_1(t_1)\} + 2Cov\left\{Z_1(t_1), \frac{U(t_1,t_2)}{H(t_1,t_2)}\right\} = \int_0^{t_1} \frac{dK_1(u)}{H_1^2(u)}.$$

Also for $Var\{Z_2(t_2)\} + 2Cov\{Z_2(t_2), U(t_1,t_2)/H(t_1,t_2)\}$, with the same arguments used for $Var\{Z_1(t_1)\} + 2Cov\{Z_1(t_1), U(t_1,t_2)/H(t_1,t_2)\}$, we obtain that

$$Var\{Z_2(t_2)\}+2Cov\left\{Z_2(t_2),\frac{U(t_1,t_2)}{H(t_1,t_2)}\right\}=\int_0^{t_2}\frac{dK_2(v)}{H_2^2(v)}.$$

It is easy to see that

$$Var\left\{\frac{U(t_1,t_2)}{H(t_1,t_2)}\right\}=Var\left\{\frac{U_n(t_1,t_2)}{H(t_1,t_2)}\right\}=\frac{1-H(t_1,t_2)}{H(t_1,t_2)}.$$

Finally for $2Cov\{Z_1(t_1),Z_2(t_2)\}$, we obtain the following 8 terms (from A to I)

A: $$2Cov\left\{\int_0^{t_1}\frac{U_1(u)}{H_1^2(u)}dK_1(u),\int_0^{t_2}\frac{U_2(v)}{H_2^2(v)}dK_2(v)\right\}$$

$$=2\int_0^{t_1}\int_0^{t_2}\frac{Cov\{U_{1n}(u),U_{2n}(v)\}}{H_1^2(u)H_2^2(v)}dK_1(u)dK_2(v)$$

$$=2\int_0^{t_1}\int_0^{t_2}\frac{H(u,v)-H_1(u)H_2(v)}{H_1^2(u)H_2^2(v)}dK_1(u)dK_2(v),$$

since $Cov\{U_{1n}(u),U_{2n}(v)\}=H(u,v)-H_1(u)H_2(v)$.

B: $$2Cov\left\{\frac{V_1(t_1)}{H_1(t_1)},\frac{V_2(t_2)}{H_2(t_2)}\right\}=2Cov\left\{\frac{V_{1n}(t_1)}{H_1(t_1)},\frac{V_{2n}(t_2)}{H_2(t_2)}\right\}$$

$$=\frac{2}{H_1(t_1)H_2(t_2)}\{K(t_1,t_2)-K_1(t_1)K_2(t_2)\}.$$

C: $$2Cov\left\{\int_0^{t_1}\frac{V_1(u)}{H_1^2(u)}dH_1(u),\int_0^{t_2}\frac{V_2(v)}{H_2^2(v)}dH_2(v)\right\}$$

$$=2Cov\left\{\int_0^{t_1}\frac{V_{1n}(u)}{H_1^2(u)}dH_1(u),\int_0^{t_2}\frac{V_{2n}(v)}{H_2^2(v)}dH_2(v)\right\}$$

$$=2\int_0^{t_1}\int_0^{t_2}\frac{K(u,v)-K_1(u)K_2(v)}{H_1^2(u)H_2^2(v)}dH_1(u)dH_2(v).$$

D: $$-2Cov\left\{\int_0^{t_1}\frac{U_1(u)}{H_1^2(u)}dK_1(u),\frac{V_2(t_2)}{H_2(t_2)}\right\}$$

$$=-\frac{2}{H_2(t_2)}\int_0^{t_1}\frac{Cov\{U_{1n}(u),V_{2n}(t_2)\}}{H_1^2(u)}dK_1(u)$$

$$=-\frac{2}{H_2(t_2)}\int_0^{t_1}\frac{K_{20}(u,t_2)-H_1(u)K_2(t_2)}{H_1^2(u)}dK_1(u)$$

since $Cov\{U_{1n}(u),V_{2n}(t_2)\}=K_{20}(u,t_2)-H_1(u)K_2(t_2)$.

E: $$-2Cov\left\{\int_0^{t_1}\frac{U_1(u)}{H_1^2(u)}dK_1(u),\int_0^{t_2}\frac{V_2(v)}{H_2^2(v)}dH_2(v)\right\}$$

$$=-2Cov\left\{\int_0^{t_1}\frac{U_{1n}(u)}{H_1^2(u)}dK_1(u),\ \int_0^{t_2}\frac{V_{2n}(v)}{H_2^2(v)}dH_2(v)\right\}$$

$$=-2\int_0^{t_1}\int_0^{t_2}\frac{K_{20}(u,v)-H_1(u)K_2(v)}{H_1^2(u)H_2^2(v)}dK_1(u)dH_2(v)$$

F: $$-2Cov\left\{\frac{V_1(t_1)}{H_1(t_1)},\ \int_0^{t_2}\frac{U_2(v)}{H_2^2(v)}dK_2(v)\right\}$$

$$=-2Cov\left\{\frac{V_{1n}(t_1)}{H_1(t_1)},\ \int_0^{t_2}\frac{U_{2n}(v)}{H_2^2(v)}dK_2(v)\right\}$$

$$=-\frac{2}{H_1(t_1)}\int_0^{t_2}\frac{Cov\{V_{1n}(t_1),\ U_{2n}(v)\}}{H_2^2(v)}dK_2(v)$$

$$=-\frac{2}{H_1(t_1)}\int_0^{t_2}\frac{K_{10}(t_1,v)-K_1(t_1)H_2(v)}{H_2^2(v)}dK_2(v)$$

since $Cov\{V_{1n}(t_1),\ U_{2n}(v)\}=K_{10}(t_1,v)-K_1(t_1)H_2(v)$.

G: $$2Cov\left\{\frac{V_1(t_1)}{H_1(t_1)},\ \int_0^{t_2}\frac{V_2(v)}{H_2^2(v)}dH_2(v)\right\}$$

$$=2Cov\left\{\frac{V_{1n}(t_1)}{H_1(t_1)},\ \int_0^{t_2}\frac{V_{2n}(v)}{H_2^2(v)}dH_2(v)\right\}$$

$$=\frac{2}{H_1(t_1)}\int_0^{t_2}\frac{K(t_1,v)-K_1(t_1)K_2(v)}{H_2^2(v)}dH_2(v).$$

H: $$-2Cov\left\{\int_0^{t_1}\frac{V_1(u)}{H_1^2(u)}dH_1(u),\ \int_0^{t_2}\frac{U_2(v)}{H_2^2(v)}dK_2(v)\right\}$$

$$=-2Cov\left\{\int_0^{t_1}\frac{V_{1n}(u)}{H_1^2(u)}dH_1(u),\ \int_0^{t_2}\frac{U_{2n}(v)}{H_2^2(v)}dK_2(v)\right\}$$

$$=-2\int_0^{t_1}\int_0^{t_2}\frac{K_{10}(u,v)-K_1(u)H_2(v)}{H_1^2(u)H_2^2(v)}dH_1(u)dK_2(v).$$

I: $$Cov\left\{\int_0^{t_1}\frac{V_1(u)}{H_1^2(u)}dH_1(u),\frac{V_2(t_2)}{H_2(t_2)}\right\}$$

$$=Cov\left\{\int_0^{t_1}\frac{V_{1n}(u)}{H_1^2(u)}dH_1(u),\frac{V_{2n}(t_2)}{H_2(t_2)}\right\}$$

$$= \frac{2}{H_2(t_2)} \int_0^{t_1} \frac{K(u, t_2) - K_1(u) K_2(t_2)}{H_1^2(u)} dH_1(u).$$

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