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Estimation for the Exponentiated Exponential Distribution Based on Multiply Type-II Censored Samples

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Abstract

It has been known that the exponentiated exponential distribution can be used as a possible alternative to the gamma distribution or the Weibull distribution in many situations. But the maximum likelihood method does not admit explicit solutions when the sample is multiply censored. So we derive the approximate maximum likelihood estimators for the location and scale parameters in the exponentiated exponential distribution that are explicit function of order statistics. We also compare the proposed estimators in the sense of the mean squared error for various censored samples.

Keywords : Approximate maximum likelihood estimator, exponentiated exponential distribution, multiply Type-II censored sample.

1. Introduction

Perhaps the most important and widely used probability distribution in life-analysis is the two-parameter exponential distribution with the probability density function (pdf) of the form

$$g_X(x;\theta,\sigma)=\frac{1}{\sigma}e^{-(x-\theta)/\sigma},\quad x\ge\theta,\ \sigma>0, \tag{1.1}$$

and the cumulative distribution function (cdf) of the form

$$G_X(x;\theta,\sigma)=1-e^{-(x-\theta)/\sigma},\quad x\ge\theta,\ \sigma>0. \tag{1.2}$$

Gupta and Kundu (1999) considered a three-parameter distribution which is a particular case of the exponentiated Weibull distribution originally proposed by Mudholkar et al. (1995) with cdf

$$F_X(x;\theta,\sigma)=[G_X(x;\theta,\sigma)]^{\lambda},\quad x>0,\ \lambda>0. \tag{1.3}$$

If X has the distribution function (1.3), then the corresponding density function is

$$f_X(x;\theta,\sigma)=\lambda g_X(x;\theta,\sigma)[G_X(x;\theta,\sigma)]^{\lambda-1},\quad x>0,\ \lambda>0. \tag{1.4}$$

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The three-parameter gamma and three-parameter Weibull distributions are the most popular distributions for analyzing any lifetime data or skewed data. Gupta and Kundu (1999) showed that many properties of this distribution are quite similar to those of a gamma or Weibull family, therefore this distribution can be used as an alternative to the gamma distribution or the Weibull distribution in many situations.

Mudholkar and Srivastava (1993) introduced the exponentiated Weibull family which is the extended Weibull. Mudholkar and Hutson (1996) discussed the exponentiated Weibull distribution and its moments. They compared the exponentiated Weibull distribution with some well known distributions. Kundu et al. (2005) considered the problem of discriminating between the log-normal and generalized exponential distributions by using the ratio of the maximized likelihood.

Gupta and Kundu (2001) studied some properties of a new family of distributions, namely exponentiated exponential distribution, discussed in Gupta et al. (1998). They considered two-parameter exponentiated exponential distribution which is special cases of the generalized exponential distribution with the location parameter $\theta=0$.

We consider the two-parameter exponentiated exponential distribution with the location and the scale parameters when the shape parameter λ is known. For example, the density function of a random variable X with $\lambda=2$ is

$$\begin{aligned} f_X(x) &= 2g_X(x)G_X(x) \\ &= \frac{2}{\sigma}\left[\exp\left\{-\frac{x-\theta}{\sigma}\right\}-\exp\left\{-\frac{2(x-\theta)}{\sigma}\right\}\right], \quad x>\theta, \quad \sigma>0, \end{aligned} \tag{1.5}$$

and cdf is

$$F_X(x)=[G_X(x)]^2=\left[1-\exp\left\{-\frac{x-\theta}{\sigma}\right\}\right]^2, \quad x>\theta, \quad \sigma>0. \tag{1.6}$$

Multiply Type-II censored sampling arises in life-testing experiments when the failure times of some units were not observed due to mechanical or experimental difficulties. Also, another multiply censored samples arise naturally when some units failed between two points of observation with exact times of failure of these units unobserved.

It has been noted that in most cases, but for a few exceptions such as exponential and Rayleigh distributions, the maximum likelihood method does not provide explicit estimators based on complete and censored samples. Especially, when the sample is multiply censored, the maximum likelihood method does not admit explicit solutions. Hence it is desirable to develop approximations to this maximum likelihood method which would provide us with estimators for the location and scale parameters that are explicit functions of order statistics.

The approximate maximum likelihood estimating method was first developed by Balakrishnan (1989) for the purpose of providing the explicit estimators of the scale parameter in Rayleigh distribution. Kang et al. (1997) proposed the approximate maximum likelihood estimators (AMLEs) of the parameters in the two-parameter exponential distribution with Type-II censoring. Kang et al. (2004) obtained the AMLE for the scale parameter of the

Weibull distribution based on multiply Type-II censored samples.

In this paper, we derive the AMLEs of the scale parameter σ and the location parameter θ based on multiply Type-II censored sample. We also compare the proposed estimators in the sense of the mean squared error (MSE) for various censored samples.

2. Approximate Maximum Likelihood Estimators

We assume that n items are put on a life test, but only a_1th, ..., a_sth failures are observed, the rest are unobserved or missing, where $a_1, \dots, a_s$ are considered to be fixed. If this censoring arises, the scheme is known as multiply Type-II censoring scheme.

Let us assume that the following multiply Type-II censored sample from a sample of size n is

$$X_{a_1:n} < X_{a_2:n} < \cdots < X_{a_s:n} \tag{2.1}$$

where $1 \le a_1 < a_2 < \cdots < a_s \le n$.

Let $a_0 = 0$, $a_{s+1} = n+1$, $F(x_{a_0:n}) = 0$, $F(x_{a_{s+1}:n}) = 1$, then the likelihood function based on the multiply Type-II censored sample (2.1) is given by

$$L = n! \prod_{j=1}^{s} f(x_{a_j:n}) \prod_{j=1}^{s+1} \frac{[F(x_{a_j:n}) - F(x_{a_{j-1}:n})]^{a_j - a_{j-1} - 1}}{(a_j - a_{j-1} - 1)!} . \tag{2.2}$$

By putting $Z_{i:n} = (X_{i:n} - \theta)/\sigma$, the log-likelihood function can be rewritten as

$$\begin{aligned} \ln L = {} & C - s\ln\sigma - \sum_{j=1}^{s+1} \ln[(a_j - a_{j-1} - 1)!] + (a_1 - 1)\ln F(Z_{a_1:n}) + (n - a_s)\ln[1 - F(Z_{a_s:n})] \\ & + \sum_{j=1}^{s} \ln f(Z_{a_j:n}) + \sum_{j=2}^{s} (a_j - a_{j-1} - 1)\ln[F(Z_{a_j:n}) - F(Z_{a_{j-1}:n})] \end{aligned} \tag{2.3}$$

where $f(z) = 2(e^{-z} - e^{-2z})$ and $F(z) = (1 - e^{-z})^2$ are the pdf and the cdf of the standard exponentiated exponential distribution, respectively.

2.1 AMLEs of the Scale Parameter When the Location Parameter is Known

We can derive the AMLEs of the scale parameter σ when the location parameter θ is known.

From equation (2.3), on differentiating with respect to σ in turn and equation to zero, we obtain the estimating equations as

$$\begin{aligned} \frac{\partial \ln L}{\partial \sigma} = {} & -\frac{1}{\sigma}\Bigg[s + (a_1 - 1)\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})} Z_{a_1:n} - (n - a_s)\frac{f(Z_{a_s:n})}{1 - F(Z_{a_s:n})} Z_{a_s:n} \\ & + \sum_{j=1}^{s} \frac{f'(Z_{a_j:n})}{f(Z_{a_j:n})} Z_{a_j:n} + \sum_{j=2}^{s} (a_j - a_{j-1} - 1)\frac{f(Z_{a_j:n})Z_{a_j:n} - f(Z_{a_{j-1}:n})Z_{a_{j-1}:n}}{F(Z_{a_j:n}) - F(Z_{a_{j-1}:n})} \Bigg] \\ = {} & 0 . \end{aligned} \tag{2.1.1}$$

Since the likelihood equations is very complicated, the equation (2.1.1) does not admit an explicit solution for σ.

Let

$$\xi_i = F^{-1}(p_i) = -\ln(1-\sqrt{p_i}) \quad \text{where} \quad p_i = \frac{i}{n+1}, \quad q_i = 1-p_i.$$

We may expand the following functions in Taylor series around the points ξ_{a_1}, ξ_{a_s}, ξ_{a_j}, and $(\xi_{a_j}, \xi_{a_{j-1}})$, respectively.

$$\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})} Z_{a_1:n} \simeq \alpha_1 + \beta_1 Z_{a_1:n} \tag{2.1.2}$$

$$\frac{f(Z_{a_s:n})}{1-F(Z_{a_s:n})} Z_{a_s:n} \simeq \kappa_1 + \delta_1 Z_{a_s:n} \tag{2.1.3}$$

$$\frac{f'(Z_{a_j:n})}{f(Z_{a_j:n})} Z_{a_j:n} \simeq \kappa_j + \delta_j Z_{a_j:n} \tag{2.1.4}$$

$$\frac{f(Z_{a_j:n})Z_{a_j:n} - f(Z_{a_{j-1}:n})Z_{a_{j-1}:n}}{F(Z_{a_j:n}) - F(Z_{a_{j-1}:n})} \simeq \alpha_j + \beta_j Z_{a_j:n} + \gamma_j Z_{a_{j-1}:n} \tag{2.1.5}$$

where

$$\alpha_1 = -\frac{\xi_{a_1}^2}{p_{a_1}}\left[f'(\xi_{a_1}) - \frac{f^2(\xi_{a_1})}{p_{a_1}}\right]$$

$$\beta_1 = \frac{1}{p_{a_1}}\left[f(\xi_{a_1}) + \xi_{a_1} f'(\xi_{a_1}) - \frac{f^2(\xi_{a_1})}{p_{a_1}}\xi_{a_1}\right]$$

$$\kappa_1 = -\frac{\xi_{a_s}^2}{q_{a_s}}\left[f'(\xi_{a_s}) + \frac{f^2(\xi_{a_s})}{q_{a_s}}\right]$$

$$\delta_1 = \frac{1}{q_{a_s}}\left[f(\xi_{a_s}) + \xi_{a_s} f'(\xi_{a_s}) - \frac{f^2(\xi_{a_s})}{q_{a_s}}\xi_{a_s}\right]$$

$$\kappa_j = -\frac{\xi_{a_j}^2}{f(\xi_{a_j})}\left[f''(\xi_{a_j}) - \frac{[f'(\xi_{a_j})]^2}{f(\xi_{a_j})}\right]$$

$$\delta_j = \frac{1}{f(\xi_{a_j})}\left[f'(\xi_{a_j}) + \xi_{a_j} f''(\xi_{a_j}) - \frac{[f'(\xi_{a_j})]^2}{f(\xi_{a_j})}\xi_{a_j}\right]$$

$$\alpha_j = K^2 - \frac{\xi_{a_j}^2 f'(\xi_{a_j}) - \xi_{a_{j-1}}^2 f'(\xi_{a_{j-1}})}{p_{a_j} - p_{a_{j-1}}}$$

$$\beta_j = \frac{1}{p_{a_j} - p_{a_{j-1}}}\left[(1-K)f(\xi_{a_j}) + \xi_{a_j} f'(\xi_{a_j})\right]$$

$$\gamma_j = -\frac{1}{p_{a_j} - p_{a_{j-1}}}\left[(1-K)f(\xi_{a_{j-1}}) + \xi_{a_{j-1}} f'(\xi_{a_{j-1}})\right]$$

$$K = \frac{\xi_{a_j} f(\xi_{a_j}) - \xi_{a_{j-1}} f(\xi_{a_{j-1}})}{p_{a_j} - p_{a_{j-1}}}$$

By substituting the equations (2.1.2), (2.1.3), (2.1.4), and (2.1.5) into the equation (2.1.1), we obtain the approximate likelihood equation for σ as follows;

$$\frac{\partial \ln L}{\partial \sigma} \simeq -\frac{1}{\sigma}\Big[s+(a_1-1)(\alpha_1+\beta_1 Z_{a_1:n}) -(n-a_s)(x_1+\delta_1 Z_{a_s:n}) + \sum_{j=1}^{s}(x_j+\delta_j Z_{a_j:n}) + \sum_{j=2}^{s}(a_j-a_{j-1}-1)(\alpha_j+\beta_j Z_{a_j:n}+\gamma_j Z_{a_{j-1}:n})\Big] = 0 . \tag{2.1.6}$$

Upon solving the equation for σ, we can derive an estimator of σ as follows;

$$\hat{\sigma}_{10} = \frac{B_1+C_1\theta}{A_1} \tag{2.1.7}$$

where

$$A_1 = s+(a_1-1)\alpha_1 -(n-a_s)x_1 + \sum_{j=1}^{s} x_j + \sum_{j=2}^{s}(a_j-a_{j-1}-1)\alpha_j$$

$$B_1 = -(a_1-1)\beta_1 X_{a_1:n} -(n-a_s)\delta_1 X_{a_s:n} - \sum_{j=1}^{s}\delta_j X_{a_j:n} - \sum_{j=2}^{s}(a_j-a_{j-1}-1)(\beta_j X_{a_j:n}+\gamma_j X_{a_{j-1}:n})$$

$$C_1 = (a_1-1)\beta_1 -(n-a_s)\delta_1 + \sum_{j=1}^{s}\delta_j + \sum_{j=2}^{s}(a_j-a_{j-1}-1)(\beta_j+\gamma_j) .$$

We can also approximate these functions by

$$\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})} \simeq \alpha_2 + \beta_2 Z_{a_1:n} \tag{2.1.8}$$

$$\frac{f(Z_{a_s:n})}{1-F(Z_{a_s:n})} \simeq x_2 + \delta_2 Z_{a_s:n} \tag{2.1.9}$$

$$\frac{f'(Z_{a_j:n})}{f(Z_{a_j:n})} \simeq x_{2j} + \delta_{2j} Z_{a_j:n} \tag{2.1.10}$$

$$\frac{f(Z_{a_j:n})}{F(Z_{a_j:n})-F(Z_{a_{j-1}:n})} \simeq \alpha_{1j} + \beta_{1j} Z_{a_j:n} + \gamma_{1j} Z_{a_{j-1}:n}, \tag{2.1.11}$$

$$\frac{f(Z_{a_{j-1}:n})}{F(Z_{a_j:n})-F(Z_{a_{j-1}:n})} \simeq \alpha_{2j} + \beta_{2j} Z_{a_j:n} + \gamma_{2j} Z_{a_{j-1}:n} \tag{2.1.12}$$

where

$$\alpha_2 = \frac{1}{p_{a_1}}\Big[f(\xi_{a_1}) - \xi_{a_1} f'(\xi_{a_1}) + \frac{f^2(\xi_{a_1})}{p_{a_1}}\xi_{a_1}\Big]$$

$$\beta_2 = \frac{1}{p_{a_1}}\Big[f'(\xi_{a_1}) - \frac{f^2(\xi_{a_1})}{p_{a_1}}\Big]$$

$$x_2 = \frac{1}{q_{a_s}}\Big[f(\xi_{a_s}) - \xi_{a_s} f'(\xi_{a_s}) - \frac{f^2(\xi_{a_s})}{q_{a_s}}\xi_{a_s}\Big]$$

$$\delta_2 = \frac{1}{q_{a_s}}\Big[f'(\xi_{a_s}) + \frac{f^2(\xi_{a_s})}{q_{a_s}}\Big]$$

$$x_{2j} = \frac{1}{f(\xi_{a_j})}\Big[f'(\xi_{a_j}) - \xi_{a_j} f''(\xi_{a_j}) + \frac{[f'(\xi_{a_j})]^2}{f(\xi_{a_j})}\xi_{a_j}\Big]$$

$$\delta_{2j} = \frac{1}{f(\xi_{a_j})}\left[f''(\xi_{a_j}) - \frac{[f'(\xi_{a_j})]^2}{f(\xi_{a_j})}\right]$$

$$\alpha_{1j} = \frac{1}{p_{a_j} - p_{a_{j-1}}}\left[(1+K)f(\xi_{a_j}) - \xi_{a_j} f'(\xi_{a_j})\right]$$

$$\beta_{1j} = \frac{1}{p_{a_j} - p_{a_{j-1}}}\left[f'(\xi_{a_j}) - \frac{f^2(\xi_{a_j})}{p_{a_j} - p_{a_{j-1}}}\right]$$

$$\gamma_{1j} = \frac{f(\xi_{a_j}) f(\xi_{a_{j-1}})}{[p_{a_j} - p_{a_{j-1}}]^2}$$

$$\alpha_{2j} = \frac{1}{p_{a_j} - p_{a_{j-1}}}\left[(1+K)f(\xi_{a_{j-1}}) - \xi_{a_{j-1}} f'(\xi_{a_{j-1}})\right]$$

$$\beta_{2j} = -\frac{f(\xi_{a_j}) f(\xi_{a_{j-1}})}{[p_{a_j} - p_{a_{j-1}}]^2} = -\gamma_{1j}$$

$$\gamma_{2j} = \frac{1}{p_{a_j} - p_{a_{j-1}}}\left[f'(\xi_{a_{j-1}}) + \frac{f^2(\xi_{a_{j-1}})}{p_{a_j} - p_{a_{j-1}}}\right]$$

By substituting the equations (2.1.8), (2.1.9), (2.1.10), (2.1.11), and (2.1.12) into the equation (2.1.1), we also obtain the approximate likelihood equation for σ. Upon solving this equation for σ, we can derive another estimator of σ as follows;

$$\hat{\sigma}_{20} = \frac{-B_2 + \sqrt{B_2^2 - 4sC_2}}{2s} \qquad (2.1.13)$$

where

$$B_2 = (a_1-1)\alpha_2 X_{a_1:n} - (n-a_s)x_2 X_{a_s:n} + \sum_{j=1}^{s} x_{2j} X_{a_j:n} + \sum_{j=2}^{s}(a_j - a_{j-1} - 1)(\alpha_{1j} X_{a_j:n} - \alpha_{2j} X_{a_{j-1}:n})$$
$$- \left[(a_1-1)\alpha_2 - (n-a_s)x_2 + \sum_{j=1}^{s} x_{2j} + \sum_{j=2}^{s}(a_j - a_{j-1} - 1)(\alpha_{1j} - \alpha_{2j})\right]\theta$$

$$C_2 = (a_1-1)\beta_2 (X_{a_1:n} - \theta)^2 - (n-a_s)\delta_2 (X_{a_s:n} - \theta)^2 + \sum_{j=1}^{s} \delta_{2j}(X_{a_j:n} - \theta)^2$$
$$+ \sum_{j=2}^{s}(a_j - a_{j-1} - 1)\left[\beta_{1j}(X_{a_j:n} - \theta)^2 + 2\gamma_{1j}(X_{a_j:n} - \theta)(X_{a_{j-1}:n} - \theta) - \gamma_{2j}(X_{a_{j-1}:n} - \theta)^2\right].$$

2.2 AMLEs of Two Parameters

Consider the exponentiated exponential distribution with the density function (1.5) when two parameters are unknown.

From equation (2.3), the likelihood equation for θ is obtained as

$$\frac{\partial \ln L}{\partial \theta} = -\frac{1}{\sigma}\left[(a_1-1)\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})} - (n-a_s)\frac{f(Z_{a_s:n})}{1-F(Z_{a_s:n})}\right.$$
$$\left. + \sum_{j=1}^{s} \frac{f'(Z_{a_j:n})}{f(Z_{a_j:n})} + \sum_{j=2}^{s}(a_j - a_{j-1} - 1)\frac{f(Z_{a_j:n}) - f(Z_{a_{j-1}:n})}{F(Z_{a_j:n}) - F(Z_{a_{j-1}:n})}\right] \qquad (2.2.1)$$
$$= 0.$$

Equation (2.2.1) does not admit an explicit solution for θ. But we can expand the following function as follows;

$$\frac{f(Z_{a_j:n})-f(Z_{a_{j-1}:n})}{F(Z_{a_j:n})-F(Z_{a_{j-1}:n})} \simeq \alpha_{3j}+\beta_{3j}Z_{a_j:n}+\gamma_{3j}Z_{a_j:n} \tag{2.2.2}$$

where

$$\alpha_{3j}=\alpha_{1j}-\alpha_{2j},\quad \beta_{3j}=\beta_{1j}-\beta_{2j},\quad \gamma_{3j}=\gamma_{1j}-\gamma_{2j}.$$

By substituting the equations (2.1.8), (2.1.9), (2.1.10), and (2.2.2) into the equation (2.2.1), we obtain the approximate likelihood equation for θ as follows;

$$\begin{aligned}\frac{\partial \ln L}{\partial \theta} &\simeq -\frac{1}{\sigma}\Big[(a_1-1)(\alpha_2+\beta_2 Z_{a_1:n})-(n-a_s)(\varkappa_2+\delta_2 Z_{a_s:n}) \\ &\quad +\sum_{j=1}^{s}(\varkappa_{2j}+\delta_{2j}Z_{a_j:n})+\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\alpha_{3j}+\beta_{3j}Z_{a_j:n}+\gamma_{3j}Z_{a_{j-1}:n})\Big] \\ &=0 .\end{aligned} \tag{2.2.3}$$

Upon solving the equations (2.1.6) and (2.2.3), we can derive an estimator of θ as follows;

$$\hat{\theta}=\frac{E}{D} \tag{2.2.4}$$

where

$$D=\frac{A_3C_1}{A_1}-C_3$$

$$E=-\frac{A_3B_1}{A_1}-B_3$$

$$A_3=(a_1-1)\alpha_2-(n-a_s)\varkappa_2+\sum_{j=1}^{s}\varkappa_{2j}+\sum_{j=2}^{s}(a_j-a_{j-1}-1)\alpha_{3j}$$

$$B_3=(a_1-1)\beta_2X_{a_1:n}-(n-a_s)\delta_2X_{a_s:n}+\sum_{j=1}^{s}\delta_{2j}X_{a_j:n}+\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\beta_{3j}X_{a_j:n}+\gamma_{3j}X_{a_{j-1}:n})$$

$$C_3=(a_1-1)\beta_2-(n-a_s)\delta_2+\sum_{j=1}^{s}\delta_{2j}+\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\beta_{3j}+\gamma_{3j}).$$

Now, we will obtain the AMLEs of the scale parameter σ.

From equations (2.1.7) and (2.2.4), we can derive an estimator of σ as follows;

$$\hat{\sigma}_{11}=\frac{B_1+C_1\hat{\theta}}{A_1}. \tag{2.2.5}$$

Second, from equations (2.1.13) and (2.2.4), we can derive another estimator of σ as follows;

$$\hat{\sigma}_{21}=\frac{-F_2+\sqrt{F_2^2-4sG_2}}{2s} \tag{2.2.6}$$

where

$$\begin{aligned}F_2 &=(a_1-1)\alpha_2X_{a_1:n}-(n-a_s)\varkappa_2X_{a_s:n}+\sum_{j=1}^{s}\varkappa_{2j}X_{a_j:n}+\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\alpha_{1j}X_{a_j:n}-\alpha_{2j}X_{a_{j-1}:n}) \\ &\quad -\Big[(a_1-1)\alpha_2-(n-a_s)\varkappa_2+\sum_{j=1}^{s}\varkappa_{2j}+\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\alpha_{1j}-\alpha_{2j})\Big]\hat{\theta}\end{aligned}$$

$$G_2 = (a_1-1)\beta_2(X_{a_1:n}-\hat{\theta})^2-(n-a_s)\delta_2(X_{a_s:n}-\hat{\theta})^2+\sum_{j=1}^{s}\delta_{2j}(X_{a_j:n}-\hat{\theta})^2$$
$$+\sum_{j=2}^{s}(a_j-a_{j-1}-1)\left[\beta_{1j}(X_{a_j:n}-\hat{\theta})^2+2\gamma_{1j}(X_{a_j:n}-\hat{\theta})(X_{a_{j-1}:n}-\hat{\theta})-\gamma_{2j}(X_{a_{j-1}:n}-\hat{\theta})^2\right].$$

3. Comparison of the Proposed Estimators

It's difficult to find the moments of all proposed estimators. So we simulate the MSEs of the proposed estimators through the Monte Carlo simulation method. The simulation procedure is repeated 10,000 times for the sample size $n=20(10)50$ and various choices of censoring. These values are given in Table 1.

From Table 1, the estimator $\hat{\sigma}_{10}$ is more efficient than $\hat{\sigma}_{20}$ in the sense of MSE when the location parameter θ is known. But the estimator $\hat{\sigma}_{21}$ is more efficient than $\hat{\sigma}_{11}$ in the sense of MSE when the location parameter θ is unknown. For fixed sample size, the MSE increases generally as k increases.

Since the estimator $\hat{\sigma}_{10}$ is the linear function of the available order statistics, we can calculate the moments by the moments of the order statistics. But $\hat{\sigma}_{20}$ is the quadratic form and nonlinear function of the available order statistics and hence we can't calculate the moments of $\hat{\sigma}_{20}$. The estimator $\hat{\sigma}_{10}$ is more simple and more efficient estimator than the estimator $\hat{\sigma}_{20}$.

When the location parameter is unknown, $\hat{\sigma}_{11}$ is also the linear function of the available order statistics and $\hat{\sigma}_{21}$ is a quadratic form. But in this case $\hat{\sigma}_{21}$ is more efficient than $\hat{\sigma}_{11}$ in the sense of MSE.

Table 1. The relative MSEs for the estimators of the location parameter θ and the scale parameter σ.

n	k	a_j	θ is known.		θ is unknown.		
			$\hat{\sigma}_{10}$	$\hat{\sigma}_{20}$	$\hat{\theta}$	$\hat{\sigma}_{11}$	$\hat{\sigma}_{21}$
20	0	1~20	0.027575	0.029978	0.042931	0.042132	0.037636
	2	1~18	0.029225	0.031639	0.044082	0.046289	0.041024
		3~20	0.027657	0.030184	0.047536	0.044759	0.044137
		2~19	0.028448	0.031036	0.040609	0.043904	0.042529
	4	2~17	0.030462	0.033129	0.042088	0.048915	0.047265
		4~19	0.028582	0.030927	0.059747	0.051219	0.050726
		3~18	0.029309	0.031818	0.049506	0.049539	0.048764
		2~4 7~14 16~20	0.027668	0.034769	0.039691	0.041774	0.041375
	5	3~17	0.030519	0.033093	0.050793	0.053033	0.052169
		4~18	0.029387	0.031728	0.061286	0.054083	0.053540
		2~6 10~19	0.028521	0.033717	0.040399	0.043882	0.042822
	6	4~17	0.030603	0.032991	0.063241	0.058221	0.057621
		1 2 6~9 12~15 17~20	0.027722	0.038166	0.040917	0.041430	0.038740

Table 1. (continued)

n	k	a_j	θ is known.		θ is unknown.		
			$\hat{\sigma}_{10}$	$\hat{\sigma}_{20}$	$\hat{\theta}$	$\hat{\sigma}_{11}$	$\hat{\sigma}_{21}$
30	0	1~30	0.018494	0.019858	0.023708	0.027286	0.025043
	2	1~28 3~30 2~29	0.019278 0.018504 0.018842	0.020671 0.019927 0.020294	0.024060 0.024775 0.021884	0.028949 0.028757 0.027802	0.026499 0.028426 0.027108
	4	2~27 4~29 3~28 2~4 7~14 16~30	0.019568 0.018872 0.019290 0.018499	0.021060 0.020246 0.020738 0.022210	0.022298 0.029434 0.025225 0.021599	0.029469 0.031350 0.030537 0.027074	0.028687 0.031114 0.030171 0.026880
	5	3~27 4~28 2~6 10~19 21~30	0.019567 0.019319 0.018520	0.021021 0.020726 0.022474	0.025549 0.029834 0.021654	0.031408 0.032545 0.027082	0.031003 0.032301 0.026939
	6	4~27 1 2 6~9 12~15 17~30	0.019597 0.018537	0.021000 0.023685	0.030290 0.022832	0.033552 0.026922	0.033282 0.025135
40	0	1~40	0.013389	0.014197	0.015262	0.019134	0.017725
	2	1~38 3~40 2~39	0.013756 0.013393 0.013570	0.014574 0.014254 0.014435	0.015453 0.015533 0.014033	0.019970 0.019863 0.019429	0.018471 0.019648 0.018982
	4	2~37 4~39 3~38 2~4 7~14 16~40	0.013953 0.013577 0.013761 0.013391	0.014842 0.014389 0.014640 0.015670	0.014196 0.017764 0.015770 0.013881	0.020221 0.021321 0.020742 0.018998	0.019747 0.021191 0.020510 0.018879
	5	3~37 4~38 2~6 10~19 21~40	0.013957 0.013763 0.013386	0.014844 0.014595 0.015701	0.015863 0.017926 0.013880	0.021162 0.021822 0.019004	0.020920 0.021690 0.018846
	6	4~37 1 2 6~9 12~15 17~40	0.013959 0.013409	0.014798 0.016464	0.018046 0.014755	0.022280 0.018913	0.022142 0.017652
50	0	1~50	0.010951	0.011503	0.011308	0.015393	0.014402
	2	1~48 3~50 2~49	0.011198 0.010957 0.011070	0.011750 0.011537 0.011657	0.011421 0.011407 0.010332	0.015935 0.015778 0.015440	0.014891 0.015601 0.015098
	4	2~47 4~49 3~48 2~4 7~14 16~50	0.011326 0.011071 0.011205 0.010955	0.011917 0.011608 0.011785 0.012388	0.010435 0.012810 0.011534 0.010240	0.015966 0.016741 0.016326 0.015173	0.015601 0.016622 0.016135 0.014911
	5	3~47 4~48 2~6 10~19 21~50	0.011329 0.011204 0.010958	0.011905 0.011748 0.012574	0.011601 0.012884 0.010251	0.016605 0.017042 0.015178	0.016403 0.016922 0.015045
	6	4~47 1 2 6~9 12~15 17~50	0.011327 0.010955	0.011867 0.012918	0.012967 0.010919	0.017344 0.015237	0.017214 0.014163

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