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AMLEs for the Exponential Distribution Based on Multiply Type-II Censored Samples

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Abstract

We propose some estimators of the location parameter and derive the approximate maximum likelihood estimators (AMLEs) of the scale parameter in the exponential distribution based on multiply Type-II censored samples. We calculate the moments for the proposed estimators of the location parameter, and the AMLEs which are the linear functions of the order statistics. We compare the proposed estimators in the sense of the mean squared error (MSE) for various censored samples.

Keywords : Approximate maximum likelihood estimator, Exponential distribution, Multiply Type-II censored sample, Location and scale parameters.

1. Introduction

The exponential distribution occupies an important position in life testing and reliability problems, especially in the area of industrial life testing. The failure time X is said to follow two-parameter exponential distribution if the probability density function (pdf) of X is

$$f(x;\theta,\sigma)=\frac{1}{\sigma}\exp\left(-\frac{x-\theta}{\sigma}\right),\quad x>\theta,\ \sigma>0 \tag{1.1}$$

and the cumulative distribution function (cdf) is

$$F(x;\theta,\sigma)=1-\exp\left(-\frac{x-\theta}{\sigma}\right),\quad x>\theta,\ \sigma>0\,. \tag{1.2}$$

The data for estimating the scale and the location parameters of the two-parameter exponential distribution are usually obtained through Type-II censored sampling scheme. The problem of estimating parameters have been investigated by many authors. Especially, the approximate maximum likelihood estimating method was first developed by Balakrishnan (1989) for the purpose of providing the explicit estimators of the scale parameter in the Rayleigh distribution. Kang (1996) obtained the AMLE for the scale parameter of the double

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exponential distribution based on Type-II censored samples and he showed that the proposed estimator is generally more efficient than the best linear unbiased estimator and the optimum unbiased absolute estimator.

For multiply Type-II censoring, Balasubramanian and Balakrishnan (1992) and Upadhyay et al. (1996) considered some estimations for the exponential distribution under multiply Type-II censoring. Kang (2003) proposed the AMLEs of the location and the scale parameters for the two-parameter exponential distribution with multiply Type-II censoring. Recently, Kang et al. (2005) derived the AMLEs of the scale and location parameters for the double exponential distribution based on Type-II censored samples.

In this paper, we obtain an unbiased estimator of the location parameter, and we derive another estimator of the location parameter which minimizes the MSE among the linear combinations of some available order statistics. we also propose the AMLEs of the scale parameter σ in the two-parameter exponential distribution with multiply Type-II censoring. The scale parameter is estimated by the approximate maximum likelihood estimation method which use Taylor series expansions of two different types. We also compare the proposed estimators in the sense of the MSE.

2. Approximate Maximum Likelihood Estimators

Let

$$X_{a_1:n} \leq X_{a_2:n} \leq \cdots \leq X_{a_s:n} \tag{2.1}$$

be the available multiply Type-II censored sample from the exponential distribution with pdf (1.1), where

$$1 \leq a_1 < a_2 < \cdots < a_s \leq n.$$

Let $a_0 = 0$, $a_{s+1} = n+1$, $F(x_{a_0:n}) = 0$, $F(x_{a_{s+1}:n}) = 1$, then the likelihood function based on the multiply Type-II censored sample (2.1) is given by

$$\begin{aligned} L &= \frac{1}{\sigma^s} \frac{n!}{\prod_{j=1}^{s+1}(a_j - a_{j-1} - 1)!} [F(Z_{a_1:n})]^{a_1 - 1} [1 - F(Z_{a_s:n})]^{n - a_s} \\ &\times \prod_{j=1}^{s} f(Z_{a_j:n}) \prod_{j=2}^{s} [F(Z_{a_j:n}) - F(Z_{a_{j-1}:n})]^{a_j - a_{j-1} - 1}, \end{aligned} \tag{2.2}$$

where $Z_{i:n} = (X_{i:n} - \theta)/\sigma$, $f(z) = e^{-z}$ is the pdf, and $F(z) = 1 - e^{-z}$ is the cdf of the standard exponential distribution.

2.1 Estimation of the Location Parameter

We consider some estimators of the location parameter θ. The following estimator is well known estimator of the location parameter.

$$\hat{\theta}_1 = X_{a_1:n}. \tag{2.1.1}$$

Since $\hat{\theta}_1$ always overestimate the location parameter θ, we consider an unbiased estimator which is the linear combination of two available smallest order statistics as follows;

$$\widehat{\theta}_2 = c_1 X_{a_1:n} + c_2 X_{a_2:n}. \tag{2.1.2}$$

where c_1 and c_2 are constants. The expectation of $\widehat{\theta}_2$ is given by

$$E(\widehat{\theta}_2) = (c_1 + c_2)\theta + \sigma\left[c_1 \sum_{j=1}^{a_1}(n-j+1)^{-1} + c_2 \sum_{j=1}^{a_2}(n-j+1)^{-1}\right]. \tag{2.1.3}$$

From the equations (2.1.2) and (2.1.3), we can easily obtain an unbiased estimator of the location parameter as follows;

$$\widehat{\theta}_2 = \frac{1}{h(a_2) - h(a_1)}\left[h(a_2) X_{a_1:n} - h(a_1) X_{a_2:n}\right] \tag{2.1.4}$$

where

$$h(a) = \sum_{j=1}^{a}(n-j+1)^{-1}.$$

Also we can derive another estimator by minimizing the MSE among the class for the estimators of the form $[1-(s-1)d]X_{a_1:n} + d\sum_{j=2}^{s} X_{a_j:n}$ where d is constant.

Let

$$\widehat{\theta}_3 = [1-(s-1)d]X_{a_1:n} + d\sum_{j=2}^{s} X_{a_j:n}. \tag{2.1.5}$$

Then the MSE of $\widehat{\theta}_3$ is given by

$$\begin{aligned} \mathrm{MSE}(\widehat{\theta}_3) = \Bigg[& [1-(s-1)d]^2[g(a_1) + h^2(a_1)] \\ & + d^2\left\{\sum_{j=2}^{s} g(a_j) + 2\sum_{j=2}^{s-1}(s-j)g(a_j) + \left(\sum_{j=2}^{s} h(a_j)\right)^2\right\} \\ & + 2d[1-(s-1)d]\left\{(s-1)g(a_1) + h(a_1)\sum_{j=2}^{s} h(a_j)\right\}\Bigg]\sigma^2. \end{aligned} \tag{2.1.6}$$

From the equation (2.1.6), we can also obtain the constant d which minimizes MSE $(\widehat{\theta}_3)$. So we propose the estimator of the location parameter as follows;

$$\widehat{\theta}_3 = [1-(s-1)d]X_{a_1:n} + d\sum_{j=2}^{s} X_{a_j:n} \tag{2.1.7}$$

where

$$d = \frac{h(a_1)\left[(s-1)\,h(a_1) - \sum_{j=2}^{s} h(a_j)\right]}{v}$$

$$\begin{aligned} v = {} & (s-1)^2[h^2(a_1) - g(a_1)] + \sum_{j=2}^{s} g(a_j) + 2\sum_{j=1}^{s-1}(s-j)g(a_j) + \left[\sum_{j=2}^{s} h(a_j)\right]^2 \\ & - 2(s-1)\,h(a_1)\sum_{j=2}^{s} h(a_j) \end{aligned}$$

and

$$g(a) = \sum_{j=1}^{a}(n-j+1)^{-2}.$$

2.2 Estimation of the Scale Parameter

We now consider the estimation of the scale parameter σ. From the equation (2.2), we can obtain the following likelihood equation for σ.

$$\begin{aligned}\frac{\partial \ln L}{\partial \sigma} &= -\frac{1}{\sigma}\Bigg[s+(a_1-1)\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})}Z_{a_1:n}-(n-a_s)\frac{f(Z_{a_s:n})}{1-F(Z_{a_s:n})}Z_{a_s:n} \\ &\quad -\sum_{j=1}^{s}Z_{a_j:n}+\sum_{j=2}^{s}(a_j-a_{j-1}-1)\frac{f(Z_{a_j:n})Z_{a_j:n}-f(Z_{a_{j-1}:n})Z_{a_{j-1}:n}}{F(Z_{a_j:n})-F(Z_{a_{j-1}:n})}\Bigg] \\ &=0. \end{aligned} \tag{2.2.1}$$

The equation (2.2.1) does not admit an explicit solution for σ. But we can expand the functions in Taylor series around the points ξ_{a_1} or $(\xi_{a_j}, \xi_{a_{j-1}})$ as follows;

$$\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})} \simeq \alpha_1+\beta_1 Z_{a_1:n} \tag{2.2.2}$$

$$\frac{f(Z_{a_j:n})}{F(Z_{a_j:n})-F(Z_{a_{j-1}:n})} \simeq \alpha_{1j}+\beta_{1j}Z_{a_j:n}+\gamma_{1j}Z_{a_{j-1}:n} \tag{2.2.3}$$

and

$$\frac{f(Z_{a_{j-1}:n})}{F(Z_{a_j:n})-F(Z_{a_{j-1}:n})} \simeq \alpha_{2j}+\beta_{2j}Z_{a_j:n}+\gamma_{2j}Z_{a_{j-1}:n} \tag{2.2.4}$$

where

$$\xi_{a_j}=F^{-1}(p_{a_j})=-\ln(1-p_{a_j})$$

$$p_i=\frac{i}{n+1}$$

$$\alpha_1=\frac{f(\xi_{a_1})}{p_{a_1}}\left[1+\xi_{a_1}+\frac{f(\xi_{a_1})}{p_{a_1}}\xi_{a_1}\right]$$

$$\beta_1=-\frac{f(\xi_{a_1})}{p_{a_1}}\left[1+\frac{f(\xi_{a_1})}{p_{a_1}}\right]$$

$$\alpha_{1j}=\frac{f(\xi_{a_j})}{p_{a_j}-p_{a_{j-1}}}\left(1+\xi_{a_j}+\frac{f(\xi_{a_j})\xi_{a_j}-f(\xi_{a_{j-1}})\xi_{a_{j-1}}}{p_{a_j}-p_{a_{j-1}}}\right)$$

$$\beta_{1j}=-\frac{f(\xi_{a_j})}{p_{a_j}-p_{a_{j-1}}}\left(1+\frac{f(\xi_{a_j})}{p_{a_j}-p_{a_{j-1}}}\right)$$

$$\gamma_{1j}=\frac{f(\xi_{a_j})f(\xi_{a_{j-1}})}{(p_{a_j}-p_{a_{j-1}})^2}$$

$$\alpha_{2j}=\frac{f(\xi_{a_{j-1}})}{p_{a_j}-p_{a_{j-1}}}\left(1+\xi_{a_{j-1}}+\frac{f(\xi_{a_j})\xi_{a_j}-f(\xi_{a_{j-1}})\xi_{a_{j-1}}}{p_{a_j}-p_{a_{j-1}}}\right)$$

$$\beta_{2j}=-\frac{f(\xi_{a_j})f(\xi_{a_{j-1}})}{(p_{a_j}-p_{a_{j-1}})^2}=-\gamma_{1j}$$

and

$$\gamma_{2j} = -\frac{f(\xi_{a_{j-1}})}{p_{a_j} - p_{a_{j-1}}}\left(1 - \frac{f(\xi_{a_{j-1}})}{p_{a_j} - p_{a_{j-1}}}\right).$$

By substituting the equations (2.2.2), (2.2.3), and (2.2.4) into the equation (2.2.1), we obtain the approximate likelihood equation of (2.2.1) as

$$\begin{aligned}\frac{\partial \ln L}{\partial \sigma} \simeq \frac{\partial \ln L^*}{\partial \sigma} &= -\frac{1}{\sigma}\Big[s + (a_1 - 1)(\alpha_1 + \beta_1 Z_{a_1:n})Z_{a_1:n} - (n - a_s)Z_{a_s:n} \\ &\quad + \sum_{j=2}^{s}(a_j - a_{j-1} - 1)\left[(\alpha_{1j} + \beta_{1j}Z_{a_j:n} + \gamma_{1j}Z_{a_{j-1}:n})Z_{a_j:n}\right. \\ &\quad \left. - (\alpha_{2j} + \beta_{2j}Z_{a_j:n} + \gamma_{2j}Z_{a_{j-1}:n})Z_{a_{j-1}:n}\right] - \sum_{j=1}^{s} Z_{a_j:n}\Big] \\ &= 0\,.\end{aligned} \tag{2.2.5}$$

The equation (2.2.5) is quadratic in σ as follows;

$$s\sigma^2 + B_{1i}\sigma + C_{1i} = 0\,, \quad i = 0, 1, 2, 3 \tag{2.2.6}$$

where

$$\begin{aligned}B_{1i} = &\ (a_1 - 1)\alpha_1 X_{a_1:n} - (n - a_s)X_{a_s:n} - \sum_{j=1}^{s} X_{a_j:n} \\ &+ \sum_{j=2}^{s}(a_j - a_{j-1} - 1)(\alpha_{1j}X_{a_j:n} - \alpha_{2j}X_{a_{j-1}:n}) \\ &- \left[(a_1 - 1)\alpha_1 - (n - a_s) - s + \sum_{j=2}^{s}(a_j - a_{j-1} - 1)(\alpha_{1j} - \alpha_{2j})\right]\hat{\theta}_i \\ C_{1i} = &\ \sum_{j=2}^{s}(a_j - a_{j-1} - 1)\left\{\beta_{1j}(X_{a_j:n} - \hat{\theta}_i)^2 + 2\gamma_{1j}(X_{a_j:n} - \hat{\theta}_i)(X_{a_{j-1}:n} - \hat{\theta}_i)\right. \\ &\left. - \gamma_{2j}(X_{a_{j-1}:n} - \hat{\theta}_i)^2\right\} + (a_1 - 1)\beta_1(X_{a_1:n} - \hat{\theta}_i)^2.\end{aligned}$$

and $\hat{\theta}_0 = \theta_0$ is known location parameter.

Upon solving the equation (2.2.6) for σ, we first derive an AMLE of σ as

$$\hat{\sigma}_{1i} = \frac{-B_{1i} + \sqrt{B_{1i}^2 - 4sC_{1i}}}{2s}, \quad i = 0, 1, 2, 3. \tag{2.2.7}$$

Second, we can expand the functions as follows

$$\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})} Z_{a_1:n} \simeq \alpha_2 + \beta_2 Z_{a_1:n} \tag{2.2.8}$$

and

$$\frac{f(Z_{a_j:n})Z_{a_j:n} - f(Z_{a_{j-1}:n})Z_{a_{j-1}:n}}{F(Z_{a_j:n}) - F(Z_{a_{j-1}:n})} \simeq \alpha_{3j} + \beta_{3j}Z_{a_j:n} + \gamma_{3j}Z_{a_{j-1}:n} \tag{2.2.9}$$

where

$$\alpha_2 = \frac{f(\xi_{a_1})}{p_{a_1}}\xi_{a_1}\left[\frac{f(\xi_{a_1})}{p_{a_1}}\xi_{a_1} + \xi_{a_1}\right]$$

$$\beta_2 = \frac{f(\xi_{a_1})}{p_{a_1}}\left[1 - \xi_{a_1} - \frac{f(\xi_{a_1})}{p_{a_1}}\xi_{a_1}\right]$$

$$\alpha_{3j} = \frac{f(\xi_{a_j})\xi_{a_j}^2 - f(\xi_{a_{j-1}})\xi_{a_{j-1}}^2}{p_{a_i} - p_{a_{i-1}}} + \left(\frac{f(\xi_{a_j})\xi_{a_j} - f(\xi_{a_{j-1}})\xi_{a_{j-1}}}{p_{a_i} - p_{a_{i-1}}}\right)^2$$

$$\beta_{3j}=\frac{f(\xi_{a_j})}{p_{a_j}-p_{a_{j-1}}}\left(1-\xi_{a_j}-\frac{f(\xi_{a_j})\xi_{a_j}-f(\xi_{a_{j-1}})\xi_{a_{j-1}}}{p_{a_j}-p_{a_{j-1}}}\right)$$

and

$$\gamma_{3j}=-\frac{f(\xi_{a_{j-1}})}{p_{a_j}-p_{a_{j-1}}}\left(1-\xi_{a_{j-1}}-\frac{f(\xi_{a_j})\xi_{a_j}-f(\xi_{a_{j-1}})\xi_{a_{j-1}}}{p_{a_j}-p_{a_{j-1}}}\right).$$

By substituting the equations (2.2.8) and (2.2.9) into the equation (2.2.1), we obtain another approximate likelihood equation of (2.2.1) as

$$\begin{aligned}\frac{\partial \ln L}{\partial \sigma}\simeq\frac{\partial \ln L^*}{\partial \sigma} &= -\frac{1}{\sigma}\Big[s+(a_1-1)(\alpha_2+\beta_2 Z_{a_1:n})-(n-a_s)Z_{a_s:n} \\ &\quad +\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\alpha_{3j}+\beta_{3j}Z_{a_j:n}+\gamma_{3j}Z_{a_{j-1}:n})-\sum_{j=1}^{s}Z_{a_j:n}\Big] \\ &= 0.\end{aligned} \qquad (2.2.10)$$

From the equation (2.2.10), we can derive simple estimator which is the linear function of the available order statistics as

$$\hat{\sigma}_{2i}=-\frac{B_{2i}}{A_2},\quad i=0,1,2,3 \qquad (2.2.11)$$

where

$$A_2=s+(a_1-1)\alpha_2+\sum_{j=2}^{s}(a_j-a_{j-1}-1)\alpha_j$$

and

$$\begin{aligned}B_{2i}= &\ (a_1-1)\beta_2 X_{a_1:n}-(n-a_s)X_{a_s:n}-\sum_{j=1}^{s}X_{a_j:n}+\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\beta_j X_{a_j:n}+\gamma_j X_{a_{j-1}:n}) \\ &-\Big[(a_1-1)\beta_2-(n-a_s)-s+\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\beta_j+\gamma_j)\Big]\hat{\theta}_i.\end{aligned}$$

Third, Johnson et al. (1994) derived the best linear unbiased estimator of the scale parameter σ when the location parameter θ is known as follows;

$$\hat{\sigma}=\left[\sum_{j=1}^{s}\left(\frac{w_{1j}}{w_{2j}}-\frac{w_{1,j+1}}{w_{2,j+1}}\right)X_{a_j:n}-\frac{w_{11}}{w_{21}}\theta\right]\left[\sum_{j=1}^{s}\frac{w_{1j}}{w_{2j}}\right]^{-1}$$

where $w_{mj}=\sum_{i=a_{j-1}}^{a_j-1}(n-i)^{-m}$, and $\frac{w_{10}}{w_{20}}$, $\frac{w_{1,s+1}}{w_{2,s+1}}$ are defined to be zero.

When the location parameter θ is unknown, we can use this estimator $\hat{\sigma}$ which θ is replaced by the proposed estimators $\hat{\theta}_i$ as follows;

$$\hat{\sigma}_{3i}=\left[\sum_{j=1}^{s}\left(\frac{w_{1j}}{w_{2j}}-\frac{w_{1,j+1}}{w_{2,j+1}}\right)X_{a_j:n}-\frac{w_{11}}{w_{21}}\hat{\theta}_i\right]\left[\sum_{j=1}^{s}\frac{w_{1j}}{w_{2j}}\right]^{-1},\quad i=0,1,2,3. \qquad (2.2.12)$$

2.3 Moments of the Proposed Estimators

It is well known that the expectation and the variance of the ith order statistic, and the covariance of the ith and jth order statistics from the two-parameter exponential distribution with pdf (1.1) as follows;

$$E(X_{i:n}) = \theta + \sigma\sum_{j=1}^{i}(n-j+1)^{-1} = \theta + \sigma h(i) \tag{2.3.1}$$

and

$$\mathrm{Var}\,(X_{i:n}) = \sigma^2\sum_{k=1}^{i}(n-k+1)^{-2} = \sigma^2 g(i) = \mathrm{Cov}\,(X_{i:n},\, X_{j:n}), \quad i\le j. \tag{2.3.2}$$

Now from the equations (2.3.1) and (2.3.2), we can obtain the expectations and the variances of the estimators of the location parameter θ.

The expectation and the variance of $\widehat{\theta}_1$ are

$$E(\widehat{\theta}_1) = \theta + \sigma h(a_1) \tag{2.3.3}$$

and

$$\mathrm{Var}\,(\widehat{\theta}_1) = \sigma^2 g(a_1). \tag{2.3.4}$$

The variance of the unbiased estimator $\widehat{\theta}_2$ is

$$\mathrm{Var}\,(\widehat{\theta}_2) = \mathrm{MSE}\,(\widehat{\theta}_2) = \frac{[\{h(a_2)-2h(a_1)\}h(a_2)g(a_1)+h^2(a_1)g(a_2)]\,\sigma^2}{[h(a_2)-h(a_1)]^2}. \tag{2.3.5}$$

The expectation and the variance of $\widehat{\theta}_3$ are

$$E(\widehat{\theta}_3) = \theta + \left[\{1-(s-1)d\}\,h(a_1) + d\sum_{j=2}^{s}h(a_j)\right] \tag{2.3.6}$$

and

$$\begin{aligned}\mathrm{Var}\,(\widehat{\theta}_3) = \Big[&\{1-(s-1)d\}^2 g(a_1) + d^2\Big\{\sum_{j=2}^{s}g(a_j) + 2\sum_{j=2}^{s-1}(s-j)g(a_j)\Big\} \\ &+ 2d\{1-(s-1)d\}(s-1)\,g(a_1)\Big]\sigma^2.\end{aligned} \tag{2.3.7}$$

We can obtain the expectation and variance of $\widehat{\sigma}_{2i}$ as follows;

If θ is known,

$$E(\widehat{\sigma}_{20}) = -\frac{E}{A_2}\sigma \tag{2.3.8}$$

and

$$\mathrm{Var}\,(\widehat{\sigma}_{20}) = \frac{F}{A_2^2}\sigma^2 \tag{2.3.9}$$

where

$$E = (a_1-1)\beta_2 h(a_1) - (n-a_s)h(a_s) - \sum_{j=1}^{s}h(a_j) + \sum_{j=2}^{s}(a_j - a_{j-1} - 1)[\beta_j h(a_j) + \gamma_j h(a_{j-1})]$$

and

$$\begin{aligned}F = \;&(a_1-1)\beta_2 g(a_1)\{(a_1-1)\beta_2 - 2(n-a_s+s)\} + (n+1-a_s)^2 g(a_s) \\ &+ \sum_{j=1}^{s-1}\{(1+2n-2a_s)g(a_j) + 2(s-j)g(a_j)\} \\ &+ \sum_{j=2}^{s}(a_j - a_{j-1}-1)^2\{\beta_j^2 g(a_j) + \gamma_j^2 g(a_{j-1}) + 2\beta_j\gamma_j g(a_{j-1})\}\end{aligned}$$

$$+2\sum_{j=2}^{s-1}(a_j-a_{j-1}-1)\{\beta_j g(a_j)+\gamma_j g(a_{j-1})\}\left[\sum_{k=j+1}^{s}(a_k-a_{k-1}-1)(\beta_k+\gamma_k)-(s-j)\right]$$
$$-2\sum_{j=1}^{s-1}g(a_j)\left[\sum_{k=j+1}^{s}(a_k-a_{k-1}-1)\{\beta_k+\gamma_k\}\right]$$
$$+2\sum_{j=2}^{s}(a_j-a_{j-1}-1)[(a_1-1)\beta_2(\beta_j+\gamma_j)g(a_1)-(n-a_s+1)\{\beta_j g(a_j)+\gamma_j g(a_{j-1})\}].$$

If we use the estimator $\widehat{\theta}_1$ of the location parameter θ, then

$$E(\widehat{\sigma}_{21})=-\frac{1}{A_2}[E-Dh(a_1)]\sigma \tag{2.3.10}$$

and

$$\mathrm{Var}(\widehat{\sigma}_{21})=\frac{1}{A_2^2}[F-D^2g(a_1)]\sigma^2 \tag{2.3.11}$$

where

$$D=(a_1-1)\beta_2-(n-a_s)-s+\sum_{j=2}^{s}(a_j-a_{j-1}-1)[\beta_j+\gamma_j].$$

If we use the estimator $\widehat{\theta}_2$ of the location parameter θ, then

$$E(\widehat{\sigma}_{22})=-\frac{E}{A_2}\sigma \tag{2.3.12}$$

and

$$\mathrm{Var}(\widehat{\sigma}_{22})=\frac{1}{A_2^2}[F+G]\sigma^2 \tag{2.3.13}$$

where

$$G=Dk_2[Dk_2+2(n-a_s+s)-2Dk_1]g(a_1)-2(a_1-1)\beta_2 g(a_1)D$$
$$+Dk_1\left[Dk_1-2(n-a_s+s-1)+2\sum_{j=3}^{s}(a_j-a_{j-1}-1)\gamma_j\right]g(a_2)$$
$$+2Dk_1[(a_2-a_1-1)\gamma_2-1]g(a_1)-2Dk_2\sum_{j=2}^{s}(a_j-a_{j-1}-1)[\beta_j+\gamma_j]g(a_1)$$
$$+2Dk_1\sum_{j=2}^{s}(a_j-a_{j-1}-1)\beta_j g(a_2)$$
$$k_1=\frac{h(a_1)}{h(a_2)-h(a_1)}$$

and

$$k_2=\frac{h(a_2)}{h(a_2)-h(a_1)}.$$

If we use the estimator $\widehat{\theta}_3$ of the location parameter θ, then

$$E(\widehat{\sigma}_{23})=-\frac{1}{A_2}\left[E+Dh(a_1)(1-sd)+Dd\sum_{j=1}^{s}h(a_j)\right]\sigma \tag{2.3.14}$$

and

$$\mathrm{Var}(\widehat{\sigma}_{23})=\frac{1}{A_2^2}[F+J]\sigma^2 \tag{2.3.15}$$

where

$$J=\{1-(s-1)d\}g(a_1)\Big[d(s-1)-2(a_1-n+a_s-s)+3-2\sum_{j=2}^{s}(a_j-a_{j-1}-1)\{\beta_j+\gamma_j\}\Big]$$
$$+d\Big[\sum_{j=2}^{s}g(a_j)(d+2+2n-2a_s)+2\sum_{j=2}^{s-1}(s-j)g(a_j)(d+2)$$
$$-2(s-1)g(a_1)\{(a_1-1)\beta_2-1\}\Big]-2d\Big[\sum_{j=2}^{s-1}(s-j)(a_j-a_{j-1}-1)\{\beta_j g(a_j)+\gamma_j g(a_{j-1})\}$$
$$+\sum_{j=2}^{s}(a_j-a_{j-1}-1)\{\beta_j g(a_j)+\gamma_j g(a_{j-1})\}+\sum_{j=2}^{s-1}g(a_j)\sum_{k=j+1}^{s}(a_k-a_{k-1}-1)\{\beta_k+\gamma_k\}\Big].$$

From equations (2.1.1), (2.1.4), (2.1.7), (2.2.7), (2.2.11), and (2.2.12), the MSEs of these estimators are simulated by Monte Carlo method for sample size $n=20, 50$, and various choices of censoring. The simulation procedure is repeated 10,000 times in multiply Type-II censored samples. These values are given in Tables 1 and 2.

From Table 1, the estimators $\hat{\theta}_3$ is more efficient than the other estimators in the sense of the MSE.

From Table 2, when the location parameter is known, the estimators $\hat{\sigma}_{20}$ and $\hat{\sigma}_{30}$ are generally more efficient than the estimator $\hat{\sigma}_{10}$ in the sense of the MSE. When the location parameter is unknown, $\hat{\theta}_3$ is more efficient than the other estimators of the location parameter but the estimators $\hat{\sigma}_{11}$ which use the estimator $\hat{\theta}_1$ is generally more efficient than the other estimators. $\hat{\sigma}_{21}$, $\hat{\sigma}_{23}$, $\hat{\sigma}_{31}$, and $\hat{\sigma}_{33}$ are also good estimators. So we can recommend the estimators $\hat{\sigma}_{l1}$ or $\hat{\sigma}_{l3}$ $(l=1,2,3)$ which use the estimators $\hat{\theta}_1$ or $\hat{\theta}_3$ to estimate the scale parameter. From the exact values of the MSE, when the location parameter is unknown, the estimator $\hat{\sigma}_{23}$ is more efficient than the other estimators. The MSEs of all the estimators generally increase as k increases.

Table 1. The relative mean squared errors for the estimators of the location parameter θ.

n	k	a_j	MSE			The exact values of MSE		
			$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$
20	0	1~20	0.0048	0.0050	0.0025	0.0050	0.0050	0.0026
	2	1~18	0.0048	0.0050	0.0025	0.0050	0.0050	0.0026
		3~20	0.0334	0.0348	0.0099	0.0334	0.0334	0.0097
		2~19	0.0158	0.0159	0.0059	0.0158	0.0158	0.0059
	5	3~17	0.0334	0.0348	0.0103	0.0334	0.0334	0.0101
		4~18	0.0591	0.0604	0.0153	0.0589	0.0589	0.0151
		2~6 10~19	0.0158	0.0159	0.0059	0.0158	0.0158	0.0059
	6	4~17	0.0591	0.0604	0.0156	0.0589	0.0589	0.0154
		1 2 6~9 12~15 17~20	0.0048	0.0050	0.0025	0.0050	0.0050	0.0026

Table 1. (continued)

n	k	a_j	MSE $\hat{\theta}_1$	MSE $\hat{\theta}_2$	MSE $\hat{\theta}_3$	The exact values of MSE $\hat{\theta}_1$	The exact values of MSE $\hat{\theta}_2$	The exact values of MSE $\hat{\theta}_3$
50	0	1~50	0.0008	0.0008	0.0004	0.0008	0.0008	0.0004
	2	1~48	0.0008	0.0008	0.0004	0.0008	0.0008	0.0004
		3~50	0.0025	0.0024	0.0009	0.0024	0.0024	0.0008
		2~49	0.0025	0.0024	0.0009	0.0024	0.0024	0.0009
	5	3~47	0.0085	0.0086	0.0019	0.0085	0.0085	0.0019
		4~48	0.0085	0.0086	0.0019	0.0085	0.0085	0.0019
		2~6 10~19 21~50	0.0025	0.0024	0.0009	0.0024	0.0024	0.0008
	6	4~47	0.0025	0.0024	0.0009	0.0024	0.0024	0.0008
		1 2 6~9 12~15 17~50	0.0008	0.0008	0.0004	0.0008	0.0008	0.0004

Table 2. The relative mean squared errors for the estimators of the scale parameter σ.

n	k	a_j	MSE $\hat{\sigma}_{10}$	$\hat{\sigma}_{11}$	$\hat{\sigma}_{12}$	$\hat{\sigma}_{13}$	$\hat{\sigma}_{20}$	$\hat{\sigma}_{21}$	$\hat{\sigma}_{22}$	$\hat{\sigma}_{23}$
20	0	1~20	0.0508	0.0505	0.0535	0.0532	0.0508	0.0505	0.0535	0.0532
	2	1~18	0.0567	0.0562	0.0600	0.0596	0.0567	0.0562	0.0600	0.0596
		3~20	0.0556	0.0565	0.0602	0.0593	0.0508	0.0653	0.0602	0.0593
		2~19	0.0577	0.0565	0.0602	0.0595	0.0537	0.0588	0.0602	0.0595
	5	3~17	0.0661	0.0677	0.0734	0.0717	0.0599	0.0797	0.0734	0.0717
		4~18	0.0629	0.0680	0.0737	0.0717	0.0567	0.0926	0.0737	0.0716
		2~6 10~19	0.0651	0.0599	0.0664	0.0655	0.0537	0.0588	0.0603	0.0596
	6	4~17	0.0666	0.0725	0.0793	0.0767	0.0599	0.1002	0.0792	0.0767
		1 2 6~9 12~15 17~20	0.0705	0.0630	0.0734	0.0728	0.0511	0.0507	0.0538	0.0536
50	0	1~50	0.0196	0.0196	0.0199	0.0199	0.0196	0.0196	0.0199	0.0199
	2	1~48	0.0208	0.0204	0.0207	0.0207	0.0208	0.0204	0.0207	0.0207
		3~50	0.0203	0.0205	0.0208	0.0208	0.0196	0.0222	0.0208	0.0208
		2~49	0.0207	0.0204	0.0207	0.0207	0.0199	0.0209	0.0207	0.0207
	5	3~47	0.0214	0.0218	0.0222	0.0221	0.0203	0.0238	0.0222	0.0221
		4~48	0.0219	0.0218	0.0221	0.0221	0.0208	0.0259	0.0221	0.0221
		2~6 10~19 21~50	0.0225	0.0213	0.0224	0.0223	0.0196	0.0206	0.0204	0.0204
	6	4~47	0.0226	0.0223	0.0227	0.0226	0.0196	0.0266	0.0227	0.0226
		1 2 6~9 12~15 17~50	0.0228	0.0216	0.0232	0.0231	0.0196	0.0196	0.0199	0.0199

n	k	a_j	MSE $\hat{\sigma}_{30}$	MSE $\hat{\sigma}_{31}$	MSE $\hat{\sigma}_{32}$	MSE $\hat{\sigma}_{33}$	The exact values of MSE $\hat{\sigma}_{20}$	The exact values of MSE $\hat{\sigma}_{21}$	The exact values of MSE $\hat{\sigma}_{22}$	The exact values of MSE $\hat{\sigma}_{23}$
20	0	1~20	0.0506	0.0503	0.0531	0.0529	0.0500	0.0500	0.0526	0.0500
	2	1~18	0.0568	0.0566	0.0601	0.0597	0.0556	0.0556	0.0588	0.0556
		3~20	0.0506	0.0651	0.0595	0.0586	0.0500	0.0650	0.0588	0.0500
		2~19	0.0531	0.0584	0.0595	0.0589	0.0526	0.0582	0.0588	0.0527
	5	3~17	0.0605	0.0803	0.0735	0.0719	0.0589	0.0795	0.0716	0.0589
		4~18	0.0569	0.0930	0.0735	0.0715	0.0556	0.0924	0.0716	0.0556
		2~6 10~19	0.0534	0.0586	0.0598	0.0592	0.0911	0.0965	0.0973	0.0911
	6	4~17	0.0605	0.1008	0.0796	0.0772	0.0589	0.1002	0.0772	0.0589
		1 2 6~9 12~15 17~20	0.0506	0.0504	0.0532	0.0529	0.0976	0.0975	0.1003	0.0976

Table 2. (continued)

			MSE				The exact values of MSE			
			$\hat{\sigma}_{30}$	$\hat{\sigma}_{31}$	$\hat{\sigma}_{32}$	$\hat{\sigma}_{33}$	$\hat{\sigma}_{20}$	$\hat{\sigma}_{21}$	$\hat{\sigma}_{22}$	$\hat{\sigma}_{23}$
50	0	1~50	0.0200	0.0200	0.0204	0.0204	0.0200	0.0200	0.0204	0.0200
	2	1~48	0.0208	0.0208	0.0213	0.0213	0.0208	0.0208	0.0213	0.0208
		3~50	0.0200	0.0224	0.0213	0.0213	0.0200	0.0224	0.0213	0.0200
		2~49	0.0203	0.0213	0.0214	0.0213	0.0204	0.0212	0.0213	0.0204
	5	3~47	0.0212	0.0239	0.0227	0.0226	0.0213	0.0240	0.0227	0.0213
		4~48	0.0208	0.0260	0.0228	0.0227	0.0208	0.0260	0.0227	0.0208
		2~6 10~19 21~50	0.0200	0.0209	0.0210	0.0209	0.0418	0.0426	0.0426	0.0418
	6	4~47	0.0212	0.0266	0.0233	0.0232	0.0213	0.0267	0.0233	0.0213
		1 2 6~9 12~15 17~50	0.0200	0.0200	0.0204	0.0204	0.0456	0.0456	0.0460	0.0456

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