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QUASISIMILARITY AND INJECTIVE p-QUASIHYPONORMAL OPERATORS

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ABSTRACT. In this paper it is proved that quasisimilar n-tuples of tensor products of injective p-quasihyponormal operators have the same spectra, essential spectra and indices, respectively. And it is also proved that a Weyl n-tuple of tensor products of injective p-quasihyponormal operators can be perturbed by an n-tuple of compact operators to an invertible n-tuple.

1. Introduction

Let $L(\mathcal{H})$ denote the Banach algebra of bounded linear operators acting on a complex infinite dimensional Hilbert space $\mathcal{H}$. Let $\mathbf{T} = (T_1, \cdots, T_n)$ denote a commuting n-tuple of operators in $L(\mathcal{H})$. Recall ([3], [9]) that $\mathbf{T}$ is said to be *invertible* if the *Koszul complex* for $\mathbf{T}$, denoted by $K(\mathbf{T}, \mathcal{H})$, is exact at every stage. Also, $\mathbf{T}$ is said to be *Fredholm* if the Koszul complex $K(\mathbf{T}, \mathcal{H})$ is Fredholm, i.e., all homologies of $K(\mathbf{T}, \mathcal{H})$ are finite dimensional. In this case the *index* of $\mathbf{T}$, denoted ind$(\mathbf{T})$, is defined as the *Euler characteristic* of $K(\mathbf{T}, \mathcal{H})$, i.e., as the alternating sum of dimensions of all homologies of $K(\mathbf{T}, \mathcal{H})$. If $\mathbf{T} \in L(\mathcal{H})$ is Fredholm with index zero, then we say that $\mathbf{T}$ is *Weyl.* We shall write $\sigma_T(\mathbf{T})$, $\sigma_{Te}(\mathbf{T})$, and $\sigma_{Tw}(\mathbf{T})$ for the *Taylor spectrum*, the *Taylor essential spectrum*, and *Taylor-Weyl spectrum* of $\mathbf{T}$, respectively : thus,

$$\sigma_T(\mathbf{T}) = \{\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n : \mathbf{T} - \lambda \text{ is not invertible}\},$$

$$\sigma_{Te}(\mathbf{T}) = \{\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n : \mathbf{T} - \lambda \text{ is not Fredholm}\},$$

and

$$\sigma_{Tw}(\mathbf{T}) = \{\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n : \mathbf{T} - \lambda \text{ is not Weyl}\}.$$

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For any open polydisk $D \subset \mathbb{C}^n$, let $\mathcal{O}(D, \mathcal{H})$ denote the Frechét space of $\mathcal{H}$-valued analytic functions on D. Then we say ([9]) that a commuting n-tuple $\mathbf{T}$ has the *single valued extension property*, shortened to SVEP, if the Koszul complex $\mathcal{K}(\mathbf{T}-\lambda, \mathcal{O}(D,\mathcal{H}))$ is exact in positive degrees and $\mathbf{T}$ has *Bishop's condition* (β) if it has the SVEP and its Koszul complex has also separated homology in degree zero. Obviously, the following implication holds:

$$\text{Bishop's condition}(\beta) \implies \text{the SVEP}.$$

For more details, see [9].

Recall [3] that $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{C}$ is said to be an eigenvalue of $\mathbf{T}$ if there exists a non-zero vector $x \in \mathcal{H}$ such that $x \in \bigcap \ker(T_i - \lambda_i)$. We denote the set of all eigenvalues if $\mathbf{T}$ by $\sigma_p(\mathbf{T})$.

$$p_{00}(\mathbf{T}) = \sigma_T(\mathbf{T}) \setminus \{\sigma_{Te}(\mathbf{T}) \cup \operatorname{acc}\sigma(\mathbf{T})\}$$

for the *Riesz points* of $\sigma_T(\mathbf{T})$.

Recall [1] that an operator $T \in L(\mathcal{H})$ is said to be *p-hyponormal* if

$$|T|^{2p} - |T^*|^{2p} \geq 0 \text{ for } p \in (0,1].$$

If $p = 1$, T is just hyponormal.

Definition 1. ([8], [13], [20]) An operator $T \in L(\mathcal{H})$ is said to be *p-quasihyponormal* if

$$T^*(|T|^{2p} - |T^*|^{2p})T \geq 0 \text{ for } p \in (0,1].$$

We denote classes of p-hyponormal, p-quasihyponormal and injective p-quasihyponormal operators by $\mathscr{H}(p)$, $\mathscr{QH}(p)$ and $\mathscr{QH}(p)^*$, respectively. It is well known that

$$\mathscr{H}(p) \subset \mathscr{QH}(p).$$

Indeed, letting $T \in \mathscr{H}(p)$ have the polar decomposition $T = U|T|$, the p-hyponormality of T implies that

$$U|T|^{2p}U^* \leq |T|^{2p} \leq U^*|T|^{2p}U,$$

which implies that

$$|T|^{2p+2} = T^*|T^*|^{2p}T \leq T^*|T|^{2p}T.$$

Hence $\mathscr{H}(p)$ operators are $\mathscr{QH}(p)$ operators.

In this paper we prove that quasisimilar n-tuples of tensor products of injective p-quasihyponormal operators have the same spectra, essential spectra and indices, respectively. Also, we prove that a Weyl n-tuple of tensor products of injective p-quasihyponormal operators can be perturbed by an n-tuple of compact operators to an invertible n-tuple. These results generalize earlier results proved in [7].

Throughout this paper, for complex infinite dimensional Hilbert spaces $\mathcal{H}_i$ $(1 \leq i \leq n)$, we let $\widehat{\mathcal{H}} = \mathcal{H}_1 \widehat{\otimes \cdots \otimes} \mathcal{H}_n$ denote the completion of $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ with respect to some crossnorm and let

$$T_i := I_1 \otimes \cdots \otimes I_{i-1} \otimes A_i \otimes \cdots \otimes I_n \text{ on } \widehat{\mathcal{H}},$$

where I_i is the identity operator on $\mathcal{H}_i$ and $A_i \in L(\mathcal{H}_i)$. Then $\mathbf{T} = (T_1, \cdots, T_n)$ is a commuting (in fact, doubly commuting) n-tuple of operators on $\widehat{\mathcal{H}}$.

2. Quasisimilarity

If T has the polar decomposition $T = U|T|$, then $T^{\frown} = |T|U$ is called to be *Duggal transform* of T. It is well known that Duggal transform is one of very useful tools to study properties of operators ([10]). As an essential tool to prove Theorem 5 below, we will use Duggal transforms of $\mathcal{QH}(p)$ operators. We begin with some lemmas.

LEMMA 2. ([17, Theorem 2.12]) *Let $A = A_1 \oplus A_2 \in L(\mathcal{H}_1 \oplus \mathcal{H}_2)$. Then $A = A_1 \oplus A_2$ has Bishop's condition (β) if and only if $A_i (i = 1, 2)$ has Bishop's condition (β).*

LEMMA 3. *Let $A \in L(\mathcal{H})$ have a kernel condition* $\ker(A) \subseteq \ker(A^*)$. *Then A has Bishop's condition (β) if and only if $A^{\frown}$ has Bishop's condition (β).*

Proof. Let A have a decomposition $A = A_1 \oplus A_2$ with respect to some decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ of $\mathcal{H}$, such that $A_1 = A|_{\mathcal{H}_1}$ is normal and $A_2 = A|_{\mathcal{H}_2}$ is pure. Let A_i have the polar decomposition $A_i = U_i|A_i|$. Then the partial isometry U_2 is an isometry, and we may choose the partial isometry U_1 to be a unitary such that the commutator $[|A_1|, U_1] = |A_1|U_1 - U_1|A_1| = 0$. Define the Duggal transform $A^{\frown}$ of A by

$$A^{\frown} = A_1 \oplus A_2^{\frown} = (|A_1| \oplus |A_2|)(U_1 \oplus U_2).$$

Then A_2 is injective operator since A_2 is injective. From [2] A_2 has Bishop's condition (β) if and only if $A_2^{\frown}$ has Bishop's condition (β). Since A_1 is normal, A_1 has Bishop's condition (β). Hence it immediately follows from Lemma 3 that A has Bishop's condition (β) if and only if $A^{\frown}$ has Bishop's condition (β). □

COROLLARY 4. *Let $A \in \mathcal{QH}(p)^*$. Then A has Bishop's condition (β) if and only if $A^\frown$ has Bishop's condition (β).*

Proof. The proof easily follows from Lemma 3 because the injectivity of A obviously implies the kernel condition $\ker(A) \subseteq \ker(A^*)$. □

THEOREM 5. *Let $A_i, B_i \in \mathcal{QH}(p)^*$. Let $\mathbf{T} = (T_1, \cdots, T_n)$ and $\mathbf{S} = (S_1, \cdots, S_n)$ be n-tuples of $T_i = I_1 \otimes \cdots \otimes I_{i-1} \otimes A_i \otimes \cdots \otimes I_n$ and $S_i = I_1 \otimes \cdots \otimes I_{i-1} \otimes B_i \otimes \cdots \otimes I_n$, respectively. If $\mathbf{T} = (T_1, \cdots, T_n)$ and $\mathbf{S} = (S_1, \cdots, S_n)$ are quasisimilar n-tuples, then they have the same spectra, essential spectra and indices, respectively.*

Proof. First, we observe that if $A \in \mathcal{QH}(p)^*$ then $A^\frown \in \mathcal{H}(p)$. We consider decompositions of $A = U|A|$ and $A^\frown = |A|U$, respectively. Then since $A \in \mathcal{QH}(p)^*$, $|A|$ is injective, and so has dense range. Thus it follows from the equivalence

$$A^*(|A|^{2p} - |A^*|^{2p})A \geq 0 \Longleftrightarrow U^*(|A|^{2p} - |A^*|^{2p})U \geq 0$$

that

$$(A^\frown A^{\frown *})^p \leq |A|^{2p} = U^*|A^*|^{2p}U \leq U^*|A|^{2p}U \leq (A^{\frown *}A^\frown)^p,$$

i.e., $A^\frown = |A|U$ is p-hyponormal. Since it is well known [22] that every $\mathcal{H}(p)$ operators has Bishop's condition (β), each $A_i^\frown$ has Bishop's condition (β). Thus it follows from Corollary 4 that for $i = 1, \cdots, n$ each $A_i \in \mathcal{QH}(p)^*$ has Bishop's condition (β). On the other hand, recall [14] that if $B_1, B_2 \in L(\mathcal{H}_i)$, then

$$B_1 \otimes B_2 \in \mathcal{QH}(p) \text{ if and only if } B_1, B_2 \in \mathcal{QH}(p).$$

Thus it follows from a finite induction argument that

$$T_i \in \mathcal{QH}(p) \text{ if and only if } A_i \in \mathcal{QH}(p) \text{ for all } i = 1, \cdots, n.$$

Thus the fact [8] that $A_i \in \mathcal{QH}(p)^*$ has Bishop's condition (β) implies that each T_i has Bishop's condition (β). Thus it follows from [21, Corollary 2.2] that the n-tuple $\mathbf{T} = (T_1, \cdots, T_n)$ has also Bishop's condition (β). Similarly, the n-tuple $\mathbf{S} = (S_1, \cdots, S_n)$ has Bishop's condition (β), too. It is well known [16, Theorem 1; Corollary 1] that if $\mathbf{T}$ and $\mathbf{S}$ are quasisimilar commuting n-tuple having Bishop's condition (β), then these n-tuples have the same spectra, essential spectra and indices, respectively. Hence the proof immediately follows from this result. □

3. Compact perturbation

Fredholm n-tuples enjoy most of the properties single Fredholm operators possess [3]. It is well known that a Fredholm operator of index zero (i.e., Weyl operator) can be perturbed by a compact operator to an invertible operator. Thus one may ask if this property holds in several variables [4, Problem 3]. As it turns out, this perturbation property fails in several variables (see [11] for an example). Despite the failure of this property for the general case, the following result gives a positive answer to the question in case of tensor products considered here.

THEOREM 6. *Let $A_i \in \mathscr{Q}\mathscr{H}^*$ and let $\mathbf{T} = (T_1, \cdots, T_n)$ be an n-tuple of operators*

$$T_i := I_1 \otimes \cdots \otimes I_{i-1} \otimes A_i \otimes \cdots \otimes I_n \text{ on } \widehat{\mathcal{H}}.$$

If $\mathbf{T}$ is Weyl but not invertible, then there exists an invertible commuting n-tuple $\mathbf{S} = (S_1, \cdots, S_n)$ such that $\mathbf{T} = \mathbf{S} + \mathbf{F}$ for some n-tuple of compact operators F_i $(i = 1, \cdots, n)$.

Proof. Since $\mathbf{T}$ is Weyl but not invertible, [15, Theorem 1] implies $0 \in p_{00}(\mathbf{T})$. Let f be the characteristic function of $0 \in \operatorname{iso}\sigma_T(\mathbf{T})$; since f is analytic in a neighborhood of $\sigma_T(\mathbf{T})$, [19, Theorem 4.8; Corollary 4.9] implies the existence of an idempotent $P_0 = f(\mathbf{T}) \in L(\widehat{\mathcal{H}})$ such that $P_0 T_i = T_i P_0$, T_i is quasinilpotent on $\operatorname{ran} P_0$, and

$$0 \notin \sigma_T(\mathbf{T}|_{\ker P_0}). \tag{3.1}$$

Since the restriction of p-quasihyponormal operator to its invariant subspace is again p-quasihyponormal [13, Theorem 1] and p-quasihyponormal operators are normaloid, we see that $T_i|_{\operatorname{ran} P_0} = 0$. Again, since $T_i|_{\operatorname{ran} P_0}$ is normal, [13, Theorem 2] implies that $\operatorname{ran} P_0$ is a reducing subspace of T_i, and so $\ker P_0 = (\operatorname{ran} P_0)^\perp$, i.e., P_0 is an orthogonal projection, and

$$T_i = 0 \oplus T_i' \text{ on } \widehat{\mathcal{H}} = \operatorname{ran} P_0 \oplus \operatorname{ran} {P_0}^\perp,$$

where T_i' is the p-hyponormal restriction of T_i to the subspace $\operatorname{ran} {P_0}^\perp$. The fact that $0 \in p_{00}(\mathbf{T})$ implies that the subspace $\operatorname{ran} P_0$ is finite dimensional, and so P_0 is a compact operator on $\widehat{\mathcal{H}}$. Considering $\mathbf{F} = (P_0, \cdots, P_0)$ and $\mathbf{S} = \mathbf{T} - \mathbf{F} = (T_1 - P_0, \cdots, T_n - P_0)$, it now follows that $\mathbf{S}$ is a commuting n-tuple. This by [3, p.39] implies that

$$\sigma_T(\mathbf{S}) = \sigma_T((\mathbf{T} - \mathbf{F})|_{\operatorname{ran} P_0}) \cup \sigma_T\left((\mathbf{T} - \mathbf{F})|_{\operatorname{ran} {P_0}^\perp}\right).$$

Obviously, $0 \notin \sigma_T((\mathbf{T}-\mathbf{F})|_{\operatorname{ran} P_0})$ and by (3.1)

$$0 \notin \sigma_T((\mathbf{T}-\mathbf{F})|_{\operatorname{ran} P_0{}^{\perp}}) = \sigma_T(\mathbf{T}|_{\ker P_0}).$$

Thus $0 \notin \sigma_T(\mathbf{S})$, i.e., $\mathbf{S} = \mathbf{T} - \mathbf{F}$ is invertible, and hence $\mathbf{T} = \mathbf{S} + \mathbf{F}$. □

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