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TOTALLY CHAIN-TRANSITIVE ATTRACTORS OF GENERIC HOMEOMORPHISMS ARE PERSISTENT

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ABSTRACT. we prove that, given any compact metric space X, there exists a residual subset $\mathcal{R}$ of $\mathcal{H}(X)$, the space of all homeomorphisms on X, such that if $f \in \mathcal{R}$ has a totally chain-transitive attractor A, then any g sufficiently close to f has a totally chain-transitive attractor A_g which is convergent to A in the Hausdorff topology.

1. Introduction

One important goal in the theory of dynamical systems is to describe the dynamics of generic sets in the space of all dynamical systems. Some of new results lead us to the study and characterization of totally chain-transitive sets that remain totally chain-transitive for all nearby systems. The notion of chain recurrence introduce by Conley[3], has remarkable connections to the structure of attractors. In fact, we can choose a weaker version of indecomposability for each attractor, that is chain-transitivity. His results tell us if the set of attractors of a flows is ordered by inclusion, then any minimal element will be chain recurrent. Hurley[4] showed that for generic flows any chain transitive attractor and chain-transitive quasi attractor persists by nearby flows. Recently Abdenur[1] showed that generic attractor (with the additional condition transitivity) are C^1-persistent. Abdenur's result also can be follows from Hurley[4] and recent remarkable work of Bonatti and Crovsier[2], in extension of C^1-connecting lemma to ϵ-pesudo orbits. Here, we extend this Hurley's result for totally chain-transitive attractors of homeomorphisms on X.

Before stating our results precisely, we introduce some definitions. Let X be a compact metric space, endowed with a metric d and $\mathcal{H}(X)$ be the space of all homeomorphisms on X with the usual C^0-topology.

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Recall that a subset is residual, if it contains a countable intersection of open and dense sets. We say that a property is generic if it holds on a residual subset.

Now, let $\mathcal{F}X$ composed of all the nonempty closed subsets of X. Let A and B be in $\mathcal{F}X$, put

$$d_H(A,B) = \inf\{\alpha > 0 : A \subset U_\alpha(B) \text{ and } B \subset U_\alpha(A)\},$$

where $U_\alpha(.)$ denotes the α-ball measured in d. By [5], d_H is a complete metric on $\mathcal{F}X$, called the Hausdorff metric.

Let (X_i, d_i) be metric spaces, $i = 1, 2$, and let X_2 be compact. Suppose that, $(\mathcal{F}X_2, d_2)$ as above, is the set of closed subsets of X_2 equipped with Hausdorff metric. A map $h : X_1 \to \mathcal{F}X_2$ is said to be lower semi-continuous at x, if each $\alpha > 0$, there is a $\beta > 0$ such that $h(x) \subseteq U_\alpha(f(y))$, whenever $d_1(x,y) < \beta$. The map h is called lower semi-continuous on X_1, if it is lower semi-continuous at x for all $x \in X_1$.

An attractor of $f \in \mathcal{H}(X)$ is a nonempty, compact f-invariant set A that has a neighborhood U satisfying $\bigcap_{n\in N} f^n(U) = A$ and $f(\overline{U}) \subset U$ (for different definition of attractors see Milnore[6]). The basin of attraction A is the set of all points in X that approach A under the forward iterations of f and denoted by $B(A)$. We say that an attractor A of f is persistent if any g sufficiently close to f has an attractor A_g such that, $A_g \to A$ in Hausdorff topology as $g \to f$.

For given $\epsilon > 0$, an ϵ-chain of f from x to y is a finite set $\{x_0, x_1, \cdots, x_n\}$ such that $x_0 = x$, $x_n = y$ and $d(f(x_{k-1}), x_k) < \epsilon$, for all $k = 1, \cdots, n$. If for each $\epsilon > 0$ there is an ϵ-chain from x to itself, then x is said to be chain-recurrent. The set of all chain recurrent points is closed and we denote it by $CR(f)$.

A nonempty compact f-invariant set Λ is f-chain transitive, if for each $x, y \in \Lambda$ and any $\epsilon > 0$, there is an ϵ-chain of f from x to y. Λ is said to be totally chain transitive, if it is f^n-chain transitive, for any $n \in N$.

For $x \in M$, let $\omega(x)$ be the set $\{\lim f^{n_i}(x) : n_i \to \infty\}$. We say that $x \in M$ is an ω-recurrent point of f, if $x \in \omega(x, f)$. The set of all recurrent points of f is denoted by $R(f)$. In general $R(f)$ is not closed.

A closed nonempty f-invariant set $\Lambda \subset X$ is minimal set, if Λ does not contain any proper closed nonempty f-invariant set. Our main result in this article is the following theorem, which says that generic totally chain transitive attractors are persistent.

THEOREM A. *There is a residual subset R of $\mathcal{H}(X)$ such that, if $f \in \mathcal{R}$ and A is a totally chain transitive attractor of f, then any g*

sufficiently close to f has a totally chain transitive attractor A_g such that $A_g \to A$ in Hausdorff topology as $g \to f$ in C^0-topology.

2. Proof of results

First, we define two binary relations which are equivalent relations and their equivalence classes present the ϵ-chain transitive components and totally ϵ-chain transitive components.

DEFINITION 2.1. Let $\epsilon \geq 0$ be given, we say that $x \in CR(f^n, \epsilon)$, ($x \in CR(f^n)$, when $\epsilon = 0$) if for every $\epsilon' > \epsilon$, there is an ϵ'-chain of f^n from x to x. We define $\overset{n}{\underset{\epsilon}{\sim}}$ on $CR(f^n, \epsilon)$ as follows: for any $x, y \in CR(f^n, \epsilon)$, we say that $x \overset{n}{\underset{\epsilon}{\sim}} y$ if and only if for any $\epsilon' > \epsilon$, there is an ϵ'-chain of f^n from x to y and conversely, from y to x.

Clearly $\overset{n}{\underset{\epsilon}{\sim}}$ defines an equivalence relation on $CR(f^n, \epsilon)$. For $x \in CR(f^n, \epsilon)$, we denote the equivalence class of x with respect to $\overset{n}{\underset{\epsilon}{\sim}}$, by $\Gamma(x, n, \epsilon, f)$. For any f, put

$$CR^\infty(f, \epsilon) = \bigcap_{n \in N} CR(f^n, \epsilon) \text{ and } CR^\infty(f) = \bigcap_{\epsilon > 0} CR^\infty(f, \epsilon).$$

DEFINITION 2.2. For $x, y \in CR^\infty(f, \epsilon)$, we say that $x \underset{\epsilon}{\sim} y$ if and only if $x \overset{n}{\underset{\epsilon}{\sim}} y$ for each $n \in N$.

Let $x \in CR^\infty(f, \epsilon)$ and $\Gamma(x, \epsilon, f)$ be the equivalence class of x with respect to equivalence relation $\underset{\epsilon}{\sim}$. Clearly

$$\Gamma(x, \epsilon, f) = \bigcap_{n \in N} \Gamma(x, n, \epsilon, f).$$

First, we show that $CR^\infty(f)$ is nonempty, compact and

$$R(f) \subseteq CR^\infty(f) \subseteq CR(f).$$

LEMMA 2.3. *For each $f \in \mathcal{H}(X)$, $R(f) \subseteq CR^\infty(f)$. In particular, $CR^\infty(f)$ is nonempty.*

Proof. By compactness of X and Zorn's Lemma, f has a minimal set. Clearly if $A \subseteq M$, is minimal set, then for any $x \in A$, $\overline{O^+(x)} = A$. Therefore, $A \subseteq R(f)$ and in particular, $R(f)$ is nonempty. Let $x \in R(f)$, then $x \in \omega(x, f)$. We claim that $x \in \omega(x, f^n)$. First, for any given $n \in N$, we find $m \in \{0, 1, \ldots, n-1\}$ such that $f^m(x) \in \omega(x, f^n)$. Let $\{n_k\}_{k \in N}$ be a sequence of positive integers such that $n_k \to \infty$ and $f^{n_k}(x) \to x$, as $k \to \infty$. For each k, we choose an integer $0 \leq r_k \leq n-1$ such that n divides $n_k + r_k$. Since the limit set of $\{f^{n_k + r_k}(x) : k \in N\}$

is contained in $\{x, \ldots, f^{n-1}(x)\}$, so for some $0 \leq m \leq n-1$, $f^m(x)$ is a limit point of $\{f^{n_k+r_k}(x) : k \in N\}$. Therefore, $f^m(x) \in \omega(x, f^n)$.

Now if $m = 0$ the claim is done, otherwise, we prove that for any $k \geq 1$, $f^{km}(x) \in \omega(x, f^n)$. By induction, let $f^{km}(x) \in \omega(x, f^n)$, we have $f^{(k+1)m}(x) = f^m(f^{km}(x)) \in f^m(\omega(x, f^n)) = \omega(f^m(x), f^n)$. Since $f^m(x) \in \omega(x, f^n)$, so $f^{(k+1)m}(x) \in \omega(x, f^n)$. In particular for $k = n$, $f^{mn}(x) \in \omega(x, f^n)$. So for any $n \in N$, there is $m \in \{0, 1, \ldots, n-1\}$ and a sequence $\{k_l\}$ of positive integers such that $f^{nk_l}(x) \longrightarrow f^{mn}(x)$ and therefore $f^{n(k_l-m)}(x) \longrightarrow x$. So the claim is done.

Now for any $\epsilon > 0$, choose $k \in N$ such that $d(f^{kn}(x), x) < \epsilon$. Clearly, the set $\{x, f^n(x), \ldots f^{(k-1)n}(x)\}$ is an ϵ-chain of f^n from x to itself and thus $x \in CR(f^n)$. Since $n \in N$ and $\epsilon > 0$ are arbitrary, this implies that $x \in CR^\infty(f)$. □

PROPOSITION 2.4. *Suppose that* $\epsilon > 0$ *is given and* $x \in R^\infty(f, \epsilon)$. *Then,* $\Gamma(x, \epsilon, f)$ *is compact.*

Proof. First we show that for any $n \in N$, $\Gamma(x, n, \epsilon, f)$ is closed. For the proof let $x_m \in \Gamma(x, n, \epsilon, f)$, be a sequence convergent to x'.

Let $\epsilon' > \epsilon$, then there is an integer $m \in N$ such that $d(x_m, x') < (\epsilon' - \epsilon)/2$. By Definition 2.1, there is an $(\epsilon' + \epsilon)/2$-chain, $x = y_0, y_1, \ldots, y_s = x_m$ of f^n from x to x_m and so $y_0, y_1, \ldots, y_{s-1}, x'$ is an ϵ'-chain of f^n from x to x'. Similarly there is an ϵ'-chain of f^n from x' to x and then, $x' \in \Gamma(x, n, \epsilon, f)$. Therefore, $\Gamma(x, n, \epsilon, f)$ is compact. Now since $\Gamma(x, \epsilon, f) = \bigcap_{n \in N} \Gamma(x, n, \epsilon, f)$, this implies that $\Gamma(x, \epsilon, f)$ is compact too. □

We note that $CR^\infty(f) \subseteq CR^\infty(f, \epsilon)$.

PROPOSITION 2.5. *There are many finitely equivalence classes* $\Gamma(x, \epsilon, f)$, *which cover* $CR^\infty(f)$.

Proof. By the argument like as the above proposition, it is easy to see that $CR^\infty(f)$ is compact. Let $x \in CR^\infty(f)$ and it's equivalence class at $CR^\infty(f, \epsilon)$ be $\Gamma(x, \epsilon, f)$. For each $n \in N$, there is an $\epsilon/2$-chain of f^n from x to x. Now, if $d(x, y) < \epsilon/2$, then any $\epsilon/2$-chain from x to x is an ϵ-chain from x to y and conversely. So $B_{\epsilon/2}(x) \subseteq \Gamma(x, \epsilon, f)$. Since $CR^\infty(f) = \bigcup_{x \in R^\infty(f)} B_{\epsilon/2}(x)$ and $CR^\infty(f)$ is compact then there exist $x_1, \ldots, x_n \in CR^\infty(f)$ such that

$$CR^\infty(f) = B_{\epsilon/2}(x_1) \cup \ldots \cup B_{\epsilon/2}(x_n) \subseteq \Gamma(x_1, \epsilon, f) \cup \ldots \cup \Gamma(x_n, \epsilon, f). \quad \square$$

PROPOSITION 2.6. *For each* $\epsilon \geq 0$ *the mapping* $f \mapsto CR^\infty(f, \epsilon)$ *is lower semi-continuous. In particular, the mapping* $f \mapsto CR^\infty(f)$ *is lower semi-continuous.*

Proof. Let $g_m \to f$ in C^0-topology, $x_m \in CR^\infty(g_m, \epsilon)$, $x_m \to x$ and let $\epsilon' > \epsilon$ be given. We show that $x \in CR(f^n, \epsilon)$, for each $n \in N$. Let $n \in N$ be fixed. Since $x_m \in CR^\infty(g_m, \epsilon)$, so for any $m \in N$, there is an $(\epsilon + 1/m)$-chain of g_m^n from x_m to x_m, say T_m. Since any T_m is compact, so there is a subsequence of $\{T_m\}_{m \in N}$ which is convergent in Hausdorff topology to a subset T_0.

Now choose $0 < \delta < (\epsilon' - \epsilon)/4$, such that if $d(x, y) < \delta$, then $d(f^n(x), f^n(y)) < (\epsilon' - \epsilon)/8$. Also, let $m \in N$ be large enough, such that $1/m < (\epsilon' - \epsilon)/2$, $d(g_m^n, f^n) < (\epsilon' - \epsilon)/8$ and $d_H(T_m, T_0) < \delta$, where as before d_H is the Hausdorff metric in the set of all compact subsets of M.

Suppose that $T_m = \{p_1^m, \dots, p_j^m\}$ is the $(\epsilon + 1/m)$-chain of g_m^n from x_m to x_m and choose $x_1, \dots, x_j \in T_0$ such that $d(x_i, p_i^m) < \delta$. Since $p_1^m = p_j^m = x_m$ and $x_m \to x$, so we can choose $x_1 = x_j = x$. Now, we have

$$\begin{aligned} d(f^n(x_i), x_{i+1}) &< d(f^n(x_i), f^n(p_i^k)) + d(f^n(p_i^k), g_m^n(p_i^k)) \\ &\quad + d(g_m^n(p_i^k), P_{i+1}^k) + d(p_{i+1}^k, x_{i+1}) \\ &< \frac{\epsilon' - \epsilon}{8} + \frac{\epsilon' - \epsilon}{8} + (\epsilon + \frac{1}{m}) + \delta \\ &< \frac{\epsilon' - \epsilon}{4} + (\epsilon' - \frac{\epsilon' - \epsilon}{2}) + \frac{\epsilon' - \epsilon}{4} \\ &= \epsilon'. \end{aligned}$$

It means that $x \in CR^\infty(f, \epsilon)$ and therefore the mapping $f \mapsto CR^\infty(f, \epsilon)$ is lower semi-continuous. The second statement follows immediately. □

In the sequel, we need the following topological lemma. For the proof see [5].

TOPOLOGICAL LEMMA I. *If X and Y are metric space with Y compact, and if $f : X \to \mathcal{F}Y$ is either upper or lower semi-continuous, then the set of continuity points of f is residual subset of X.*

By Topological Lemma I, we have the following result.

COROLLARY 2.7. *There is a residual subset $\mathcal{R}^\infty \subseteq \mathcal{H}(X)$ such that the mapping $f \mapsto CR^\infty(f)$ is continuous on it.*

PROPOSITION 2.8. *There is a residual set $\mathcal{R}_1$ such that, if $f \in \mathcal{R}_1$ and A is an attractor of f, then each $g \in \mathcal{H}(X)$ sufficiently close to f, has an attractor A_g such that, $A_g \to A$ in Hausdorff topology as $g \to f$ in C^0-topology.*

Proof. Let $\{U_n\}_{n\in N}$ be a countable basis for X and $\mathcal{C}$ be the family of all finite union of U_n's. Then $\mathcal{C}$ is countable and denote it by $\mathcal{C} = \{V_n\}_{n\in N}$. For each $V_k \in \mathcal{C}$, we define the mapping $A_{V_k} : \mathcal{H}(X) \to 2^X$ by $A_{V_k}(f) = \bigcap_{n\in N} f^n(\overline{V_k})$. We show that, this mapping is lower semi-continuous. For this, let $A_{V_k}(f) \subseteq U$, where U is an open subset of X. Then $\bigcap_{n\in N} f^n(\overline{V_k}) \subseteq U$ and hence

$$U^c \subseteq \bigcup_{n\in N} (f^n(\overline{V_k}))^c.$$

Since U^c is compact, then there is an integer $N_0 \in N$ such that

$$U^c \subseteq \bigcup_{n=1}^{N_0} (f^n(\overline{V_k}))^c.$$

Now, if g sufficiently close to f, then

$$U^c \subseteq \bigcup_{n=1}^{N_0} (g^n(\overline{V_k}))^c,$$

and thus

$$A_{V_k}(g) = \bigcap_{n\in N} g^n(\overline{V_k}) \subseteq \bigcap_{n=1}^{N_0} g^n(\overline{V_k}) \subseteq U.$$

This implies that the mapping $f \mapsto A_{V_k}(f)$ is lower semi-continuous and so by topological lemma I, there is residual subset $\mathcal{R}_{V_k}$ such that A_{V_k} is continuous on $\mathcal{R}_{V_k}$. Put

$$\mathcal{R}_1 = \bigcap_{k\in N} \mathcal{R}_{V_k}.$$

Let $f \in \mathcal{R}_1$ and A be an attractor of f. Then, there is an open set U such that $A = \bigcap_{n\in N} f^n(U)$ and $f(\overline{U}) \subseteq U$. Let $V_k \in \mathcal{C}$ be such that $A \subseteq V_k \subseteq \overline{V_k} \subseteq U$. Then $A = \bigcap_{n\in N} f^n(\overline{V_k})$, therefore $A = A_{V_k}(f)$. By continuity of A_{V_k} at f, $A_{V_k}(g) \to A_{V_k}(f)$ in Hausdorff topology and so

$$A_g = \bigcap_{n\in N} g^n(\overline{V_k}) = A_{V_k}(g) \to A. \qquad \square$$

Now, for every $\epsilon > 0$ and open set U, take $N_\epsilon(f, U)$ the number of distinct equivalence class $\Gamma(x, \epsilon, f)$ that intersect U, where $x \in CR^\infty(f)$. Note that by Proposition 2.5, $N_\epsilon(f, U)$ is finite.

LEMMA 2.9. *For any $\epsilon > 0$ and open set $U \subseteq M$, the mapping $f \mapsto N_\epsilon(f, U)$ is lower semi-continuous on $\mathcal{R}^\infty$.*

Proof. Let $f \in \mathcal{R}^\infty$ and $N_\epsilon(f,U) = n$. Let $\Gamma(x_1,\epsilon,f),\ldots,\Gamma(x_n,\epsilon,f)$, $x_i \in CR^\infty(f)$, be the equivalence classes which intersect U. Since $\Gamma(x_i,\epsilon,f)$'s are closed and disjoint, there exists an $\delta > 0$, such that for each $i \neq j$, $U_\delta(\Gamma(x_i,\epsilon,f)) \cap U_\delta(\Gamma(x_j,\epsilon,f)) = \emptyset$, where $U_\delta(\Gamma(x_i,\epsilon,f))$ and $U_\delta(\Gamma(x_j,\epsilon,f))$ are δ-neighborhoods of $\Gamma(x_i,\epsilon,f)$ and $\Gamma(x_j,\epsilon,f)$, respectively. If $g \in \mathcal{R}^\infty$ is sufficiently close to f, then $CR^\infty(g)$ is close enough to $CR^\infty(f)$. Thus

$$CR^\infty(g,\epsilon) \subseteq U_\delta(\Gamma(x_1,\epsilon,f)),\ldots,U_\delta(\Gamma(x_n,\epsilon,f)).$$

These two facts imply that $N_\epsilon(g,U) \geq n$ and therefore, the mapping $f \mapsto N_\epsilon(g,U)$ is lower semi-continuous on $\mathcal{R}^\infty$. □

Note that $\mathcal{R}^\infty$ is a Baire space. Now, we recall the second topological lemma. For the proof see [5].

TOPOLOGICAL LEMMA II. *Let X be a Baire topological space and $\Gamma : X \to N$ be a lower semi-continuous map, then there exists a residual set $\mathcal{N}$ of X such that $\Gamma|_{\mathcal{N}}$ is locally constant on each point of $\mathcal{N}$.*

By Lemma 2.8 and Topological Lemma II, we obtain the following result immediately.

COROLLARY 2.10. *There is a residual subset $\mathcal{R}_U(\epsilon)$ of $\mathcal{R}^\infty$ such that the mapping $f \mapsto N_\epsilon(f,U)$ is locally constant on it.* □

Note that $\mathcal{R}_U(\epsilon)$ is residual in $\mathcal{H}(X)$. Now we are ready to prove the main result.

Proof of Theorem A. Let $\{U_n\}_{n\in N}$ be a countable basis for X and suppose that the family $\mathcal{C} = \{V_m\}_{m\in N}$ is the all of the finite union of it's element. By Proposition 2.7, associate to each V_m, there is a residual set $\mathcal{R}_{V_m}(\epsilon)$ of $\mathcal{H}(X)$, such that the mapping $N_\epsilon(-,V_m)$ is locally constant on $\mathcal{R}_{V_m}(\epsilon)$. Put $\mathcal{R}_2 = \bigcap_{m,n\in N} \mathcal{R}_{V_m}(\frac{1}{n})$. Then, $\mathcal{R}_2$ is residual set of $\mathcal{H}(X)$. Finally, we put

$$\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2,$$

where $\mathcal{R}_1$ is given by Proposition 2.7. Now, let $f \in \mathcal{R}$ and A be a totally chain-transitive attractor of f. Let g be sufficiently close to f, since $f \in \mathcal{R}_1$, g has an attractor A_g, which is close to A in Hausdorff topology.

Now, let $V \in \mathcal{C}$ be such that $A \subset V \subset \overline{V} \subset B(A)$, if $k \in N$ is large enough then the equivalence class of $CR^\infty(f,\frac{1}{k})$ which intersect A is contained in V, thus $N_{1/k}(f,V) = 1$. If g is sufficiently close to f, by continuity of $N_{1/k}(f, V)$ at f, we have $N_{1/k}(g, V) = 1$ and so for each $s \geq k$,

$$N_{1/s}(g, V) = 1.$$

This implies that A_g is totally chain-transitive. □

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