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INCLUSION RELATIONS FOR k-UNIFORMLY STARLIKE AND RELATED FUNCTIONS UNDER CERTAIN INTEGRAL OPERATORS

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Abstract. Inclusion relations under certain integral operators are proved for k-uniformly starlike functions. These results are also extended to k-uniformly convex, close-to-convex, and quasi-convex functions

1. Introduction

Let A denote the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in the open unit disc $U = \{z : |z| < 1\}$. A function $f \in A$ is said to be in $UST(k,\gamma)$, the class of k-uniformly starlike functions of order γ, $0 \leq \gamma < 1$, if f satisfies the condition

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right| + \gamma, \quad k \geq 0. \tag{1.1}$$

Replacing f in (1.1) by zf' we obtain the condition

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > k\left|\frac{zf''(z)}{f'(z)}\right| + \gamma, \quad k \geq 0 \tag{1.2}$$

required for function f to be in the subclass $UCV(k,\gamma)$ of k-uniformly convex functions of order γ. Uniformly starlike and convex functions were first introduced by Goodman[2] and then studied by various authors. For a wealth of reference, see Ronning[5]. Setting $\Omega_{k,\gamma} = \{u + iv; u > k\sqrt{(u-1)^2 + v^2} + \gamma\}$, with $p(z) = \frac{zf'(z)}{f(z)}$ or $p(z) = 1 + \frac{zf''(z)}{f'(z)}$ and considering the functions which maps U on to the conic domain

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$\Omega_{k,\gamma}$, such that $1 \in \Omega_{k,\gamma}$, we may rewrite the conditions (1.1) or (1.2) in the form

$$p(z) \prec q_{k,\gamma}(z). \tag{1.3}$$

Note that the explicit forms of function $q_{k,\gamma}$ for $k = 0$ and $k = 1$ are

$$q_{0,\gamma}(z) = \frac{1 + (1 - 2\gamma)z}{1 - z}, \quad \text{and} \quad q_{1,\gamma}(z) = 1 + \frac{2(1-\gamma)}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2.$$

For $0 < k < 1$ we obtain

$$q_{k,\gamma}(z) = \frac{1-\gamma}{1-k^2}\cos\left\{\frac{2}{\pi}(\cos^{-1} k)\, i \log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right\} - \frac{k^2-\gamma}{1-k^2},$$

and if $k > 1$, then $q_{k,\gamma}$ has the form

$$q_{k,\gamma}(z) = \frac{1-\gamma}{k^2-1}\sin\left(\frac{\pi}{2K(k)}\int_0^{\frac{u(z)}{\sqrt{k}}} \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}\right) + \frac{k^2-\gamma}{k^2-1},$$

where $u(z) = \frac{z-\sqrt{k}}{1-\sqrt{k}z}$ and K is so that $k = \cosh \frac{\pi K'(z)}{4K(z)}$.
By virtue of (1.3) and the properties of the domains $\Omega_{k,\gamma}$ we have

$$\Re(p(z)) > \Re(q_{k,\gamma}(z)) > \frac{k+\gamma}{k+1}. \tag{1.4}$$

Define $UCC(k, \gamma, \beta)$ to be the family of functions $f \in A$ so that

$$\Re\left(\frac{zf'(z)}{g(z)}\right) \geq k\left|\frac{zf'(z)}{g(z)} - 1\right| + \gamma, \quad k \geq 0, \ 0 \leq \gamma < 1$$

for some $g \in UST(k, \beta)$.
Similarly, we define $UQC(k, \gamma, \beta)$ to be the family of function $f \in A$ so that

$$\Re\left(\frac{(zf'(z))'}{g'(z)}\right) \geq k\left|\frac{(zf'(z))'}{g'(z)} - 1\right| + \gamma, \quad k \geq 0, \ 0 \leq \gamma < 1$$

for some $g \in UCV(k, \beta)$.

If $k = 0$ then $UCC(0,\gamma,\beta)$ is the class of close-to-convex functions of order γ and type β and $UQC(0,\gamma,\beta)$ is the class of quasi-convex functions of order γ and type β.

The aim of this note is to study the inclusion properties of the above mentioned classes of functions under the following one-parameter family of integral operator (see Jung, Kim, and Srivastava[3])

$$I^\alpha = I^\alpha f(z) = \frac{2^\alpha}{z\Gamma(\alpha)} \int_0^z (\log \frac{z}{t})^{\alpha-1} f(t)dt, \quad \alpha > 0,$$

and the generalized Bernardi-Libera-Livingston integral operator

$$L_c(f) = L_c(f(z)) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)dt, \ c > -1.$$

2. Main results

First we state and prove an inclusion theorem for $UST(k,\gamma)$ under I^α.

THEOREM 1. *If $I^\alpha \in UST(k,\gamma)$ then $I^{\alpha+1} \in UST(k,\gamma)$.*

In order to prove the above theorem we shall need the following lemma which is due to Eenigenburg, Miller, Mocanu, and Read[1].

LEMMA A. *Let β, γ be complex constants and h be univalently convex in the unit disk U with $h(0) = c$ and $\Re(\beta h(z) + \gamma) > 0$. Let $g(z) = c + \sum_{n=1}^{\infty} p_n z^n$ be analytic in U. Then*

$$g(z) + \frac{zg'(z)}{\beta g(z) + \gamma} \prec h(z) \Rightarrow g(z) \prec h(z).$$

Proof of Theorem 1. Since $I^\alpha \in UST(k,\gamma)$, by definition, we have

$$z(I^{\alpha+1}f(z))' = 2I^\alpha f(z) - I^{\alpha+1}f(z). \tag{2.1}$$

Setting $p(z) = z(I^{\alpha+1}f(z))'/(I^{\alpha+1}f(z))$ in (2.1) we can write

$$\frac{I^\alpha f(z)}{I^{\alpha+1}f(z)} = \frac{1}{2}\left(\frac{z(I^{\alpha+1}f(z))'}{I^{\alpha+1}f(z)} + 1\right) = \frac{1}{2}(p(z)+1). \tag{2.2}$$

Differentiating (2.2) yields

$$\frac{z(I^\alpha f(z))'}{I^\alpha f(z)} = \frac{z(I^{\alpha+1}f(z))'}{I^{\alpha+1}f(z)} + \frac{zp'(z)}{p(z)+1} = p(z) + \frac{zp'(z)}{p(z)+1}. \tag{2.3}$$

From this and the argument given in Section 1 we may write

$$p(z) + \frac{zp'(z)}{p(z)+1} \prec q_{k,\gamma}(z).$$

Therefore the theorem follows by Lemma A and the condition (1.4) since $q_{k,\gamma}$ is univalent and convex in U and $\Re(q_{k,\gamma}) > \frac{k+\gamma}{k+1}$.
Using a similar argument we can prove

THEOREM 2. *If* $I^\alpha \in UCV(k,\gamma)$ *then* $I^{\alpha+1} \in UCV(k,\gamma)$.

We next prove

THEOREM 3. *If* $I^\alpha \in UCC(k,\gamma,\beta)$ *then* $I^{\alpha+1} \in UCC(k,\gamma,\beta)$.

We shall need the following lemma which is due to Miller and Mocanu[4].

LEMMA B. *Let h be convex in the unit disk U and let $A \geq 0$. Suppose $B(z)$ is analytic in U with $\Re(B(z)) \geq A$. If g is analytic in U and $g(0) = h(0)$. Then*

$$Az^2g''(z) + B(z)zg'(z) + g(z) \prec h(z) \Rightarrow g(z) \prec h(z).$$

Proof of Theorem 3. Since $I^\alpha \in UCC(k,\gamma,\beta)$, by definition, we can write

$$\frac{z(I^\alpha f(z))'}{k(z)} \prec q_{k,\gamma}(z)$$

for some $k(z) \in UST(k,\beta)$. For g so that $I^\alpha g(z) = k(z)$, we have

$$\frac{z(I^\alpha f(z))'}{I^\alpha g(z)} \prec q_{k,\gamma}(z). \tag{2.4}$$

Letting $h(z) = \frac{z(I^{\alpha+1}f(z))'}{I^{\alpha+1}g(z)}$ and $H(z) = \frac{z(I^{\alpha+1}g(z))'}{I^{\alpha+1}g(z)}$ we observe that h and H are analytic in U and $h(0) = H(0) = 1$. Now, by Theorem 1, $I^{\alpha+1}g(z) \in UST(k,\beta)$ and so $\Re(H(z)) > \frac{k+\beta}{k+1}$. Also, note that

$$z(I^{\alpha+1}f(z))' = (I^{\alpha+1}g(z))h(z). \tag{2.5}$$

Differentiating both sides of (2.5) yields

$$\frac{z(I^{\alpha+1}(zf'(z)))'}{I^{\alpha+1}g(z)} = \frac{z(I^{\alpha+1}g(z))'}{I^{\alpha+1}g(z)}h(z) + zh'(z) = H(z)h(z) + zh'(z).$$

Now using the identity (2.1) we obtain

$$
\begin{aligned}
\frac{z(I^{\alpha}f(z))'}{I^{\alpha}g(z)} &= \frac{I^{\alpha}(zf'(z))}{I^{\alpha}g(z)} \\
&= \frac{z(I^{\alpha+1}(zf'(z)))' + I^{\alpha+1}(zf'(z))}{z(I^{\alpha+1}g(z))' + I^{\alpha+1}g(z)} \\
&= \frac{\frac{z(I^{\alpha+1}(zf'(z)))'}{I^{\alpha+1}g(z)} + \frac{I^{\alpha+1}(zf'(z))}{I^{\alpha+1}g(z)}}{\frac{z(I^{\alpha+1}g(z))'}{I^{\alpha+1}g(z)} + 1} \qquad (2.6) \\
&= \frac{H(z)h(z) + zh'(z) + h(z)}{H(z)+1} \\
&= h(z) + \frac{1}{H(z)+1}zh'(z).
\end{aligned}
$$

From (2.4), (2.5), and (2.6) we conclude that

$$h(z) + \frac{1}{H(z)+1}zh'(z) \prec q_{k,\gamma}(z).$$

For letting $A = 0$ and $B(z) = \frac{1}{H(z)+1}$, we obtain

$$\Re(B(z)) = \frac{1}{|1+H(z)|^2}\Re(1+H(z)) > 0.$$

The above inequality satisfies the conditions required by Lemma B. Hence $h(z) \prec q_{k,\gamma}(z)$ and so the proof is complete.

Using a similar argument we can prove

THEOREM 4. *If $I^{\alpha} \in UQC(k,\gamma,\beta)$ then $I^{\alpha+1} \in UQC(k,\gamma,\beta)$.*

Now we examine the closure properties of the integral operator L_c.

THEOREM 5. *Let $c > \frac{-(k+\gamma)}{k+1}$. If $I^{\alpha} \in UST(k,\gamma)$ so is $L_c(I^{\alpha})$.*

Proof. From definition of $L_c(f)$ and the linearity of operator I^{α} we have

$$z(I^{\alpha}L_c(f))' = (c+1)I^{\alpha}f(z) - cI^{\alpha}L_c(f). \qquad (2.7)$$

Substituting $\frac{z(I^{\alpha}L_c(f))'}{I^{\alpha}L_c(f)} = p(z)$ in (2.7) we may write

$$p(z) = (c+1)\frac{I^{\alpha}f(z)}{I^{\alpha}L_c(f)} - c. \qquad (2.8)$$

Differentiating (2.8) gives

$$\frac{z(I^\alpha f(z))'}{I^\alpha f(z)} = \frac{z(I^\alpha L_c(f))'}{I^\alpha L_c(f)} + \frac{zp'(z)}{p(z)+c} = p(z) + \frac{zp'(z)}{p(z)+c}.$$

Therefore, the theorem follows by Lemma A, since $\Re(q_{k,\gamma}(z)+c) > 0$. A similar argument leads to

THEOREM 6. *Let* $c > \frac{-(k+\gamma)}{k+1}$. *If* $I^\alpha \in UCV(k,\gamma)$ *so is* $L_c(I^\alpha)$.

THEOREM 7. *Let* $c > \frac{-(k+\gamma)}{k+1}$. *If* $I^\alpha \in UCC(k,\gamma,\beta)$ *so is* $L_c(I^\alpha)$.

Proof. By definition, there exists a function

$$k(z) = I^\alpha g(z) \in UST(k,\beta)$$

so that

$$\frac{z(I^\alpha f(z))'}{I^\alpha g(z)} \prec q_{k,\gamma}(z) \quad (z \in U). \tag{2.9}$$

Now from (2.7) we have

$$\begin{aligned} \frac{z(I^\alpha f)'}{I^\alpha g} &= \frac{z(I^\alpha L_c(zf'))' + cI^\alpha L_c(zf')}{z(I^\alpha L_c(g(z)))' + cI^\alpha L_c(g(z))} \\ &= \frac{\frac{z(I^\alpha L_c(zf'))'}{I^\alpha L_c(g)} + \frac{cI^\alpha L_c(zf')}{I^\alpha L_c(g)}}{\frac{z(I^\alpha L_c(g(z)))'}{I^\alpha L_c(g)} + c}. \end{aligned} \tag{2.10}$$

Since $I^\alpha g \in UST(k,\beta)$, by Theorem(5), we have $L_c(I^\alpha g) \in UST(k,\beta)$. Letting $\frac{z(I^\alpha L_c(g(z)))'}{I^\alpha L_c(g(z))} = H(z)$, we note that $\Re(H(z)) > \frac{k+\beta}{k+1}$. Now for $h(z) = \frac{z(I^\alpha L_c(f(z)))'}{I^\alpha L_c(g(z))}$ we obtain

$$z(I^\alpha L_c(f(z)))' = h(z) I^\alpha L_c(g(z)). \tag{2.11}$$

Differentiating both sides of (2.11) yields

$$\begin{aligned} \frac{z(I^\alpha (zL_c(f))')'}{I^\alpha L_c(g)} &= zh'(z) + h(z)\frac{z(I^\alpha L_c(g))'}{I^\alpha L_c(g)} \\ &= zh'(z) + H(z)h(z). \end{aligned} \tag{2.12}$$

Therefore from (2.10) and (2.12) we obtain

$$\frac{z(I^\alpha f(z))'}{I^\alpha g} = \frac{zh'(z) + h(z)H(z) + ch(z)}{H(z) + c}.$$

This in conjunction with (2.9) leads to

$$h(z) + \frac{zh'(z)}{H(z) + c} \prec q_{k,\gamma}(z). \tag{2.13}$$

Letting $B(z) = \frac{1}{H(z)+c}$ in (2.13) we note that $\Re(B(z)) > 0$ if $c > -\frac{k+\beta}{k+1}$. Now for $A = 0$ and B as described we conclude the proof since the required conditions of Lemma B are satisfied.

A similar argument yields

THEOREM 8. *Let* $c > \frac{-(k+\gamma)}{k+1}$. *If* $I^\alpha \in UQC(k, \gamma, \beta)$, so is $L_c(I^\alpha)$.

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