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A RECURRENCE RELATION FOR BERNOULLI NUMBERS

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Abstract. In this paper, using Gauss multiplication formula, a recurrence relation for Bernoulli numbers, generalizing Namias' results, is given.

1. Introduction

Stirling's formula, with many applications in probability theory and statistical physics, is given in the simpler form by

$$n! \simeq n^n e^{-n}, \tag{1.1}$$

is quite sufficient when n is large. From a mathematical point of view, the more accurate expression

$$n! = n^n e^{-n}\sqrt{2\pi n}\exp\left\{\frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} + \cdots\right\}, \tag{1.2}$$

is, as is well known, an example of divergent asymptotic series whose performance gets worse as the number of terms is increased beyond a certain value.

The Bernoulli numbers, denoted by B_m, are given by the formal power series expansion

$$\frac{x}{e^x - 1} = \sum_{m=0}^{\infty} B_m \frac{x^m}{m!}, \quad (|x| < 2\pi). \tag{1.3}$$

It is well known that $B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, \ldots$, and $B_{2k+1} = 0$ for $k > 0$. Some properties and relations for these numbers have been extensively studied by several mathematicians such as Howard[4], Deeba and Rodriguez[3] and Namias[5].

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In [5], Namias, using Gauss and Gauss-Legendre formulas for the gamma function

$$\Gamma(2n) = \frac{1}{\sqrt{\pi}}\, e^{2n-1}\Gamma(n)\Gamma\left(n+\frac{1}{2}\right),$$
$$\Gamma(3n) = \frac{1}{2\pi}\, 3^{3n-\frac{1}{2}}\Gamma(n)\Gamma\left(n+\frac{1}{3}\right)\Gamma\left(n+\frac{2}{3}\right),$$

gave the following recurrence relations for Bernoulli numbers

$$\text{(1.4)}\qquad \begin{aligned} B_s &= \frac{1}{2\left(1-2^s\right)}\sum_{m=0}^{s-1}\binom{s}{m}2^m B_m, \\ B_s &= \frac{1}{3\left(1-3^s\right)}\sum_{m=0}^{s-1}\binom{s}{m}2^m B_m\left(1+2^{s-m}\right), \end{aligned}$$

respectively. He conjectured that an infinite number of recurrence relations such as (1.4) for Bernoulli numbers can be obtained using Gauss multiplication formula. He also conjectured that in all cases the same Stirling's series is obtained even though the recurrence relation for the coefficients of the series is different.

In 1991, Deeba and Rodriguez[3] presented an elementary procedure, without use of Gauss multiplication formula, to obtain an infinite number of recurrence relations for the Bernoulli numbers and exhibited distinct recurrence relations for the coefficients of the Stirling series, as conjectured by Namias[5]. In particular, using the generating function of Bernoulli numbers, for any positive integer s and any positive integer $k>1$, Deeba and Rodriguez[3] obtained

$$\text{(1.5)}\qquad B_s = \frac{1}{k(1-k^s)}\sum_{m=0}^{s-1}\binom{s}{m}k^m B_m\sum_{j=1}^{k-1}j^{s-m},$$

which is a generalization of (1.4). In 1995, Howard[4] proved that (1.5) is an easy consequence of the multiplication formula for Bernoulli polynomials.

In this paper, we show that, same results can be obtained by using Gauss multiplication formula, what is exactly conjectured by Namias. We also obtain different relations for the coefficients of the Stirling series and in all cases the same Stirling series is obtained.

2. Main relation

For the first step, we note that the gamma function $\Gamma(z)$ satisfies the Gauss multiplication formula (see [2, p.250])

$$\Gamma(nz)\, n^{\frac{1}{2}-nz}(2\pi)^{\frac{1}{2}(n-1)} = \prod_{i=0}^{n-1} \Gamma\left(z+\frac{i}{n}\right). \tag{2.1}$$

Let $k,\ n \in \mathbb{Z}^+$ and $k > 1$. Putting k and n instead of n and z in (2.1) respectively, we get

$$\Gamma(kn)\, k^{\frac{1}{2}-kn}(2\pi)^{\frac{1}{2}(k-1)} = \prod_{i=0}^{k-1} \Gamma\left(n+\frac{i}{k}\right). \tag{2.2}$$

On the other hand, since

$$\begin{aligned}
\Gamma\left(n+\frac{1}{k}\right) &= \left(n-\frac{k-1}{k}\right)\Gamma\left(n-\frac{k-1}{k}\right)\\
&\vdots\\
\Gamma\left(n+\frac{k-1}{k}\right) &= \left(n-\frac{1}{k}\right)\Gamma\left(n-\frac{1}{k}\right),
\end{aligned}$$

(2.2) becomes

$$\frac{\Gamma(kn)(2\pi)^{\frac{k-1}{2}}}{k^{kn-\frac{1}{2}}} = \Gamma(n)\prod_{i=1}^{k-1}\left(n-\frac{i}{k}\right)\Gamma\left(n-\frac{i}{k}\right). \tag{2.3}$$

Since we have $\Gamma(n) = (n-1)!$, Stirling's formula (1.1) can be written as

$$\Gamma(n) = n^{n-1}e^{-n}\sqrt{2\pi n}\, F(n), \tag{2.4}$$

where $F(n)$ is a function to be determined. Substituting (2.4) in (2.3), after necessary operations, we get

$$\begin{aligned}
&\frac{F(kn)}{\prod_{i=0}^{k-1} F\left(n-\frac{i}{k}\right)}\\
&= n^{n-1}e^{-n}\sqrt{2\pi n}\left(n-\frac{1}{k}\right)^{n-\frac{1}{k}}\sqrt{2\pi\left(n-\frac{1}{k}\right)}\, e^{-\left(n-\frac{1}{k}\right)}\cdots
\end{aligned} \tag{2.5}$$

$$\times\left(n-\frac{k-1}{k}\right)^{n-\frac{k-1}{k}}\sqrt{2\pi\left(n-\frac{k-1}{k}\right)}\,e^{-\left(n-\frac{k-1}{k}\right)}$$
$$\times(2\pi)^{-\frac{k-1}{2}}k^{kn-\frac{1}{2}}(kn)^{-(kn-1)}e^{kn}(2\pi kn)^{-\frac{1}{2}}$$
$$=e^{\frac{k-1}{2}}\left(1-\frac{1}{kn}\right)^{n+\frac{1}{2}-\frac{1}{k}}\cdots\left(1-\frac{k-1}{kn}\right)^{n+\frac{1}{2}-\frac{k-1}{k}}$$
$$=\exp\left\{\frac{k-1}{2}+\sum_{j=1}^{k-1}\left(n+\frac{1}{2}-\frac{j}{k}\right)\ln\left(1-\frac{j}{kn}\right)\right\}.$$

Since

$$\ln(1-x)=-\sum_{s=1}^{\infty}\frac{x^s}{s},$$

we have, for $1\le j\le k-1$,

$$\left(n+\frac{1}{2}-\frac{j}{k}\right)\ln\left(1-\frac{j}{kn}\right)$$
$$=-\sum_{s=1}^{\infty}\frac{j^s}{sk^sn^{s-1}}-\sum_{s=1}^{\infty}\frac{j^s}{(kn)^s}\left(\frac{1}{2s}-\frac{j}{ks}\right)$$
$$=-\sum_{s=0}^{\infty}\frac{j^{s+1}}{(s+1)k^{s+1}n^s}+\sum_{s=1}^{\infty}\frac{j^{s+1}}{sk^{s+1}n^s}-\sum_{s=1}^{\infty}\frac{j^s}{2sk^sn^s}$$
$$=-\frac{j}{k}+\sum_{s=1}^{\infty}\frac{1}{n^s}\left\{\frac{j^{s+1}}{k^{s+1}s(s+1)}-\frac{j^s}{2sk^s}\right\}.$$

Inserting these equations in the right hand-side of the expression (2.5), we get

$$\text{(2.6)}\quad \exp\left\{\sum_{s=1}^{\infty}\frac{1}{n^s}\left(\frac{1^{s+1}+\cdots+(k-1)^{s+1}}{s(s+1)k^{s+1}}-\frac{1^s+\cdots+(k-1)^s}{2sk^s}\right)\right\},$$

after some manipulation. This result implies that $F(n)$ can be written as

$$\text{(2.7)}\qquad F(n)=\exp\left\{\sum_{s=1}^{\infty}\frac{a_s}{n^s}\right\},$$

where the a_s are constant coefficients to be determined. Therefore, (2.4) becomes the series

$$\text{(2.8)}\qquad \Gamma(n)=n^{n-1}e^{-n}\sqrt{2\pi n}\exp\left\{\sum_{s=1}^{\infty}\frac{a_s}{n^s}\right\}.$$

Using the left hand-side of the expression (2.5) and (2.7), we obtain

$$\frac{F(kn)}{\prod_{i=0}^{k-1} F\left(n-\frac{i}{k}\right)} = \exp\left\{\sum_{s=1}^{\infty} \frac{a_s}{n^s}\left[\frac{1}{k^s} - 1 - \sum_{j=1}^{k-1} \frac{1}{\left(1-\frac{j}{kn}\right)^s}\right]\right\}. \tag{2.9}$$

We now use the expansion

$$\frac{1}{(1-x)^m} = \sum_{t=0}^{\infty} \binom{t}{m-1} x^{t-m+1}, \quad (|x|<1).$$

Letting $x = \frac{1}{kn}, \ldots, \frac{k-1}{kn}$ and inserting in the right hand-side of the expression (2.9), we find

$$\exp\left\{\sum_{s=1}^{\infty} \frac{a_s}{n^s}\left(\frac{1}{k^s} - 1\right) - \sum_{m=1}^{\infty}\sum_{t=0}^{\infty} \binom{t}{m-1} \frac{a_m}{n^m (kn)^{t+1-m}} \sum_{j=1}^{k-1} j^{t+1-m}\right\}.$$

In the double summation, let $t+1=s$. Then we obtain

$$\exp\left\{\sum_{s=1}^{\infty} \frac{a_s}{n^s}\left(\frac{1}{k^s} - 1\right) - \sum_{m=1}^{\infty}\sum_{s=1}^{\infty} \binom{s-1}{m-1} \frac{a_m}{n^s k^{s-m}} \sum_{j=1}^{k-1} j^{s-m}\right\}.$$

For $m > s$ this expression becomes

$$\exp\left\{\sum_{s=1}^{\infty} \frac{1}{n^s}\left[\left(\frac{1}{k^s} - 1\right) a_s - \sum_{m=1}^{s} \binom{s-1}{m-1} \frac{a_m}{k^{s-m}} \sum_{j=1}^{k-1} j^{s-m}\right]\right\}. \tag{2.10}$$

Equating the coefficients of $\frac{1}{n^s}$ in (2.6) and (2.10), we obtain the coefficients of Stirling's series as

$$a_s = \frac{1}{1-k^{s+1}} \sum_{j=1}^{k-1} j^s \left\{\frac{j}{s(s+1)k} - \frac{1}{2s} + \sum_{m=1}^{s-1} \binom{s-1}{m-1} a_m \left(\frac{k}{j}\right)^m\right\}. \tag{2.11}$$

Notice that for each positive integer k, equation (2.11) gives a recurrence formula for the sequence a_s. Furthermore, each recurrence formula generates (1.2), as conjectured by Namias[5].

On the other hand, Stirling's formula can be expressed in terms of Bernoulli numbers as (see [1, p.246]),

$$\Gamma(n) = n^{n-1} e^{-n} \sqrt{2\pi n} \exp\left\{\sum_{s=2}^{\infty} \frac{B_s}{s(s-1)n^{s-1}}\right\}. \tag{2.12}$$

Comparing equations (2.8) and (2.12), we find

$$a_s = \frac{B_{s+1}}{s(s+1)}. \tag{2.13}$$

Using (2.11) and (2.13), we obtain

(2.14)

$$\begin{aligned} B_{s+1} &= \frac{1}{k(1-k^{s+1})} \sum_{j=1}^{k-1} j^s \left(j - \frac{1}{2}k(s+1) + \sum_{m=1}^{s-1} \frac{k^{m+1}(s+1)!\,B_{m+1}}{(s-m)!\,m!\,j^m} \right) \\ &= \frac{1}{k(1-k^{s+1})} \sum_{m=0}^{s} \binom{s+1}{m} k^m B_m \sum_{j=1}^{k-1} j^{s+1-m}. \end{aligned}$$

In (2.14), writing s instead of $s+1$, we get (1.5).

Relation (2.14) generalizes both the expressions in (1.4) using Gauss multiplication formula, which conjectured by Namias[5].

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