Bull. Korean Math. Soc. **42** (2005), No. 3, pp. 609–616

PURITY OF POLYNOMIAL MODULES AND INVERSE POLYNOMIAL MODULES

Sangwon Park and Eunha Cho

Abstract. In this paper we show that we can extend the purity of left R-modules to the case of polynomial modules, bipolynomial modules, and also inverse polynomial modules.

1. Introduction

An exact sequence of left R-modules $0 \to A' \xrightarrow{\lambda} A \to A'' \to 0$ is pure exact if, for every right R-module B, we have exactness of

$$0 \longrightarrow B \otimes A' \xrightarrow{1\otimes\lambda} B \otimes A \longrightarrow B \otimes A'' \longrightarrow 0.$$

We say that $\lambda A'$ is a pure submodule of A in this case ([7]). For example, a split exact sequence $0 \to A' \to A \to A'' \to 0$ is a pure exact sequence. Let M be an R-module, then the character module M^+ of M is defined by $\mathrm{Hom}(M,\ \mathbb{Q}/\mathbb{Z})$ and we denote $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M,\ \mathbb{Q}/\mathbb{Z})$. In this paper we show that if $M \subset N$ is pure as left R-modules, then $M[x] \subset N[x]$ is pure as left $R[x]$-modules. We also prove that if $M \subset N$ is pure as R-module, then $M[x, x^{-1}] \subset N[x, x^{-1}]$ is pure as left $R[x]$-module. We can extend this result to the inverse polynomial modules so that if $M \subset N$ is pure, then $M[x^{-1}] \subset N[x^{-1}]$ is also pure as left $R[x]$-modules. Inverse polynomial modules were studied in ([1], [2], [3], [4]) and recently in ([5], [6]).

Definition 1.1. [4] Let R be a ring and M be a left R-module, then $M[x^{-1}]$ is a left $R[x]$-module defined by

$$x(m_0 + m_1x^{-1} + \cdots + m_ix^{-i}) = m_1 + m_2x^{-1} + \cdots + m_ix^{-i+1}$$

Received July 2, 2004.

2000 Mathematics Subject Classification: Primary 16E30; Secondary 13C11, 16D80.

Key words and phrases: module, pure, pure submodule, inverse polynomial module.

This paper was supported by Dong-A University Research Fund in 2003.

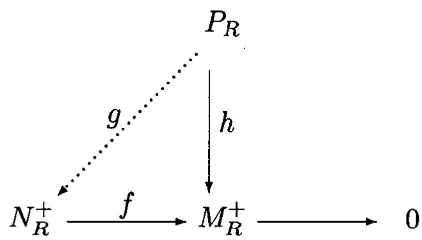
and such that

$$r(m_0 + m_1x^{-1} + \cdots + m_nx^{-n}) = rm_0 + rm_1x^{-1} + \cdots + rm_nx^{-n},$$

where $r \in R$. We call $M[x^{-1}]$ as an inverse polynomial module.

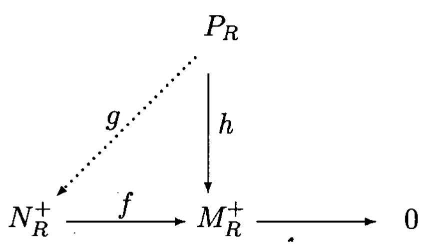
Similarly, we can define $M[[x^{-1}]], M[x, x^{-1}], M[[x, x^{-1}]], M[x, x^{-1}]]$ and $M[[x, x^{-1}]$ as left $R[x]$-modules where, for example, $M[[x, x^{-1}]$ is the set of Laurent series in x with coefficients in M, i.e. the set of all formal sums $\sum_{k \geq n_0} m_k x^k$ with n_0 any element of $\mathbb{Z}$ ($\mathbb{Z}$ is the set of all integers).

LEMMA 1.2. *$M \subset N$ is pure as left R-modules if and only if for any right R-module P the following diagram*



can be completed as an commutative diagram.

Proof. Since $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator as $\mathbb{Z}$-module, $0 \to M \to N$ if and only if $f : N^+ \to M^+$ is surjective. Since $M \subset N$ is pure as left R-modules, for any right R-module P, $0 \to P \otimes M \to P \otimes N$. So $(P \otimes N)^+ \to (P \otimes M)^+ \to 0$. Then by the adjoint theorem, $\phi : \mathrm{Hom}_R(P, N^+) \to \mathrm{Hom}_R(P, M^+)$ is surjective. So for any $h \in \mathrm{Hom}_R(P, M^+)$ there exist $g \in \mathrm{Hom}_R(P, N^+)$ such that $\phi(g) = f$. Therefore, we can complete the following diagram



as a commutative diagram. Conversely, let $M \subset N$ and P be any right R-module and suppose we can complete the above diagram as a following diagram. Then

$$\mathrm{Hom}(P,\ N^+) \to \mathrm{Hom}(P,\ M^+) \to 0.$$

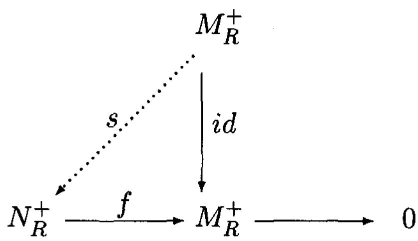
So by the adjoint theorem, $(P \otimes N)^+ \to (P \otimes M)^+ \to 0$ so that $0 \to (P \otimes M) \to (P \otimes N)$. Thus, $M \subset N$ is pure. Hence $M \subset N$ is pure as R-modules if and only if the diagram can be completed to a commutative diagram for any right R-module P. □

The following theorem is originally due to R. Warfield.

DEFINITION 1.3. Let M, N be left R-modules, then $f : N^+ \to M^+$ has a section means there exist $s : M^+ \to N^+$ such that $f \circ s = id_{M^+}$.

THEOREM 1.4. *$M \subset N$ is pure as left R-modules if and only if $f : N^+ \to M^+$ has a section.*

Proof. Suppose $M \subset N$ is pure and P be any right R-module and let $P = M^+$ with $id : M^+ \to M^+$, then by the above lemma we have the following diagram



as a commutative diagram. Thus $f \circ s = id_{M^+}$. Therefore, $f : N^+ \to M^+$ has a section. Conversely, let $M \subset N$ and P be any right R-module and suppose $N^+ \to M^+ \to 0$ has a section . Then

$$\mathrm{Hom}(P,\ N^+) \to \mathrm{Hom}(P,\ M^+) \to 0.$$

So $(P \otimes N)^+ \to (P \otimes M)^+ \to 0$ and then $0 \to P \otimes M \to P \otimes N$. Thus $M \subset N$ is pure. □

2. Purity of polynomial modules

Some part of the results of the following theorem are well-known, for the completeness of the paper we will give the proof.

THEOREM 2.1. *Let M, N be left R-modules. Then*

$$\mathrm{Hom}_R(M[x], N) \cong \mathrm{Hom}_R(M, N)[[x^{-1}]].$$

Proof. Let $\phi \in \mathrm{Hom}_R(M[x], N)$ and define $d_{Mx^n} : M \to Mx^n$ by $d_{Mx^n}(m) = mx^n$ and $\phi|_{Mx^n} : Mx^n \to N$. Let $f_n = \phi|_{Mx^n} \circ d_{Mx^n}$ for each $n = 0, 1, 2, 3, \cdots$. Define

$$\psi : \mathrm{Hom}_R(M[x], N) \to \mathrm{Hom}_R(M, N)[[x^{-1}]]$$

by $\psi(\phi) = f_0 + f_1x^{-1} + f_2x^{-2} + f_3x^{-3} + \cdots$. Then easily ψ is a well-defined group homomorphism. And $\ker(\psi) = 0$, so that ψ is injective. Let

$$f_0 + f_1x^{-1} + f_2x^{-2} + f_3x^{-3} + \cdots \in \mathrm{Hom}_R(M, N)[[x^{-1}]].$$

Choose $\phi \in \mathrm{Hom}_R(M[x], N)$ such that $\phi(m_0+m_1x+m_2x^2+\cdots+m_ix^i) = f_0(m_0) + f_1(m_1) + f_2(m_2) + \cdots + f_i(m_i)$. Then

$$\begin{aligned}\psi(\phi) &= (\phi|_{Mx^0} \circ d_{Mx^0}) + (\phi|_{Mx^1} \circ d_{Mx^1})x^{-1} + (\phi|_{Mx^2} \circ d_{Mx^2})x^{-2} \\ &\quad + (\phi|_{Mx^3} \circ d_{Mx^3})x^{-3} + \cdots\cdots\cdots\cdots \\ &= f_0 + f_1x^{-1} + f_2x^{-2} + f_3x^{-3} + \cdots .\end{aligned}$$

Therefore, ψ is surjective. □

We note that if N is a left R-module, then since $R[x]$ is R-$R[x]$ bimodule $\mathrm{Hom}_R(R[x], N)$ is a left $R[x]$-module, and $N[[x^{-1}]]$ is also a left $R[x]$-module. So we have the following Theorem 2.2.

THEOREM 2.2. *Let R be a ring and N be a left R-module. Then*

$$\mathrm{Hom}_R(R[x], N) \cong N[[x^{-1}]]$$

as $R[x]$-modules.

Proof. Define $\phi : \mathrm{Hom}_R(R[x], N) \to N[[x^{-1}]]$ by

$$\phi(f) = f(1) + f(x)x^{-1} + f(x^2)x^{-2} + \cdots\cdots .$$

Then ϕ is an isomorphism of left $R[x]$-modules. □

We now have one of our main theorems.

THEOREM 2.3. *If $M \subset N$ is pure as left R-modules, then $M[x] \subset N[x]$ is pure as left $R[x]$-modules.*

Proof. Since $M \subset N$ is pure as left R-modules, then by Theorem 1.3, $f : N^+ \to M^+$ has a section $g : M^+ \to N^+$ such that $f \circ g = id_{M^+}$. And by Theorem 2.1, $(M[x])^+ \cong M^+[[x^{-1}]]$. So let $f^* : N^+[[x^{-1}]] \to M^+[[x^{-1}]]$ be

$$f^*(\psi_0 + \psi_1x^{-1} + \psi_2x^{-2} + \cdots) = f(\psi_0) + f(\psi_1)x^{-1} + f(\psi_2)x^{-2} + \cdots .$$

Define $g^* : M^+[[x^{-1}]] \to N^+[[x^{-1}]]$ by

$$g^*(\phi_0 + \phi_1 x^{-1} + \phi_2 x^{-2} + \cdots) = g(\phi_0) + g(\phi_1)x^{-1} + g(\phi_2)x^{-2} + \cdots .$$

Then

$$\begin{aligned}(f^* \circ g^*)&(\phi_0 + \phi_1 x^{-1} + \phi_2 x^{-2} + \cdots)\\ &= f^*(g^*(\phi_0 + \phi_1 x^{-1} + \phi_2 x^{-2} + \cdots))\\ &= f^*(g(\phi_0) + g(\phi_1)x^{-1} + g(\phi_2)x^{-2} + \cdots)\\ &= f(g(\phi_0)) + f(g(\phi_1))x^{-1} + f(g(\phi_2))x^{-2} + \cdots\\ &= (f \circ g)(\phi_0) + (f \circ g)(\phi_1)x^{-1} + (f \circ g)(\phi_2)x^{-2} + \cdots\\ &= \phi_0 + \phi_1 x^{-1} + \phi_2 x^{-2} + \cdots .\end{aligned}$$

Therefore, f^* has a section g^* such that $f^* \circ g^* = id_{M^+[[x^{-1}]]}$. □

3. Purity of bipolynomial modules and inverse polynomial modules

THEOREM 3.1. *Let M, N be left R-modules. Then*

$$\mathrm{Hom}_R(M[x^{-1}, x], N) \cong \mathrm{Hom}_R(M, N)[[x^{-1}, x]].$$

Proof. Let $\phi \in \mathrm{Hom}_R(M[x^{-1}, x], N)$ and define $d_{Mx^n} : M \to Mx^n$ by $d_{Mx^n}(m) = mx^n$ and $\phi|_{Mx^n} : Mx^n \to N$, for $n \in \mathbb{Z}$ ($\mathbb{Z}$ is the set of integers). Let $f_n = \phi|_{Mx^n} \circ d_{Mx^n}$ for each $n \in \mathbb{Z}$. Define

$$\psi : \mathrm{Hom}_R(M[x^{-1}, x],\ N) \to \mathrm{Hom}_R(M,\ N)[[x^{-1}, x]]$$

by

$$\psi(\phi) = \sum_{n \in \mathbb{Z}} f_n x^{-n}.$$

Then easily ψ is an isomorphism. □

If N is a left R-module, then since $R[x^{-1}, x]$ is R-$R[x]$ bimodule, $\mathrm{Hom}_R(R[x^{-1}, x], N)$ is a left $R[x]$-module, and also $N[[x^{-1}, x]]$ is a left $R[x]$-module. So we have the following theorem.

THEOREM 3.2. *Let R be a ring and N be a left R-module. Then*

$$\mathrm{Hom}_R(R[x^{-1}, x], N) \cong N[[x^{-1}, x]]$$

as $R[x]$-modules.

Proof. Define $\phi : \mathrm{Hom}_R(R[x^{-1}, x], N) \to N[[x^{-1}, x]]$ by
$\phi(f) = \cdots + f(x^2)x^{-2} + f(x)x^{-1} + f(1) + f(x^{-1})x^1 + f(x^{-2})x^2 + \cdots$.
Then ϕ is an $R[x]$-module isomorphism. □

THEOREM 3.3. *If $M \subset N$ is pure as left R-modules, then $M[x, x^{-1}] \subset N[x, x^{-1}]$ is pure as left $R[x]$-modules.*

Proof. Since $M \subset N$ is pure as left R-modules, $f : N^+ \to M^+$ has a section $g : M^+ \to N^+$ such that $f \circ g = id_{M^+}$. And by Theorem 3.1,

$$(M[x, x^{-1}])^+ \cong M^+[[x, x^{-1}]].$$

So let $f^* : N^+[[x, x^{-1}]] \to M^+[[x, x^{-1}]]$ be

$$\begin{aligned} &f^*(\cdots + \psi_{-1}x^{-1} + \psi_0 + \psi_1 x^1 + \psi_2 x^2 + \cdots) \\ &\quad = \cdots + f(\psi_{-1})x^{-1} + f(\psi_0) + f(\psi_1)x^1 + f(\psi_2)x^2 + \cdots . \end{aligned}$$

Define $g^* : M^+[[x, x^{-1}]] \to N^+[[x, x^{-1}]]$ by

$$\begin{aligned} &g^*(\cdots + \phi_{-1}x^{-1} + \phi_0 + \phi_1 x^1 + \phi_2 x^2 + \cdots) \\ &\quad = \cdots + g(\phi_{-1})x^{-1} + g(\phi_0) + g(\phi_1)x^1 + g(\phi_2)x^2 + \cdots . \end{aligned}$$

Then $(f^* \circ g^*) = id_{M^+[[x,x^{-1}]]}$. Therefore, f^* has a section g^*. □

THEOREM 3.4. *Let M, N be left R-modules. Then*

$$\mathrm{Hom}_R(M[x^{-1}], N) \cong \mathrm{Hom}_R(M, N)[[x]].$$

Proof. Let $\phi \in \mathrm{Hom}_R(M[x^{-1}], N)$ and define $d_{Mx^{-n}} : M \to Mx^{-n}$ by $d_{Mx^{-n}}(m) = mx^{-n}$ and $\phi|_{Mx^{-n}} : Mx^{-n} \to N$. Let $f_n = \phi|_{Mx^{-n}} \circ d_{Mx^{-n}}$ for each $n = 0, 1, 2, 3, \cdots$. Define $\psi : \mathrm{Hom}_R(M[x^{-1}], N) \to \mathrm{Hom}_R(M, N)[[x]]$ by

$$\psi(\phi) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \cdots .$$

Thus ψ is an isomorphism. □

If N is a left R-module, then since $R[x]$ is R-$R[x]$ bimodule

$$\mathrm{Hom}_R(R[x^{-1}], N)$$

is a left $R[x]$-module, and $N[[x]]$ is also a left $R[x]$-module. So we have the following theorem.

THEOREM 3.5. *Let R be a ring and N be a left R-module. Then*

$$\mathrm{Hom}_R(R[x^{-1}], N) \cong N[[x]]$$

as $R[x]$-modules.

Proof. Define $\phi : \mathrm{Hom}_R(R[x^{-1}],\ N) \to N[[x]]$ by

$$\phi(f) = f(1) + f(x^{-1})x + f(x^{-2})x^2 + \cdots\cdots .$$

Then ϕ is an $R[x]$-module isomorphism. □

THEOREM 3.6. *If $M \subset N$ is pure as left R-modules, then $M[x^{-1}] \subset N[x^{-1}]$ is pure as left $R[x]$-modules.*

Proof. Since $M \subset N$ is pure as left R-modules, $f : N^+ \to M^+$ has a section $g : M^+ \to N^+$ such that $f \circ g = id_{M^+}$. And by the Theorem 3.4, $(M[x^{-1}])^+ \cong M^+[[x]]$. So let $f^* : N^+[[x]] \to M^+[[x]]$ be

$$f^*(\psi_0 + \psi_1 x + \psi_2 x^2 + \cdots) = f(\psi_0) + f(\psi_1)x + f(\psi_2)x^2 + \cdots .$$

Define $g^* : M^+[[x]] \to N^+[[x]]$ by

$$g^*(\phi_0 + \phi_1 x + \phi_2 x^2 + \cdots) = g(\phi_0) + g(\phi_1)x + g(\phi_2)x^2 + \cdots .$$

Then $(f^* \circ g^*) = id_{M^+[[x]]}$. Therefore, f^* has a section g^*. □

EXAMPLE 3.7. Let R be a commutative ring and $I = (r)$ be a principal ideal generalized by r. If $M \subset N$ is pure as R-modules, then

$$0 \to R/(r) \otimes_R M \to R/(r) \otimes_R N.$$

But since $R \otimes_R M \cong M$, we have $R/(r) \otimes_R M \cong M/rM$. Therefore, $M \subset N$ is pure implies that if r divides a in N, then r divides a in M. By the above result we see that $M[x] \subset M[x, x^{-1}]$ is not pure because x divides p in $M[x, x^{-1}]$ but x does not divide p in $M[x]$ as $R[x]$-modules.

Let S be a submonoid of $\mathbb{N}$ ($\mathbb{N}$ is the set of natural numbers) and consider S such that S contain all n in N larger than some n_0 in $\mathbb{N}$. Then the *conductor* of S is the largest element of $\mathbb{Z}$ not in S. If the conductor of S is c, then S is said to be *symmetric* if and only if S satisfies: n is in S if and only if $c - n$ is not in S. We can easily generalize the previous results so that we can get

1. $M \subset N$ is pure as left R-modules, then $M[x^s] \subset N[x^s]$ is pure as left $R[x^s]$-modules.
2. $M \subset N$ is pure as left R-modules, then $M[x^{-s}, x^s] \subset N[x^{-s}, x^s]$ is pure as left $R[x^s]$-modules.
3. $M \subset N$ is pure as left R-modules, then $M[x^{-s}] \subset N[x^{-s}]$ is pure as left $R[x^s]$-modules.

References

[1] A. S. McKerrow, *On the Injective Dimension of Modules of Power Series*, Quart J. Math. Oxford **25** (1974), no. 3, 359–368.
[2] D. G. Northcott, *Injective Envelopes and Inverse Polynomials*, London Math. Soc. **2** (1974), 290–296.
[3] S. Park, *Inverse Ploynomials and Injective Covers*, Comm. Algebra **21** (1993), 4599–4613.
[4] ———, *The Macaulay-Northcott Functor*, Arch. Math. (Basel) **63** (1994), 225–230.
[5] ———, *Gorenstein Rings and Inverse Polynomials*, Comm. Algebra **28** (2000), no. 2, 785–789.
[6] ———, *The General Structure of Inverse Polynomial Modules*, Czechoslovak Math. J. **51** (2001), no. 126, 343–349.
[7] J. Rotman, *An Introduction to Homological Algebra*, Academic Press Inc. New York, 1979.

SANGWON PARK AND EUNHA CHO, DEPARTMENT OF MATHEMATICS, DONG-A UNIVERSITY, PUSAN 604-714, KOREA
E-mail: swpark@donga.ac.kr
choeh@donga.ac.kr