

## SOME CRITERIA FOR STARLIKENESS AND STRONGLY STARLIKENESS

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ABSTRACT. In this paper we derive certain sufficient conditions for starlikeness and strongly starlikeness of analytic functions in the unit disk. Our results generalize and refine the related results due to Li and Owa[1, 2] and Ramesha et al.[5]. Some other new results are also given.

### 1. Introduction

Let  $A$  be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disk  $E = \{z : |z| < 1\}$ . A function  $f(z)$  in  $A$  is said to be starlike of order  $\alpha$  in  $E$  if it satisfies

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha \quad (z \in E)$$

for some  $\alpha$  ( $0 \leq \alpha < 1$ ). We denote by  $S^*(\alpha)$  ( $0 \leq \alpha < 1$ ) the subclass of  $A$  consisting of all starlike functions of order  $\alpha$  in  $E$ . A function  $f(z)$  in  $A$  is said to be strongly starlike of order  $\alpha$  in  $E$  if it satisfies

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\alpha\pi}{2} \quad (z \in E)$$

for some  $\alpha$  ( $0 < \alpha \leq 1$ ). We denote by  $\tilde{S}^*(\alpha)$  ( $0 < \alpha \leq 1$ ) the subclass of  $A$  consisting of all functions which are strongly starlike of order  $\alpha$  in  $E$ . Also we denote by  $S^*(0) = \tilde{S}^*(1) = S^*$ .

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Let  $f(z)$  and  $g(z)$  be analytic in  $E$ . Then the function  $f(z)$  is said to be subordinate to  $g(z)$ , written  $f(z) \prec g(z)$ , if there exists an analytic function  $w(z)$  with  $w(0) = 0$  and  $|w(z)| < 1$  ( $z \in E$ ) such that  $f(z) = g(w(z))$  for  $z \in E$ . If  $g(z)$  is univalent in  $E$ , then  $f(z) \prec g(z)$  is equivalent to  $f(0) = g(0)$  and  $f(E) \subset g(E)$ .

A function  $f(z)$  in  $A$  is said to be in the class  $S^*(a, b)$  if it satisfies

$$\frac{zf'(z)}{f(z)} \prec \frac{1+az}{1+bz}$$

for some  $a$  and  $b$  ( $-1 \leq b < a \leq 1$ ). The class  $S^*(a, b)$  ( $-1 \leq b < a \leq 1$ ) can be reduced to several well known classes of starlike functions by selecting special values for  $a$  and  $b$ . Note that  $S^*(1 - 2\alpha, -1) \equiv S^*(\alpha)$  ( $0 \leq \alpha < 1$ ).

Recently, Li and Owa proved the following two results which are the main theorems of [1].

**THEOREM A.** *If  $f(z) \in A$  satisfies  $f(z) \neq 0$  in  $0 < |z| < 1$  and*

$$\operatorname{Re} \left\{ \alpha \frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \right\} > -\frac{\alpha}{2} \quad (z \in E)$$

for some  $\alpha$  ( $\alpha > 0$ ), then  $f(z) \in S^*$ .

**THEOREM B.** *If  $f(z) \in A$  satisfies  $f(z) \neq 0$  in  $0 < |z| < 1$  and*

$$\operatorname{Re} \left\{ 2\alpha \frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \right\} > -\alpha^2(1 - 2\alpha) \quad (z \in E)$$

for some  $\alpha$  ( $0 < \alpha < 1$ ), then  $f(z) \in S^*(\alpha)$ .

In [2], Li and Owa have derived

**THEOREM C.** *If  $f(z) \in A$  satisfies  $f(z)f'(z) \neq 0$  in  $0 < |z| < 1$  and*

$$\left| \frac{z^2 f''(z)}{f(z)} - \frac{zf''(z)}{f'(z)} \right| < \frac{3}{2} \quad (z \in E),$$

then  $f(z) \in S^*$ .

For Theorems A and B in [1], the condition  $f(z) \neq 0$  ( $0 < |z| < 1$ ) was not assumed, but it is necessary to complete the proof, because  $p(z) = zf'(z)/f(z)$  ( $z \in E$ ) must be analytic in the proof. Also the condition  $f(z)f'(z) \neq 0$  ( $0 < |z| < 1$ ) is necessary to complete the proof of Theorem C in [2].

Ramesha et al.[5] have given

THEOREM D. If  $f(z) \in A$  satisfies  $f(z) \neq 0$  in  $0 < |z| < 1$  and

$$\operatorname{Re} \left\{ \frac{z^2 f''(z)}{f(z)} + \frac{z f'(z)}{f(z)} \right\} > 0 \quad (z \in E),$$

then  $f(z) \in \tilde{S}^*(\frac{1}{2})$ .

The object of this paper is to derive some sufficient conditions for starlikeness and strongly starlikeness of functions in  $A$ . In particular, we extend and refine Theorems A (with  $0 < \alpha \leq 1$ ), B, C, and D.

To derive our results, we need the following lemmas due to Miller and Mocanu.

LEMMA 1. [3] Let  $g(z)$  be analytic and univalent in  $E$  and let  $\theta(w)$  and  $\varphi(w)$  be analytic in a domain  $D$  containing  $g(E)$ , with  $\varphi(w) \neq 0$  when  $w \in g(E)$ . Set

$$Q(z) = z g'(z) \varphi(g(z)), \quad h(z) = \theta(g(z)) + Q(z)$$

and suppose that

- (i)  $Q(z)$  is univalent and starlike in  $E$ , and
- (ii)  $\operatorname{Re} \frac{z h'(z)}{Q(z)} = \operatorname{Re} \left\{ \frac{\theta'(g(z))}{\varphi(g(z))} + \frac{z Q'(z)}{Q(z)} \right\} > 0 \quad (z \in E)$ .

If  $p(z)$  is analytic in  $E$ , with  $p(0) = g(0)$ ,  $p(E) \subset D$  and

$$\theta(p(z)) + z p'(z) \varphi(p(z)) \prec \theta(g(z)) + z g'(z) \varphi(g(z)) = h(z),$$

then  $p(z) \prec g(z)$  and  $g(z)$  is the best dominant of the subordination.

LEMMA 2. [4] Let  $p(z)$  be analytic in  $E$  with  $p(0) = 1$  and  $p(z) \neq 1$ . If  $0 < |z_0| < 1$  and  $\operatorname{Re} p(z_0) = \min_{|z| \leq |z_0|} \operatorname{Re} p(z)$ , then

$$z_0 p'(z_0) \leq - \frac{|1 - p(z_0)|^2}{2(1 - \operatorname{Re} p(z_0))}.$$

## 2. Conditions for starlikeness

THEOREM 1. If  $f(z) \in A$  satisfies  $f(z) \neq 0$  in  $0 < |z| < 1$  and

$$(1) \quad \frac{z^2 f''(z)}{f(z)} + \lambda \frac{z f'(z)}{f(z)} \prec h(z),$$

where

$$(2) \quad h(z) = \frac{a(a - b + \lambda b)z^2 + (2(a - b) + \lambda(a + b))z + \lambda}{(1 + bz)^2},$$

$$(3) \quad -1 \leq b < a \leq 1 \quad \text{and} \quad \lambda \geq 2 \left( \frac{|b|}{1 + |b|} - \frac{1 - a}{1 - b} \right),$$

then  $f(z) \in S^*(a, b)$ .

*Proof.* Let us define the analytic function  $p(z)$  in  $E$  by

$$p(z) = \frac{zf'(z)}{f(z)}.$$

Then

$$\begin{aligned} (4) \quad \frac{z^2 f''(z)}{f(z)} + \lambda \frac{zf'(z)}{f(z)} &= \frac{zf'(z)}{f(z)} \left( \frac{zf''(z)}{f'(z)} + \lambda \right) \\ &= zp'(z) + (p(z))^2 + (\lambda - 1)p(z). \end{aligned}$$

From (1) and (4) we have

$$(5) \quad zp'(z) + (p(z))^2 + (\lambda - 1)p(z) \prec h(z).$$

Let  $a$ ,  $b$  and  $\lambda$  satisfy (3) and choose

$$(6) \quad g(z) = \frac{1+az}{1+bz}, \quad \theta(w) = w^2 + (\lambda - 1)w, \quad \varphi(w) = 1.$$

Then  $g(z)$  is analytic and univalent in  $E$ ,  $g(0) = p(0) = 1$ ,  $\theta(w)$  and  $\varphi(w)$  are analytic with  $\varphi(w) \neq 0$  in the  $w$ -plane. The function

$$(7) \quad Q(z) = zg'(z)\varphi(g(z)) = \frac{(a-b)z}{(1+bz)^2}$$

is univalent and starlike in  $E$  because

$$\operatorname{Re} \frac{zQ'(z)}{Q(z)} = \operatorname{Re} \frac{1-bz}{1+bz} > 0 \quad (z \in E).$$

Further, we have

$$\begin{aligned} (8) \quad \theta(g(z)) + Q(z) &= \left( \frac{1+az}{1+bz} \right)^2 + (\lambda - 1) \left( \frac{1+az}{1+bz} \right) + \frac{(a-b)z}{(1+bz)^2} \\ &= \frac{a(a-b+\lambda b)z^2 + (2(a-b) + \lambda(a+b))z + \lambda}{(1+bz)^2} \\ &= h(z) \end{aligned}$$

and

$$\begin{aligned} (9) \quad \operatorname{Re} \frac{zh'(z)}{Q(z)} &= 2 \operatorname{Re} \frac{1+az}{1+bz} + \lambda - 1 + \operatorname{Re} \frac{1-bz}{1+bz} \\ &> \frac{2(1-a)}{1-b} + \lambda - 1 + \frac{1-|b|}{1+|b|} \\ &\geq 0 \end{aligned}$$

for  $z \in E$ . The inequality (9) shows that the function  $h(z)$  is close-to-convex and univalent in  $E$ . Now it follows from (5)–(8) that

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(g(z)) + zg'(z)\varphi(g(z)) = h(z).$$

Therefore, by virtue of Lemma 1, we conclude that  $p(z) \prec g(z)$ , that is,  $f(z) \in S^*(a, b)$ .

Making use of Theorem 1, we can obtain a number of useful consequences.

**COROLLARY 1.** *If  $f(z) \in A$  satisfies  $f(z) \neq 0$  in  $0 < |z| < 1$  and*

$$(10) \quad \frac{z^2 f''(z)}{f(z)} + \lambda \frac{z f'(z)}{f(z)} \prec a^2 z^2 + a(\lambda + 2)z + \lambda,$$

where  $0 < a \leq 1$  and  $\lambda \geq -2(1 - a)$ , then

$$(11) \quad \left| \frac{z f'(z)}{f(z)} - 1 \right| < a \quad (z \in E)$$

and the bound  $a$  in (11) is sharp.

*Proof.* Letting  $b = 0$  in Theorem 1 and using (10), we have the inequality (11). If we take  $f(z) = ze^{az}$ , then

$$\frac{z^2 f''(z)}{f(z)} + \lambda \frac{z f'(z)}{f(z)} = a^2 z^2 + a(\lambda + 2)z + \lambda$$

and

$$\left| \frac{z f'(z)}{f(z)} - 1 \right| = a|z| \rightarrow a$$

as  $|z| \rightarrow 1$ . The proof of the corollary is complete.

**COROLLARY 2.** *If  $f(z) \in A$  satisfies  $f(z) \neq 0$  in  $0 < |z| < 1$  and*

$$(12) \quad \left| \frac{z^2 f''(z)}{f(z)} + \lambda \left( \frac{z f'(z)}{f(z)} - 1 \right) \right| < a(\lambda + 2 - a) \quad (z \in E),$$

where  $0 < a \leq 1$  and  $\lambda \geq -2(1 - a)$ , then

$$\left| \frac{z f'(z)}{f(z)} - 1 \right| < a \quad (z \in E).$$

*Proof.* Since the function  $a^2 z^2 + a(\lambda + 2)z$  is univalent in  $E$  and

$$|a^2 z^2 + a(\lambda + 2)z| \geq a(\lambda + 2 - a) > 0 \quad (|z| = 1),$$

the subordination (10) holds true by using (12). Hence the corollary follows immediately from Corollary 1. □

COROLLARY 3. If  $f(z) \in A$  satisfies  $f(z) \neq 0$  in  $0 < |z| < 1$  and

$$(13) \quad \frac{z^2 f''(z)}{f(z)} + \lambda \frac{z f'(z)}{f(z)} \prec h(z),$$

where

$$(14) \quad h(z) = \frac{(1 - 2\alpha)(2 - 2\alpha - \lambda)z^2 + 2(2 - 2\alpha - \alpha\lambda)z + \lambda}{(1 - z)^2},$$

$0 \leq \alpha < 1$  and  $\lambda \geq 1 - 2\alpha$ , then  $f(z) \in S^*(\alpha)$  and the order  $\alpha$  is sharp.

*Proof.* Setting  $a = 1 - 2\alpha$ ,  $0 \leq \alpha < 1$  and  $b = -1$  in Theorem 1, it follows from (13) and (14) that  $f(z) \in S^*(\alpha)$ .

To show that the order  $\alpha$  cannot be increased, we consider

$$f(z) = \frac{z}{(1 - z)^{2(1-\alpha)}}.$$

It is easy to verify that the function  $f(z)$  satisfies

$$\frac{z^2 f''(z)}{f(z)} + \lambda \frac{z f'(z)}{f(z)} = h(z)$$

and

$$\operatorname{Re} \frac{z f'(z)}{f(z)} \rightarrow \alpha \quad \text{as } z \rightarrow -1.$$

The proof of the corollary is complete.  $\square$

For the univalent function  $h(z)$  given by (14), we now find the image  $h(E)$  of the unit disk  $E$ .

For  $\lambda = 1 - 2\alpha$ , we can write

$$h(z) = 1 - 2\alpha + 2(1 - \alpha)(3 - 2\alpha) \frac{z}{(1 - z)^2}.$$

So  $w = h(z)$  maps  $E$  onto the  $w$ -plane slit along the negative real axis from  $w = -(1 - \alpha + 2\alpha^2)/2$ , that is,

$$(15) \quad h(E) = \left\{ w : \left| \arg \left( w + \frac{1 - \alpha + 2\alpha^2}{2} \right) \right| < \pi \right\}.$$

For  $\lambda > 1 - 2\alpha$ , let  $h(e^{i\theta}) = u + iv$ , where  $u$  and  $v$  are real. We have

$$u = -\frac{((1 - \alpha)(1 - 2\alpha) + \alpha\lambda) \cos \theta + 2(1 - \alpha) - \alpha\lambda}{1 - \cos \theta}$$

and

$$v = \frac{(1 - \alpha)(\lambda - 1 + 2\alpha) \sin \theta}{1 - \cos \theta}.$$

Elimination of  $\theta$  yields  $v^2 = -a_0(u - b_0)$ , where

$$a_0 = \frac{2(1 - \alpha)(\lambda - 1 + 2\alpha)^2}{3 - 2\alpha}, \quad b_0 = \frac{2\alpha^2 - \alpha(1 - 2\lambda) - 1}{2}.$$

Therefore, we conclude that

$$h(E) = \{w = u + iv : v^2 > -a_0(u - b_0)\},$$

which properly contains the half plane  $\operatorname{Re} w > b_0$ .

Setting  $\lambda = 1/(2\alpha)$  ( $0 < \alpha < 1$ ), Corollary 3 reduces to

**COROLLARY 4.** *If  $f(z) \in A$  satisfies  $f(z) \neq 0$  in  $0 < |z| < 1$  and*

$$2\alpha \frac{z^2 f''(z)}{f(z)} + \frac{z f'(z)}{f(z)} \prec h(z),$$

where

$$(16) \quad h(z) = \frac{(2\alpha - 1)^3 z^2 + 2\alpha(3 - 4\alpha)z + 1}{(1 - z)^2}$$

and  $0 < \alpha < 1$ , then  $f(z) \in S^*(\alpha)$  and the order  $\alpha$  is sharp.

**REMARK 1.** For the function  $h(z)$  given by (16), we have

$$h(E) = \left\{ w = u + iv : v^2 > -\frac{(1 - \alpha)(1 - 2\alpha + 4\alpha^2)^2}{\alpha(3 - 2\alpha)}(u - \alpha^2(2\alpha - 1)) \right\},$$

which properly contains the half plane  $\operatorname{Re} w > -\alpha^2(1 - 2\alpha)$ . Hence Corollary 4 refines Theorem B by Li and Owa.

Taking  $\alpha = 0$  in Corollary 3, we have

**COROLLARY 5.** *If  $f(z) \in A$  satisfies  $f(z) \neq 0$  in  $0 < |z| < 1$  and*

$$\frac{z^2 f''(z)}{f(z)} + \lambda \frac{z f'(z)}{f(z)} \prec h(z),$$

where  $\lambda \geq 1$  and

$$h(z) = \frac{(2 - \lambda)z^2 + 4z + \lambda}{(1 - z)^2},$$

the  $f(z) \in S^*$  and the order 0 is sharp.

**REMARK 2.** For  $\lambda = 1/\alpha$  ( $0 < \alpha \leq 1$ ), it is easy to find that Corollary 5 is better than Theorem A (with  $0 < \alpha \leq 1$ ) by Li and Owa.

Letting  $\lambda = 1 - 2\alpha$  in Corollary 3 and using (15), we obtain

COROLLARY 6. If  $f(z) \in A$  satisfies  $f(z) \neq 0$  in  $0 < |z| < 1$  and

$$\left| \arg \left\{ \frac{z^2 f''(z)}{f(z)} + (1 - 2\alpha) \frac{z f'(z)}{f(z)} + \frac{1 - \alpha + 2\alpha^2}{2} \right\} \right| < \pi \quad (z \in E)$$

for some  $\alpha$  ( $0 \leq \alpha < 1$ ), then  $f(z) \in S^*(\alpha)$  and the order  $\alpha$  is sharp.

For  $\lambda = 0$ , Corollary 3 yields

COROLLARY 7. If  $f(z) \in A$  satisfies  $f(z) \neq 0$  in  $0 < |z| < 1$  and

$$\frac{z^2 f''(z)}{f(z)} < 2(1 - \alpha) \frac{z(2 + (1 - 2\alpha)z)}{(1 - z)^2}$$

for some  $\alpha$  ( $1/2 \leq \alpha < 1$ ), then  $f(z) \in S^*(\alpha)$  and the order  $\alpha$  is sharp.

THEOREM 2. If  $f(z) \in A$  satisfies  $f(z)f'(z) \neq 0$  in  $0 < |z| < 1$  and

$$(17) \quad \left| \operatorname{Im} \left\{ \frac{z^2 f''(z)}{f(z)} - \lambda \frac{z f''(z)}{f'(z)} \right\} \right| < \sqrt{\lambda(2 + 3\lambda)} \quad (z \in E)$$

for some  $\lambda$  ( $\lambda > 0$ ), then  $f(z) \in S^*$ .

*Proof.* Let us define the analytic function  $p(z)$  in  $E$  by

$$p(z) = \frac{z f'(z)}{f(z)}.$$

Then  $p(0) = 1$ ,  $p(z) \neq 0$  and

$$(18) \quad \begin{aligned} \frac{z^2 f''(z)}{f(z)} - \lambda \frac{z f''(z)}{f'(z)} &= \frac{z f''(z)}{f'(z)} \left( \frac{z f'(z)}{f(z)} - \lambda \right) \\ &= (p(z) - \lambda) \left( \frac{z p'(z)}{p(z)} + p(z) - 1 \right) \quad (z \in E). \end{aligned}$$

Suppose that there exists a point  $z_0$  ( $0 < |z_0| < 1$ ) such that

$$(19) \quad \operatorname{Re} p(z) > 0 \quad (|z| < |z_0|), \quad p(z_0) = i\beta,$$

where  $\beta$  is real and  $\beta \neq 0$ . Then, applying Lemma 2, we get

$$(20) \quad z_0 p'(z_0) \leq -\frac{1 + \beta^2}{2}.$$

Thus it follows from (18), (19) and (20) that

$$(21) \quad I_0 = \operatorname{Im} \left\{ \frac{z_0^2 f''(z_0)}{f(z_0)} - \lambda \frac{z_0 f''(z_0)}{f'(z_0)} \right\} = -\beta(1 + \lambda) + \frac{\lambda}{\beta} z_0 p'(z_0).$$

In view of  $\lambda > 0$ , from (20) and (21) we obtain

$$(22) \quad I_0 \geq -\frac{\lambda + (2 + 3\lambda)\beta^2}{2\beta} \geq \sqrt{\lambda(2 + 3\lambda)} \quad (\beta < 0)$$



and

$$(23) \quad I_0 \leq -\frac{\lambda + (2 + 3\lambda)\beta^2}{2\beta} \leq -\sqrt{\lambda(2 + 3\lambda)} \quad (\beta > 0).$$

But both (22) and (23) contradict the assumption (17). Therefore, we must have  $\operatorname{Re} p(z) > 0$  for  $z \in E$ , that is,  $f(z) \in S^*$ .

Taking  $\lambda = 1$  in Theorem 2, we have the following result which is an improvement of Theorem C by Li and Owa[2].

COROLLARY 8. *If  $f(z) \in A$  satisfies  $f(z)f'(z) \neq 0$  in  $0 < |z| < 1$  and*

$$\left| \operatorname{Im} \left\{ \frac{z^2 f''(z)}{f(z)} - \frac{z f''(z)}{f'(z)} \right\} \right| < \sqrt{5} \quad (z \in E),$$

then  $f(z) \in S^*$ .

Note that Corollary 8 is comparable to the result of [1, Theorem 3].

### 3. Conditions for strongly starlikeness

THEOREM 3. *Let  $m$  be an integer,  $0 < \alpha \leq 1$ ,  $\alpha|m - 1| \leq 1$ ,  $m\lambda \geq 0$  and  $\operatorname{Re} \mu \geq 0$ . If  $f(z)$  in  $A$  satisfies  $f(z) \neq 0$  in  $0 < |z| < 1$ ,  $f'(z) \neq 0$  when  $m < 0$ , and*

$$(24) \quad \lambda \left( \frac{z f'(z)}{f(z)} \right)^m + \frac{z^2 f''(z)}{f(z)} + (\mu + 1) \frac{z f'(z)}{f(z)} - \left( \frac{z f'(z)}{f(z)} \right)^2 \prec h(z),$$

where

$$(25) \quad h(z) = \lambda \left( \frac{1+z}{1-z} \right)^{m\alpha} + \mu \left( \frac{1+z}{1-z} \right)^\alpha + \frac{2\alpha z}{(1+z)^{1-\alpha}(1-z)^{1+\alpha}},$$

then  $f(z) \in \tilde{S}^*(\alpha)$  and the order  $\alpha$  is sharp.

*Proof.* Let us define the function  $p(z)$  in  $E$  by

$$p(z) = \frac{z f'(z)}{f(z)}.$$

Then  $p(z)$  is analytic with  $p(z) \neq 0$  when  $m < 0$ . In view of

$$\begin{aligned} & \lambda \left( \frac{z f'(z)}{f(z)} \right)^m + \frac{z^2 f''(z)}{f(z)} + (\mu + 1) \frac{z f'(z)}{f(z)} - \left( \frac{z f'(z)}{f(z)} \right)^2 \\ &= \lambda (p(z))^m + \mu p(z) + z p'(z), \end{aligned}$$

from (24) we have

$$(26) \quad z p'(z) + \lambda (p(z))^m + \mu p(z) \prec h(z).$$

Set

$$(27) \quad g(z) = \left(\frac{1+z}{1-z}\right)^\alpha, \quad \theta(w) = \lambda w^m + \mu w, \quad \varphi(w) = 1$$

and

$$D = \begin{cases} C & (m \geq 0), \\ C \setminus \{0\} & (m < 0). \end{cases}$$

It is easy to verify that  $g(z)$ ,  $\theta(w)$  and  $\varphi(w)$  satisfy the conditions of Lemma 1. The function

$$(28) \quad Q(z) = zg'(z)\varphi(g(z)) = \frac{2\alpha z}{(1+z)^{1-\alpha}(1-z)^{1+\alpha}}$$

is univalent and starlike in  $E$  because

$$\operatorname{Re} \frac{zQ'(z)}{Q(z)} = 1 + (1-\alpha) \operatorname{Re} \left(-\frac{z}{1+z}\right) + (1+\alpha) \operatorname{Re} \frac{z}{1-z} > 0 \quad (z \in E)$$

for  $0 < \alpha \leq 1$ . Also

$$(29) \quad \begin{aligned} & \theta(g(z)) + Q(z) \\ &= \lambda \left(\frac{1+z}{1-z}\right)^{m\alpha} + \mu \left(\frac{1+z}{1-z}\right)^\alpha + \frac{2\alpha z}{(1+z)^{1-\alpha}(1-z)^{1+\alpha}} \\ &= h(z). \end{aligned}$$

Since  $0 \leq \alpha|m-1| \leq 1$ ,  $m\lambda \geq 0$  and  $\operatorname{Re} \mu \geq 0$ , we have

$$|\arg(g(z))^{m-1}| < \frac{\pi}{2} \quad (z \in E)$$

and so

$$\operatorname{Re} \frac{zh'(z)}{Q(z)} = \operatorname{Re} \left\{ m\lambda(g(z))^{m-1} + \mu + \frac{zQ'(z)}{Q(z)} \right\} > 0$$

for  $z \in E$ . Now  $p(0) = g(0) = 1$ ,  $p(E) \subset D$ , and it follows from (26)–(29) that

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(g(z)) + zg'(z)\varphi(g(z)) = h(z).$$

Consequently, an application of Lemma 1 yields  $p(z) \prec g(z)$ , which implies that  $f(z) \in \tilde{S}^*(\alpha)$ .

For the function

$$f(z) = z \exp \int_0^z \frac{1}{t} \left( \left(\frac{1+t}{1-t}\right)^\alpha - 1 \right) dt \in A,$$

it is easy to verify that

$$\lambda \left(\frac{zf'(z)}{f(z)}\right)^m + \frac{z^2 f''(z)}{f(z)} + (\mu + 1) \frac{zf'(z)}{f(z)} - \left(\frac{zf'(z)}{f(z)}\right)^2 = h(z)$$

and

$$\left| \arg \frac{zf'(z)}{f(z)} \right| = \alpha \left| \arg \frac{1+z}{1-z} \right| \rightarrow \frac{\alpha\pi}{2} \text{ as } z \rightarrow i.$$

This completes the proof of the theorem.

By using Theorem 3, we have several useful consequences.

Letting  $m = 2$ , Theorem 3 leads to .

**COROLLARY 9.** *If  $f(z) \in A$  satisfies  $f(z) \neq 0$  in  $0 < |z| < 1$  and*

$$(\lambda - 1) \left( \frac{zf'(z)}{f(z)} \right)^2 + \frac{z^2 f''(z)}{f(z)} + (\mu + 1) \frac{zf'(z)}{f(z)} \prec h(z),$$

where

$$h(z) = \lambda \left( \frac{1+z}{1-z} \right)^{2\alpha} + \mu \left( \frac{1+z}{1-z} \right)^\alpha + \frac{2\alpha z}{(1+z)^{1-\alpha}(1-z)^{1+\alpha}},$$

$0 < \alpha \leq 1$ ,  $\lambda \geq 0$  and  $\operatorname{Re} \mu \geq 0$ , then  $f(z) \in \tilde{S}^*(\alpha)$  and the order  $\alpha$  is sharp.

**REMARK 3.** For  $\lambda = 1$ ,  $\mu = 0$  and  $\alpha = 1/2$ , Corollary 9 refines Theorem D by Ramesha et al.[5].

Putting  $m = 1$  and  $\mu = 0$ , Theorem 3 yields

**COROLLARY 10.** *If  $f(z) \in A$  satisfies  $f(z) \neq 0$  in  $0 < |z| < 1$  and*

$$(\lambda + 1) \frac{zf'(z)}{f(z)} + \frac{z^2 f''(z)}{f(z)} - \left( \frac{zf'(z)}{f(z)} \right)^2 \prec h(z),$$

where

$$h(z) = \lambda \left( \frac{1+z}{1-z} \right)^\alpha + \frac{2\alpha z}{(1+z)^{1-\alpha}(1-z)^{1+\alpha}},$$

$0 < \alpha \leq 1$  and  $\lambda \geq 0$ , then  $f(z) \in \tilde{S}^*(\alpha)$  and the order  $\alpha$  is sharp.

For  $m = -1$ , Theorem 3 reduces to

**COROLLARY 11.** *If  $f(z) \in A$  satisfies  $f(z)f'(z) \neq 0$  in  $0 < |z| < 1$  and*

$$\lambda \frac{f(z)}{zf'(z)} + \frac{z^2 f''(z)}{f(z)} + (\mu + 1) \frac{zf'(z)}{f(z)} - \left( \frac{zf'(z)}{f(z)} \right)^2 \prec h(z),$$

where

$$h(z) = \lambda \left( \frac{1-z}{1+z} \right)^\alpha + \mu \left( \frac{1+z}{1-z} \right)^\alpha + \frac{2\alpha z}{(1+z)^{1-\alpha}(1-z)^{1+\alpha}},$$

$0 < \alpha \leq 1/2$ ,  $\lambda \leq 0$  and  $\operatorname{Re} \mu \geq 0$ , then  $f(z) \in \tilde{S}^*(\alpha)$  and the order  $\alpha$  is sharp.

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