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k-TH ROOTS OF p-HYPONORMAL OPERATORS

BHAGWATI P. DUGGAL, IN HO JEON(*), AND EUNGIL KO

ABSTRACT. In this paper we prove that if T is a k-th root of a p-hyponormal operator when T is compact or T^n is normal for some integer $n > k$, then T is (generalized) scalar, and that if T is a k-th root of a semi-hyponormal operator and have the property $\sigma(T)$ is contained in an angle $< \frac{2\pi}{k}$ with vertex in the origin, then T is subscalar.

1. Introduction

Let H and K be complex Hilbert spaces and let $\mathcal{L}(H, K)$ denote the space of all bounded linear operators from H to K. If $H = K$, we write $\mathcal{L}(H)$ in place of $\mathcal{L}(H, K)$.

A bounded linear operator S on H is called *scalar* of order m if it has a spectral distribution of order m, i.e., if there is a continuous unital morphism of topological algebras

$$\Phi : C_0^m(\mathbb{C}) \longrightarrow \mathcal{L}(H)$$

such that $\Phi(z) = S$, where as usual z stands for the identity function on $\mathbb{C}$ and $C_0^m(\mathbb{C})$ stands for the space of compactly supported functions on $\mathbb{C}$, continuously differentiable of order m, $0 \leq m \leq \infty$. An operator is *subscalar* if it is similar to the restriction of a scalar operator to a closed invariant subspace.

Let $d\mu(z)$, or simply $d\mu$, denote the planar Lebesgue measure. Let D be a bounded open disc in $\mathbb{C}$. We shall denote by $L^2(D, H)$ the Hilbert

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space of measurable functions $f : D \to H$, such that

$$\|f\|_{2,D} = \left(\int_D \|f(z)\|^2 \, d\mu(z) \right)^{\frac{1}{2}} < \infty.$$

The space of functions $f \in L^2(D, H)$ which are analytic functions in D (i.e., $\bar{\partial} f = 0$) is defined by

$$A^2(D, H) = L^2(D, H) \cap \mathcal{O}(D, H),$$

where $\mathcal{O}(D, H)$ denotes the Fréchet space of H-valued analytic functions on D with respect to uniform topology. $A^2(D, H)$ is called the Bergman space for D. Let us define a Sobolev type space, denoted $W^2(D, H)$. $W^2(D, H)$ will be the space of those functions $f \in L^2(D, H)$ whose derivatives $\bar{\partial} f, \bar{\partial}^2 f$ in the sense of distributions still belong to $L^2(D, H)$. Endowed with the norm $\|f\|^2_{W^2} = \sum_{i=0}^2 \|\bar{\partial}^i f\|^2_{2,D}$, $W^2(D, H)$ becomes a Hilbert space contained continuously in $L^2(D, H)$. Now, for $f \in C_0^2(\mathbb{C})$, let M_f denote the operator on $W^2(D, H)$ given by multiplication by f. This has a spectral distribution of order 2, defined by the functional calculus

$$\Phi_M : C_0^2(\mathbb{C}) \longrightarrow \mathcal{L}(W^2(D, H)), \ \Phi_M(f) = M_f.$$

Therefore M_f is a scalar operator of order 2. Consider a bounded open disk D which contains $\sigma(T)$ and the quotient space

$$H(D) = W^2(D, H) / \overline{(T - z) W^2(D, H)} \tag{1.1}$$

endowed with the Hilbert space norm. We denote the class of a vector f or an operator A on $H(D)$ by $\widehat{f}$, respectively $\widehat{A}$. Let M_z be the operator of multiplication by z on $W^2(D, H)$. As noted above, M_z is a scalar of order 2 and has a spectral distribution Φ. Let $S \equiv \widehat{M_z}$. Since $\overline{(T - z) W^2(D, H)}$ is invariant under every operator $M_f, f \in C_0^2(\mathbb{C})$, we infer that S is a scalar operator of order 2 with spectral distribution $\widehat{\Phi}$. Consider the natural map $V : H \longrightarrow H(D)$ defined by $Vh = \widehat{1 \otimes h}$, for $h \in H$, where $1 \otimes h$ denotes the constant function identically equal to h. In [11], Putinar showed that if $T \in \mathcal{L}(H)$ is a hyponormal operator then V is one-to-one and has closed range such that $VT = SV$, and so T is subscalar of order 2.

An operator $T \in \mathcal{L}(H)$ is said to be p-hyponormal, $0 < p \leq 1$, if $(T^*T)^p \geq (TT^*)^p$ where T^* is the adjoint of T. If $p = 1$, T is hyponormal and if $p = \frac{1}{2}$, T is called *semi-hyponormal*. Semi-hyponormal operators were introduced by Xia (see [12]) and there are many works on general p-hyponormal operators ([1], [3], [5], [6], [9]).

LÖWNER-HEINZ'S INEQUALITY. *Let $A, B \in \mathcal{L}(H)$ be $A \geq B \geq 0$ and $p \in (0,1]$. Then*

$$A^p \geq B^p.$$

This inequality gives the following implications:

$$\begin{aligned} \text{hyponormality} &\Rightarrow p\text{-hyponormality } (\frac{1}{2} < p < 1) \\ &\Rightarrow \text{semi-hyponormality} \\ &\Rightarrow p\text{-hyponormality } (0 < p < \frac{1}{2}). \end{aligned}$$

It is well known that all the above implications are strict(see [6] and [12]).

In this paper we prove that if T is a k-th root of a p-hyponormal operator when T is compact or T^n is normal for some integer $n > k$, then T is (generalized) scalar, and that if T is a k-th root of a semi-hyponormal operator and has the property $\sigma(T)$ is contained in an angle $< 2\pi/k$ with vertex in the origin, then T is subscalar. These results extend [8, Theorem 4.3].

2. Results

THEOREM 2.1. *Let T be a k-th root of a p-hyponormal operator. If T is compact or T^n is normal for some integer $n > k$, then T is a* (generalized) *scalar operator.*

Proof. First, we claim that T^k is normal. If T is compact, then that is straightforward, since T^k is compact and a compact p-hyponormal operator is normal ([5, Theorem 2]). If T^n is normal for some integer $n > k$, then there exists an n-nilpotent operator T_0 and an operator T_1 which is quasi-similar to a normal operator N with $\sigma(T_1) = \sigma(N)$ such that $T = T_0 \oplus T_1$ [7, Theorem 3.1]. Consider $T^k = T_0^k \oplus T_1^k$. Clearly, T_0^k is nilpotent. Since the only quasi-nilpotent p-hyponormal operator is the zero operator, $T_0^k = 0$. Let X be a quasi-affinity such that $T_1^k X = X N^k$. Applying the Putnam-Fuglede theorem for p-hyponormal operators ([3, Theorem 7]), it follows that T_1^k is normal. Hence T^k is normal. Now it follows from [2] and [7, Remark, p.141] that T is a (generalized) scalar operator. □

COROLLARY 2.2. *Let T be a k-th root of a p-hyponormal operator. If T is compact or T^n is normal for some integer $n > k$, then T has hyperinvariant subspaces.*

Proof. Since T is a (generalized) scalar operator by Theorem 2.1, T is decomposable. Hence T has hyperinvariant subspaces. □

THEOREM 2.3. *Let T be a k-th root of a semi-hyponormal operator and have the property $\sigma(T)$ is contained in an angle $< 2\pi/k$ with vertex in the origin. Then T is subscalar of order 2.*

We need the following lemmas to prove Theorem 2.3.

LEMMA 2.4. ([11, Proposition 2.1]) *For every bounded disk D in $\mathbb{C}$ there is a constant C_D, such that for an arbitrary operator $T \in \mathcal{L}(H)$ and $f \in W^2(D, H)$ we have*

$$\|(I-P)f\|_{2,D} \leq C_D \left(\|(T-z)^* \bar{\partial} f\|_{2,D} + |(T-z)^* \bar{\partial}^2 f\|_{2,D} \right),$$

where P denotes the orthogonal projection of $L^2(D, H)$ onto the Bergman space $A^2(D, H)$.

LEMMA 2.5. ([9, Lemma 4]) *Let T be a semi-hyponormal. Then for a $z \in \mathbb{C}$ and a sequence $f_n \in L^2(D, H)$*

$$\lim_{n\to\infty} \|(T-z)f_n\|_{2,D} = 0 \implies \lim_{n\to\infty} \|(T-z)^* f_n\|_{2,D} = 0.$$

Proof of Theorem 2.3. Consider a bounded disk D which contains $\sigma(T)$ and $H(D)$ as in (1.1). Then we define the map $V : H \to H(D)$ by

$$Vh = \widehat{1 \otimes h} \left(\equiv 1 \otimes h + \overline{(T-z)W^2(D, H)} \right),$$

where $1 \otimes h$ denotes the constant function sending any $z \in D$ to h. As mentioned in section 1, to prove Theorem 2.3 it suffices to show that V is one-to-one and has closed range.

Let $h_n \in H$ and $f_n \in W^2(D, H)$ be sequences such that

$$\lim_{n\to\infty} \|(T-z)f_n + 1 \otimes h_n\|_{W^2} = 0. \tag{2.1}$$

Then equation (2.1) implies

$$\lim_{n\to\infty} \|(T-z)\bar{\partial}^i f_n\|_{2,D} = 0 \quad \text{for} \quad i = 1, 2. \tag{2.2}$$

From (2.2), we get

$$\lim_{n\to\infty} \|(T^k - z^k)\bar{\partial}^i f_n\|_{2,D} = 0 \quad \text{for} \quad i = 1, 2.$$

Since T^k is semi-hyponormal, by Lemma 2.5 we have

$$\lim_{n\to\infty} \|(T^{*k} - \bar{z}^k)\bar{\partial}^i f_n\|_{2,D} = 0. \tag{2.3}$$

Now we claim that

$$\lim_{n\to\infty} \|(T - z)^* \bar{\partial}^i f_n\|_{2,D} = 0. \tag{2.4}$$

Indeed, since $T - z$ is invertible for $z \in D \setminus \sigma(T)$, the equation (2.2) implies that

$$\lim_{n\to\infty} \|\bar{\partial}^i f_n\|_{2,D\setminus\sigma(T)} = 0.$$

Therefore,

$$\lim_{n\to\infty} \|(T - z)^* \bar{\partial}^i f_n\|_{2,D\setminus\sigma(T)} = 0.$$

Also, since $\sigma(T)$ is contained in an angle $< \frac{2\pi}{k}$ with vertex in the origin, it is clear from the equation (2.3) that

$$\lim_{n\to\infty} \|(T - z)^* \bar{\partial}^i f_n\|_{2,D} = 0.$$

Thus Lemma 2.4 and equation (2.4) imply

$$\lim_{n\to\infty} \|(I - P) f_n\|_{2,D} = 0,$$

where P denotes the orthogonal projection of $L^2(D, H)$ onto $A^2(D, H)$. Then by (2.1)

$$\lim_{n\to\infty} \|(T - z) P f_n + 1 \otimes h_n\|_{2,D} = 0.$$

Let Γ be a curve in D surrounding $\sigma(T)$. Then for $z \in \Gamma$

$$\lim_{n\to\infty} \|P f_n(z) + (T - z)^{-1}(1 \otimes h_n)\| = 0, \text{ uniformly}.$$

Hence

$$\lim_{n\to\infty} \left\| \frac{1}{2\pi i} \int_\Gamma P f_n(z) dz + h_n \right\| = 0.$$

But by Cauchy's theorem,

$$\int_\Gamma P f_n(z) dz = 0.$$

Thus $\lim_{n\to\infty} h_n = 0$. Hence V is one-to-one and has closed range. This completes the proof. □

COROLLARY 2.6. *Let T be a k-th root of a semi-hyponormal operator and have the property that $\sigma(T)$ is contained in an angle $< 2\pi/k$ with vertex in the origin. If $\sigma(T)$ has interior in the plane, then T has a non-trivial invariant subspace.*

Proof. The corollary follows from Theorem 2.3 and [4]. □

We say that an operator $T - z$ on the space $\mathcal{O}(D, H)$ has *Bishop's property* (β) if $T - z$ is one-to-one and has closed range for every disc D. Since every subscalar operator has Bishop's property (β) ([10]), from Theorem 2.3 we have the following.

COROLLARY 2.7. *Let T be as in Corollary 2.6. Then T has Bishop's property (β).*

Does Theorem 2.3 hold for k-th roots of arbitrary p-hyponormal operators? A partial answer is given by the following corollary.

COROLLARY 2.8. *Let T be the k-th root of a p-hyponormal operator A, $0 < p < \frac{1}{2}$, such that $0 \notin \sigma(|A|^{\frac{1}{2}})$. If $\sigma(T)$ is contained in angle $< 2\pi/k$ with vertex in the origin, T is subscalar of order 2.*

Proof. Letting A have the polar decomposition $A = U|A|$, it is seen that the operator $S = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$ is a semi-hyponormal operator such that $S = |A|^{\frac{1}{2}}A|A|^{-\frac{1}{2}}$. Since $S = |A|^{\frac{1}{2}}T^k|A|^{-\frac{1}{2}} = (|A|^{\frac{1}{2}}T|A|^{-\frac{1}{2}})^k$, S has a k-th root $T_\circ = |A|^{\frac{1}{2}}T|A|^{-\frac{1}{2}}$ with spectrum contained in an angle $< 2\pi/k$ with vertex in the origin. Hence $T_\circ$, and so also T, is subscalar of order 2 by Theorem 2.3. □

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Bhagwati P. Duggal, 8 Redwood Grove, Northfields Avenue Ealing, London W5 4SZ, United Kingdom
E-mail: bpduggal@yahoo.co.uk

In Ho Jeon, Department of Mathematics, Ewha Women's University, Seoul 120-750, Korea
Recent Address: Department of Mathematics, Seoul National University, Seoul 151-747, Korea
E-mail: jih@math.ewha.ac.kr

Eungil Ko, Department of Mathematics, Ewha Women's University, Seoul 120-750, Korea
E-mail: eiko@ewha.ac.kr