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ON THE ANALYTIC PART OF HARMONIC UNIVALENT FUNCTIONS

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Abstract. In [2], Jahangiri studied the harmonic starlike functions of order α, and he defined the class $\mathcal{T}_{\mathcal{H}}(\alpha)$ consisting of functions $f = h + \bar{g}$ where h and g are the analytic and the co-analytic part of the function f, respectively. In this paper, we introduce the class $\mathcal{T}_{\mathcal{H}}(\alpha, \beta)$ of analytic functions and prove various coefficient inequalities, growth and distortion theorems, radius of convexity for the function h, if the function f belongs to the classes $\mathcal{T}_{\mathcal{H}}(\alpha)$ and $\mathcal{T}_{\mathcal{H}}(\alpha, \beta)$.

1. Introduction

A continuous complex valued function $f = u + iv$ defined in a simply connected complex domain $\mathcal{D}$ is said to be harmonic in $\mathcal{D}$ if both u and v are real harmonic in $\mathcal{D}$. In any simply connected domain we can write $f = h + \bar{g}$, where h and g are analytic in $\mathcal{D}$. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient for f to be locally univalent and sense preserving in $\mathcal{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathcal{D}$.

Let $\mathcal{H}$ denote the family of functions $f = h + \bar{g}$ that are harmonic univalent and sense preserving in the unit disk $\mathcal{U} = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \bar{g} \in \mathcal{H}$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1. \tag{1.1}$$

The harmonic function $f = h + \bar{g}$ for $g \equiv 0$ reduces to an analytic function $f = h$.

In 1984, Clunie and Sheil-Small[1] investigated the class $\mathcal{H}$ as well as its geometric subclasses and obtained some coefficient bounds. Since

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then, there has been several papers related on $\mathcal{H}$ and its subclasses. Jahangiri[2], Silverman[3], Silverman and Silvia[4] studied the harmonic starlike functions. Jahangiri[4] defined the class $\mathcal{T}_{\mathcal{H}}(\alpha)$ consisting of functions $f = h + \bar{g}$ such that h and g are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad g(z) = \sum_{n=1}^{\infty} |b_n| z^n \tag{1.2}$$

which satisfy the condition

$$\frac{\partial}{\partial \theta}\left(\arg f\left(re^{i\theta}\right)\right) \geq \alpha, \quad 0 \leq \alpha < 1, \quad |z| = r < 1. \tag{1.3}$$

Also Jahangiri[2] proved that if $f = h + \bar{g}$ is given by (1.1) and if

$$\sum_{n=1}^{\infty}\left(\frac{n-\alpha}{1-\alpha}|a_n| + \frac{n+\alpha}{1-\alpha}|b_n|\right) \leq 2, \quad 0 \leq \alpha < 1, \quad a_1 = 1, \tag{1.4}$$

then f is harmonic, univalent, and starlike of order α in $\mathcal{U}$. This condition is proved to be also necessary if $f \in \mathcal{T}_{\mathcal{H}}(\alpha)$. The case when $\alpha = 0$ is given in [4] and for $\alpha = b_1 = 0$, see [3].

A function $f = h + \bar{g} \in \mathcal{T}_{\mathcal{H}}(\alpha)$ is said to be in the class $\mathcal{T}_{\mathcal{H}}(\alpha, \beta)$ if the analytic functions h and g satisfies the condition

$$\operatorname{Re}\left\{\alpha z h''(z) + \frac{g(z)}{z}\right\} > 1 - |\beta| \qquad (\beta \in \mathbb{C},\ \alpha \geq 0,\ z \in \mathcal{U}). \tag{1.5}$$

In the present paper and for $f = h + \bar{g} \in \mathcal{T}_{\mathcal{H}}(\alpha, \beta)$, we prove various coefficient inequalities, growth and distortion theorems, radii of close-to-convexity, starlikeness and convexity for the function h, the analytic part of f .

2. Coefficient inequalities

THEOREM 1. *Let the function $f = h + \bar{g}$ be so that h and g are given by* (1.2). *If $f \in \mathcal{T}_{\mathcal{H}}(\alpha, \beta)$, then*

$$\sum_{n=2}^{\infty}\left[\alpha n(n-1)|a_n| - \frac{1-3\alpha}{n+\alpha}\right] \leq |\beta|, \tag{2.1}$$

where $a_1 = b_1 = 1, 0 \leq \alpha \leq 1/3$ and $\beta \in \mathbb{C}$. The result (2.1) *is sharp.*

Proof. Let $f(z) \in \mathcal{T}_{\mathcal{H}}(\alpha, \beta)$. From (1.5) we have

$$\operatorname{Re}\left\{-\sum_{n=2}^{\infty} \alpha n(n-1)|a_n| z^{n-1} + 1 + \sum_{n=2}^{\infty} |b_n| z^{n-1}\right\} > 1 - |\beta|.$$

Choose z to be real and let $z \to 1^-$, we get

$$1 - \left[\sum_{n=2}^{\infty} \alpha n(n-1)|a_n| - \sum_{n=2}^{\infty} |b_n|\right] \geq 1 - |\beta|$$

or, equivalently

$$\sum_{n=2}^{\infty} \{\alpha n(n-1)|a_n| - |b_n|\} \leq |\beta| . \tag{2.2}$$

Since $f(z) \in \mathcal{T}_{\mathcal{H}}(\alpha)$, from (1.4) we obtain

$$\sum_{n=1}^{\infty} \left(\frac{n+\alpha}{1-\alpha}|b_n|\right) \leq \sum_{n=1}^{\infty} \left(\frac{n-\alpha}{1-\alpha}|a_n| + \frac{n+\alpha}{1-\alpha}|b_n|\right) \leq 2$$

or

$$\sum_{n=2}^{\infty}(n+\alpha)|b_n| \leq 1-3\alpha, \tag{2.3}$$

that is,

$$|b_n| \leq \frac{1-3\alpha}{n+\alpha} \qquad (n \geq 2). \tag{2.4}$$

A substitution of (2.4) into (2.2) yields the inequality (2.1). □

COROLLARY 1. *Let the function $f = h + \bar{g}$ be so that h and g are given by (1.2). If $f \in \mathcal{T}_{\mathcal{H}}(\alpha, \beta)$, then*

$$|a_n| \leq \frac{(n+\alpha)|\beta| + 1 - 3\alpha}{\alpha n(n+\alpha)(n-1)} \quad (0 \leq \alpha \leq 1/3,\ \beta \in \mathbb{C},\ n \geq 2\ ). \tag{2.5}$$

The result (2.5) is sharp for the functions

$$h(z) = z - \frac{(n+\alpha)|\beta| + 1 - 3\alpha}{\alpha n(n+\alpha)(n-1)} z^n \quad (n \geq 2), \tag{2.6}$$

and

$$g(z) = z + \frac{1-3\alpha}{n+\alpha} z^n \quad (n \geq 2). \tag{2.7}$$

THEOREM 2. *Let the function $f = h+\bar{g}$ be so that h and g are given by (1.2). If $|\beta_1| \leq |\beta_2|$. Then $\mathcal{T}_{\mathcal{H}}(\alpha, \beta_1) \subset \mathcal{T}_{\mathcal{H}}(\alpha, \beta_2)$, where $0 \leq \alpha \leq 1/3$.*

Proof. Let $f(z) \in \mathcal{T}_{\mathcal{H}}(\alpha, \beta_1)$. Then

$$\sum_{n=2}^{\infty} \left[\alpha n(n-1)|a_n| - \frac{1-3\alpha}{n+\alpha}\right] \leq |\beta| \leq |\beta_2| ,$$

which completes the proof of Theorem 2. □

3. Growth and distortion theorems

THEOREM 3. *Let the function $f = h + \bar{g}$ be so that h and g are given by* (1.2). *If $f \in \mathcal{T}_{\mathcal{H}}(\alpha, \beta)$, then for $|z| = r < 1$, we have*

$$r - \frac{|\beta|(2+\alpha) + 1 - 3\alpha}{4\alpha + 2\alpha^2} r^2 \le |h(z)| \le r + \frac{|\beta|(2+\alpha) + 1 - 3\alpha}{4\alpha + 2\alpha^2} r^2 \tag{3.1}$$

and

$$1 - \frac{|\beta|(2+\alpha) + 1 - 3\alpha}{2\alpha + \alpha^2} r \le |h'(z)| \le 1 + \frac{|\beta|(2+\alpha) + 1 - 3\alpha}{2\alpha + \alpha^2} r \tag{3.2}$$

The results (3.1) *and* (3.2) *are sharp.*

Proof. Let $f(z) \in \mathcal{T}_{\mathcal{H}}(\alpha, \beta)$, then from (2.2) we have

$$2\alpha \sum_{n=2}^{\infty} |a_n| - \sum_{n=2}^{\infty} |b_n| \le |\beta| \quad \text{for } |z| = r < 1. \tag{3.3}$$

Since $f(z) \in \mathcal{T}_{\mathcal{H}}(\alpha)$, from (2.3) we obtain

$$\sum_{n=2}^{\infty} |b_n| \le \frac{1 - 3\alpha}{2 + \alpha} \tag{3.4}$$

so that (3.3) reduces to

$$\sum_{n=2}^{\infty} |a_n| < \frac{|\beta|(2+\alpha) + 1 - 3\alpha}{4\alpha + 2\alpha^2}. \tag{3.5}$$

Consequently,

$$\begin{aligned} |h(z)| &\ge r - \sum_{n=2}^{\infty} |a_n| |r|^n \ge r - r^2 \sum_{n=2}^{\infty} |a_n| \\ &\ge r - \frac{|\beta|(2+\alpha) + 1 - 3\alpha}{4\alpha + 2\alpha^2} r^2 \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} |h(z)| &\le r + \sum_{n=2}^{\infty} |a_n| |r|^n \le r + r^2 \sum_{n=2}^{\infty} |a_n| \\ &\le r + \frac{|\beta|(2+\alpha) + 1 - 3\alpha}{4\alpha + 2\alpha^2} r^2. \end{aligned} \tag{3.7}$$

Furthermore, we note from (2.2) that

$$\alpha \sum_{n=2}^{\infty} n\,|a_n| - \sum_{n=2}^{\infty} |b_n| \le |\beta|. \tag{3.8}$$

A substitution of (3.4) into (3.8) yields

$$\sum_{n=2}^{\infty} n\,|a_n| \le \frac{|\beta|\,(2+\alpha)+1-3\alpha}{2\alpha+\alpha^2}. \tag{3.9}$$

Thus we have

$$\left|h'(z)\right| \ge 1 - |r| \sum_{n=2}^{\infty} n\,|a_n| \ge 1 - \frac{|\beta|\,(2+\alpha)+1-3\alpha}{2\alpha+\alpha^2} r \tag{3.10}$$

and

$$\left|h'(z)\right| \le 1 + |r| \sum_{n=2}^{\infty} n\,|a_n| \le 1 + \frac{|\beta|\,(2+\alpha)+1-3\alpha}{2\alpha+\alpha^2} r. \tag{3.11}$$

Finally, the equality in (3.1) and (3.2) are attained for the functions $h(z)$ and $g(z)$ given by

$$h(z) = z - \frac{|\beta|\,(2+\alpha)+1-3\alpha}{4\alpha+2\alpha^2} z^2 \tag{3.12}$$

and

$$g(z) = z + \frac{1-3\alpha}{2+\alpha} z^2. \tag{3.13}$$

This completes the proof of Theorem 3. □

4. Radii of close-to-convexity, starlikeness and convexity

THEOREM 4. *Let the function $f = h + \bar{g}$ be so that h and g are given by (1.2). If $f \in \mathcal{T}_{\mathcal{H}}(\alpha, \beta)$ then $h(z)$ is starlike of order $\rho (0 \le \rho < 1)$in $|z| < r_1$, where*

$$\begin{aligned} r_1 &= r_1(\alpha, \beta, \rho) \\ &= \inf_n \left[\frac{(2\alpha+\alpha^2)(1-\rho)n}{(|\beta|\,(2+\alpha)+1-3\alpha)(n-\rho)}\right]^{1/(n-1)} \quad (n \ge 2). \end{aligned} \tag{4.1}$$

The result is sharp for the function $h(z)$ given by (2.6).

Proof. It is sufficient to show that

$$\left|\frac{zh'(z)}{h(z)} - 1\right| \le 1 - \rho$$

for $|z| < r_1$, where r_1 is given by (4.1). From (1.2) we find that

$$\left|\frac{zh'(z)}{h(z)} - 1\right| \le \frac{\sum\limits_{n=2}^{\infty}(n-1)\,|a_n|\,|z|^{n-1}}{1 - \sum\limits_{n=2}^{\infty}|a_n|\,|z|^{n-1}}.$$

Thus $\left|\frac{zh'(z)}{h(z)} - 1\right| \le 1 - \rho$ if

$$\sum_{n=2}^{\infty}\left(\frac{n-\rho}{1-\rho}\right)|a_n|\,|z|^{n-1} \le 1. \tag{4.2}$$

With the aid of (3.9), (4.2) will be true if

$$\left(\frac{n-\rho}{1-\rho}\right)|z|^{n-1} \le \frac{(2\alpha+\alpha^2)n}{|\beta|\,(2+\alpha)+1-3\alpha},$$

that is, if

$$|z| \le \left[\frac{(2\alpha+\alpha^2)(1-\rho)n}{(|\beta|\,(2+\alpha)+1-3\alpha)(n-\rho)}\right]^{1/(n-1)} \quad (n \ge 2). \tag{4.3}$$

Theorem 4 follows easily from (4.3). □

Corollary 2. *Let the function $f = h + \bar{g}$ be so that h and g are given by (1.2). If $f \in \mathcal{T}_{\mathcal{H}}(\alpha, \beta)$ then $h(z)$ is convex of order $\rho(0 \le \rho < 1)$ in $|z| < r_2$, where*

$$\begin{aligned} r_2 &= r_2(\alpha,\beta,\rho) \\ &= \inf_n \left[\frac{(2\alpha+\alpha^2)(1-\rho)}{(|\beta|\,(2+\alpha)+1-3\alpha)(n-\rho)}\right]^{1/(n-1)} \quad (n \ge 2). \end{aligned} \tag{4.4}$$

The result is sharp for the function $h(z)$ given by (2.6)

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