

THE RESIDUAL FINITENESS OF CERTAIN HNN EXTENSIONS

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ABSTRACT. In this note we give characterizations for certain HNN extensions with central associated subgroups to be residually finite. We then apply our results to HNN extensions of polycyclic-by-finite groups.

1. Introduction

A group G is called residually finite if for each $x \in G$, $x \neq 1$, there exists a normal subgroup N of finite index in G such that $x \notin N$. Many groups, including the free groups and polycyclic groups, are known to be residually finite. Finite extensions of residually finite groups are again residually finite. However the HNN extensions of residually finite groups need not be residually finite. Indeed one of the simplest type of HNN extensions, the Baumslag-Solitar group, $\langle h, t \mid t^{-1}h^2t = h^3 \rangle$ is not residually finite (see [6]).

Baumslag and Treutkoff[5] began formally the study of the residual finiteness of HNN extensions. Then Allenby and Tang[1] proved the residual finiteness of some one-relator groups with torsion by considering them as HNN extensions. In [9], Shirvani considered necessary conditions for HNN extensions to be residually finite. Around the mid 1980s, Andreadakis, Raptis and Varsos began their study of the residual finiteness of HNN extensions of finitely generated abelian groups. In a series of papers ([2], [3], [4], and [8]), they gave characterizations for these HNN extensions to be residually finite. Then Raptis and Varsos[8]

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extended these results to HNN extensions of finitely generated nilpotent groups.

In this note we investigate the residual finiteness of HNN extensions with central associated subgroups. We shall show that if $G = \langle t, A \mid t^{-1}Ht = K, \varphi \rangle$ is an HNN extension where H and K are subgroups in the center of A , $H \neq A \neq K$ and A is central subgroup separable, then G is residually finite if and only if its subgroup (HNN extension) $G_1 = \langle t, HK \mid t^{-1}Ht = K, \varphi \rangle$ is residually finite. Thus we are able to extend the results of Andreadakis, Raptis and Varsos by giving characterizations for HNN extensions of polycyclic-by-finite groups with central associated subgroups to be residually finite.

More importantly, our result shows that the study of the residual finiteness of HNN extensions with central associated subgroups can be reduced to that of the residual finiteness of HNN extensions of abelian groups. Thus the characterizations given in the papers [2], [3], [4], and [8] of Andreadakis, Raptis and Varsos can be applied to these HNN extensions.

2. Preliminaries

The notation used here is standard. In addition, the following will be used for any group G :

$N \triangleleft_f G$ means N is a normal subgroup of finite index in G .

$G = \langle t, A \mid t^{-1}Ht = K, \varphi \rangle$ denotes an HNN extension, where A is the base group, H, K are the associated subgroups and $\varphi : H \rightarrow K$ is the associated isomorphism.

DEFINITION 2.1. A group G is called H -separable for the subgroup H if for each $x \in G \setminus H$, there exists $N \triangleleft_f G$ such that $x \notin HN$.

G is termed subgroup separable if G is H -separable for every finitely generated subgroup H .

G is termed central subgroup separable if G is H -separable for every finitely generated subgroup H in the center of G .

The followings lemmas and theorem will be used to prove our main result.

LEMMA 2.2. [2] *Let $G = \langle t, A \mid t^{-1}Ht = K, \varphi \rangle$ be an HNN extension. If A is finitely generated abelian and either $H = A$ or $K = A$, then G is residually finite.*

Proof. Theorem 1 of [2]. □

Since abelian groups satisfy the identity $w(x, y) = x^{-1}y^{-1}xy$, the following theorem can be derived from Theorem 3' of [9] and Theorem 4.2 of [5].

THEOREM 2.3. *Let $G = \langle t, A \mid t^{-1}Ht = K, \varphi \rangle$ be an HNN extension where H and K are subgroups in the center of A and $H \neq A \neq K$. Let $\Delta = \{N \triangleleft_f A \mid \varphi(N \cap H) = N \cap K\}$. Then G is residually finite if and only if $\bigcap_{N \in \Delta} N = 1$, $\bigcap_{N \in \Delta} NH = H$ and $\bigcap_{N \in \Delta} NK = K$.*

The following lemma is similar to Corollary 2.1 of [8].

LEMMA 2.4. *Let $G = \langle t, A \mid t^{-1}Ht = K, \varphi \rangle$ be an HNN extension where H and K are subgroups in the center of A and $H \neq A \neq K$. Suppose G is residually finite. If $H \subseteq K$ or $K \subseteq H$ then $H = K$.*

3. The main results

LEMMA 3.1. *Let $G = \langle t, A \mid t^{-1}Ht = H, \varphi \rangle$ be an HNN extension where H is a finitely generated subgroup in the center of A and $H \neq A$. If A is H^n -separable for every positive integer n , then G is residually finite.*

Proof. Let $\Delta = \{N \triangleleft_f A \mid \varphi(N \cap H) = N \cap H\}$. By Theorem 2.3, it is sufficient to show that $\bigcap_{N \in \Delta} N = 1$ and $\bigcap_{N \in \Delta} NH = H$.

First we show that $\bigcap_{N \in \Delta} NH = H$. Let $a \in A - H$. Since A is H -separable, there exists $M_a \triangleleft_f A$ such that $a \notin M_aH$. Then $M_aH \triangleleft_f A$ and $M_aH \in \Delta$. This implies that $\bigcap_{a \in A - H} M_aH = H$ and hence that $\bigcap_{N \in \Delta} NH = H$.

Next we show that $\bigcap_{N \in \Delta} N = 1$. But first we construct a subgroup $N_n \in \Delta$ for each $n \geq 2$. Let $h_0 = 1, h_1, \dots, h_m$ be coset representatives of H^n in H where $n \geq 2$. Since A is H^n -separable, there exists $M_n \triangleleft_f A$ such that $h_i \notin M_nH^n$ for all $h_i, 1 \leq i \leq m$. Let $N_n = M_nH^n$. Then $N_n \triangleleft_f A$. We claim that $N_n \in \Delta$, that is, $N_n \cap H = H^n$. Clearly we only need to show that $N_n \cap H \subseteq H^n$. Suppose $a \in (N_n \cap H) - H^n$. Since $a \notin H^n$, we have $a = h_i\bar{h}$ where $h_i \neq 1$ is a coset representative of H^n in H and $\bar{h} \in H^n$. On the other hand, since $a \in N_n = M_nH^n$, we have $a = m\tilde{h}$, where $m \in M_n$ and $\tilde{h} \in H^n$. But this implies that $h_i \in M_nH^n$, a contradiction. Hence $N_n \cap H \subseteq H^n$. Therefore $N_n \in \Delta$ for each $n \geq 2$.

Let $a \in A$, $a \neq 1$. If $a \notin H$, then as above, there exists $M_a \triangleleft_f A$ such that $a \notin M_a H$. If $a \in H$, then $a \notin H^n$ for some n , since $\bigcap_{n \geq 1} H^n = 1$. This implies that $a \notin N_n$ since $N_n \cap H = H^n$. Therefore $\bigcap_{a \in A-H} M_a H \bigcap_{n \geq 2} N_n = 1$ and hence that $\bigcap_{N \in \Delta} N = 1$. The theorem now follows by Theorem 2.3. \square

THEOREM 3.2. *Let $G = \langle t, A \mid t^{-1}Ht = K, \varphi \rangle$ be an HNN extension where H and K are subgroups in the center of A and $H \neq A \neq K$. Suppose $H \not\subseteq K$, $K \not\subseteq H$ and A is M -separable for every subgroup $M \triangleleft_f HK$. Then G is residually finite if and only if $G_1 = \langle t, HK \mid t^{-1}Ht = K, \varphi \rangle$ is residually finite.*

Proof. Suppose G is residually finite. Since G_1 is a subgroup of G , G_1 is residually finite.

Suppose G_1 is residually finite. Since $H \neq HK \neq K$, then by Theorem 2.3, $\bigcap_{M \in \Delta_1} M = 1$, $\bigcap_{M \in \Delta_1} MH = H$ and $\bigcap_{M \in \Delta_1} MK = K$, where $\Delta_1 = \{M \triangleleft_f HK \mid \varphi(M \cap H) = M \cap K\}$.

Let $\Delta = \{N \triangleleft_f A \mid \varphi(N \cap H) = N \cap K\}$. By Theorem 2.3, it is sufficient to show that $\bigcap_{N \in \Delta} N = 1$, $\bigcap_{N \in \Delta} NH = H$ and $\bigcap_{N \in \Delta} NK = K$. But first we construct a subgroup $N_M \in \Delta$ for each $M \in \Delta_1$. Let $M \in \Delta_1$. Let $h_0 = 1, h_1, \dots, h_m$ be coset representatives of M in HK . Since A is M -separable, there exists $P_M \triangleleft_f A$ such that $h_i \notin P_M M$ for all h_i , $1 \leq i \leq m$. Let $N_M = P_M M$. Then $N_M \triangleleft_f A$. Next we show $N_M \cap HK = M$. Clearly we only need to show that $N_M \cap HK \subseteq M$. Suppose $a \in (N_M \cap HK) - M$. Since $a \notin M$, then $a = h_i m_1$ where $h_i \neq 1$ is a coset representative of M in HK and $m_1 \in M$. On the other hand, since $a \in N_M = P_M M$, we have $a = pm_2$ where $p \in P_M$ and $m_2 \in M$. But this implies that $h_i \in P_M M$, a contradiction. Therefore $N_M \cap HK = M$. Similarly we can show that $N_M \cap H = M \cap H$ and $N_M \cap K = M \cap K$. Hence $N_M \in \Delta$.

Now we show that $\bigcap_{N \in \Delta} NH = H$. Let $a \in A - H$. Suppose $a \notin HK$. Since A is HK -separable, there exists $M_a \triangleleft_f A$ such that $a \notin M_a HK$. Then $M_a HK \triangleleft_f A$ and $M_a HK \in \Delta$. Next suppose $a \in HK$. Since $a \notin H$ and $\bigcap_{M \in \Delta_1} MH = H$, there exists $M \in \Delta_1$ such that $a \notin MH$. We claim that $a \notin N_M H$, where $N_M = P_M M$ is as defined above. Suppose $a \in N_M H$. Then $a = nh$ for some $n \in N_M$ and $h \in H$. This implies that $n \in HK \cap N_M = M$ and thus $a \in MH$, a contradiction. So $a \notin N_M H$. Therefore $\bigcap_{N \in \Delta} NH = H$. Similarly we can show that $\bigcap_{N \in \Delta} NK = K$.

Finally, we show that $\bigcap_{N \in \Delta} N = 1$. Let $a \in A$, $a \neq 1$. Suppose $a \notin HK$. Since A is HK -separable, there exists $M_a \triangleleft_f A$ such that

$a \notin M_aHK$. Then $M_aHK \triangleleft_f A$ and $M_aHK \in \Delta$. Suppose $a \in HK$. Since $\bigcap_{M \in \Delta_1} M = 1$, there exists $M \in \Delta_1$ such that $a \notin M$. We claim that $a \notin N_M$ where $N_M = P_M M$ is as defined above. Suppose $a \in N_M$. Since $a \notin M$, we have $a = h_i m_1$ where $h_i \neq 1$ is a coset representative of M in HK and $m_1 \in M$. On the other hand, since $a \in N_M = P_M M$, we have $a = pm_2$ where $p \in P_M$ and $m_2 \in M$. But this implies that $h_i \in P_M M$, a contradiction. So $a \notin N_M$. Therefore $\bigcap_{N \in \Delta} N = 1$. The theorem now follows from Theorem 2.3. \square

THEOREM 3.3. *Let $G = \langle t, A \mid t^{-1}Ht = K, \varphi \rangle$ be an HNN extension where H and K are finitely generated subgroups in the center of A . Suppose A is central subgroup separable. Then G is residually finite if and only if one of the following holds:*

- (a) $H = A$ or $K = A$;
- (b) $H = K$;
- (c) $H \not\subseteq K, K \not\subseteq H$ and there exists a torsion free normal subgroup $N \triangleleft_f HK$ such that $\varphi(N \cap H) = N \cap K$ and $N \cap K, N \cap H$ are isolated in N .

Proof. Suppose G is residually finite and suppose $H \neq A \neq K$ and $H \neq K$. If $H \subset K$, then by Lemma 2.4, $H = K$, a contradiction. Therefore $H \neq HK$ and similarly $K \neq HK$. So by Theorem 3.2, $G_1 = \langle t, HK \mid t^{-1}Ht = K, \varphi \rangle$ is residually finite. Then by Theorem 5* of [8], there exists a torsion free normal subgroup $N \triangleleft_f HK$ such that $\varphi(N \cap H) = N \cap K$ and $N \cap K, N \cap H$ are isolated in N .

If $H = A$ or $K = A$, then A is abelian. Hence by Lemma 2.2, G is residually finite. If $H \neq A \neq K$ and $H = K$, then by Lemma 3.1, G is residually finite.

Suppose $H \not\subseteq K, K \not\subseteq H$ and there exists a torsion free normal subgroup $N \triangleleft_f HK$ such that $\varphi(N \cap H) = N \cap K$ and $N \cap K, N \cap H$ are isolated in N . Then $H \neq HK \neq K$. So by Theorem 5* of [8], $G_1 = \langle t, HK \mid t^{-1}Ht = K, \varphi \rangle$ is residually finite. Therefore by Theorem 3.2, G is residually finite. \square

COROLLARY 3.4. *Let $G = \langle t, A \mid t^{-1}Ht = K, \varphi \rangle$ be an HNN extension where A is a polycyclic-by-finite group. Suppose H and K are finitely generated subgroup in the center of A . Then G is residually finite if and only if one of the following holds:*

- (a) $H = A$ or $K = A$;
- (b) $H = K$;

- (c) $H \not\subseteq K, K \not\subseteq H$ and there exists a torsion free normal subgroup $N \triangleleft_f HK$ such that $\varphi(N \cap H) = N \cap K$ and $N \cap K, N \cap H$ are isolated in N .

Proof. Since polycyclic-by-finite groups are subgroup separable, they are also central subgroup separable. Therefore the corollary follows from Theorem 3.3. □

COROLLARY 3.5. *Let $G = \langle t, A \mid t^{-1}Ht = K, \varphi \rangle$ be an HNN extension where A is a polycyclic-by-finite group. Suppose H and K are finitely generated subgroup in the center of A such that $H \cap K$ is finite. Then G is residually finite.*

Proof. Let $T(H)$ and $T(K)$ be the torsion subgroups of H and K respectively. Since H is finitely generated abelian, there exists $S_H \triangleleft_f H$ such that $S_H \cap T(H) = 1$. Let $S_K = \varphi(S_H)$. Then $S_K \cap T(K) = 1$. This also implies that $S_H \cap K = 1 = S_K \cap H$ since $H \cap K$ is finite. Clearly $S_H S_K \triangleleft_f HK$. We claim that $S_H S_K$ is torsion free. Let $(hk)^r = 1$, where $h \in S_H, k \in S_K$ and r is a positive integer. Then $h^r = k^{-r} \in S_H \cap K = 1$. This implies that $h \in S_H \cap T(H) = 1$. Similarly $k = 1$. Thus $S_H S_K$ is torsion free.

Let $i(H)$ and $i(K)$ be the isolated closures of H and K respectively in HK . Since $i(H)$ is finitely generated and every element in $i(H)/H$ has finite order, $i(H)/H$ is finite. Let u_0, u_1, \dots, u_m be a complete set of coset representatives of H in $i(H)$, where $u_0 = 1$. Since HK is finitely generated abelian and thus subgroup separable, there exists $M_1 \triangleleft_f HK$ such that $u_i \notin M_1 H$ for $i \geq 1$. Similarly there exists $M_2 \triangleleft_f HK$ such that $v_i \notin M_2 K$ for $i \geq 1$, where $1 = v_0, v_1, \dots, v_n$ form a complete set of coset representatives of K in $i(K)$. Clearly $M_1 \cap i(H) \subseteq H$ and $M_2 \cap i(K) \subseteq K$. Let $M_H = M_2 \cap H \cap \varphi^{-1}(M_1 \cap K)$, $M_K = \varphi(M_2 \cap H) \cap M_1 \cap K$ and $N = (M_H \cap S_H)(M_K \cap S_K)$. Then $\varphi(M_H) = M_K$ and $N \triangleleft_f HK$. Furthermore $N \cap i(H) = (M_H \cap S_H)(M_K \cap S_K) \cap i(H) = (M_H \cap S_H)(M_K \cap S_K \cap i(H)) \subseteq (M_H \cap S_H)(M_1 \cap i(H)) \subseteq H$ since $M_K \cap S_K \subseteq M_1$. Similarly $N \cap i(K) \subseteq K$.

We claim that N is a torsion free normal subgroup of finite index in HK such that $\varphi(N \cap H) = N \cap K$ and $N \cap K, N \cap H$ are isolated in N .

Clearly N is torsion free. Observe that $N \cap H = (S_H \cap M_H)(S_K \cap M_K) \cap H = (S_H \cap M_H)(S_K \cap M_K \cap H) = S_H \cap M_H$ since $S_K \cap H = 1$. Similarly we have $N \cap K = S_K \cap M_K$. Therefore $\varphi(N \cap H) = \varphi(S_H \cap M_H) = \varphi(S_H) \cap \varphi(M_H) = S_K \cap M_K = N \cap K$. Next we show that $N \cap H$ is isolated in N . Let $n^r \in N \cap H$, where $n \in N$ and r is a positive

integer. We need only to show that $n \in H$. Since $n^r \in H$ then $n \in i(H)$. Hence $n \in N \cap i(H) \subseteq H$. Similarly we can show that $N \cap K$ is isolated in N . The corollary now follows from Corollary 3.4. \square

REMARKS. One of the easy and natural applications of Corollary 3.4 and Corollary 3.5 is the HNN extensions of abelian groups with cyclic associated subgroups. Residual finiteness of such HNN extensions is known to Kim and Tang[7].

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