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REMARKS ON SPECTRAL PROPERTIES OF p-HYPONORMAL AND LOG-HYPONORMAL OPERATORS

Bhagwati P. Duggal and In Ho Jeon

Abstract. In this paper it is proved that for p-hyponormal or log-hyponormal operator A there exist an associated hyponormal operator T, a quasi-affinity X and an injection operator Y such that $TX = XA$ and $AY = YT$. The operator A and T have the same spectral picture. We apply these results to give brief proofs of some well known spectral properties of p-hyponormal and log-hyponormal operators, amongst them that the spectrum is a continuous function on these classes of operators.

1. Introduction

Let H be a complex separable infinite dimensional Hilbert space and let $B(H)$ denote the Banach algebra of bounded linear operators acting on H. For an operator $A \in B(H)$, we write $\ker A$ and ran$A$ for the null space and the range of A. An operator $A \in B(H)$ is called *semi-Fredholm*, denoted $A \in \rho_{SF}$, if ran$A$ is closed and either $\alpha(A) := \dim(\ker A)$ or $\beta(A) := \dim(\ker A^*)$ is finite; in this case the *index* of A, denoted $i(A)$, is defined by $i(A) = \alpha(A) - \beta(A)$. If $\alpha(A)$ and $\beta(A)$ are both finite, then A is called *Fredholm*. If $A \in \rho_{SF}$ and $i(A) = 0$, then A is called *Weyl*, denoted $A \in \rho_{SF}^{\circ}$. Also, A is called *Browder* if A is Fredholm and $A - \lambda\,(:= A - \lambda I_H)$ is invertible for sufficiently small $\lambda \neq 0 \in \mathbb{C}$. Recall([12]) that the *ascent*, denoted asc(A), and the *descent*, denoted des(A), of $A \in B(H)$ are the extended integers given by asc$(A) = \inf\{n \geq 0 : \ker A^n = \ker A^{n+1}\}$ and des$(A) = \inf\{n \geq 0 :$

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$\operatorname{ran} A^n = \operatorname{ran} A^{n+1}\}$, respectively. The infimum over the empty set is taken to be ∞. If the ascent and the descent of A are both finite, then $\operatorname{asc}(A) = \operatorname{des}(A)$([12, Proposition 38.4]). Let $K(H)$ denote the ideal of all compact operators in $B(H)$ and let $C(H) = B(H)/K(H)$ be the Calkin algebra with the Calkin canonical projection $\pi : B(H) \to C(H)$.

Throughout this paper we shall use the following standard notations for various spectra:

$$\sigma(A) = \{\lambda \in \mathbb{C} : \ A - \lambda \text{ is not invertible}\} \text{ for the spectrum of } A$$
$$\sigma_\ell(A) = \{\lambda \in \mathbb{C} : \ A - \lambda \text{ is not left invertible}\} \text{ for the left spectrum of } A$$
$$\sigma_r(A) = \{\lambda \in \mathbb{C} : \ A - \lambda \text{ is not right invertible}\} \text{ for the right spectrum of } A$$
$$\sigma_a(A) = \{\lambda \in \mathbb{C} : \ A - \lambda \text{ is not bounded below}\} \text{ for the approximate point spectrum of } A$$
$$\sigma_\delta(A) = \{\lambda \in \mathbb{C} : \ A - \lambda \text{ is not surjective}\} \text{ for the approximate defect spectrum of } A$$
$$\sigma_e(A) = \{\lambda \in \mathbb{C} : \ A - \lambda \text{ is not Fredholm}\} \text{ for the essential spectrum of} A$$
$$\sigma_w(A) = \{\lambda \in \mathbb{C} : \ A - \lambda \text{ is not Weyl}\} \text{ for the Weyl spectrum of } A$$
$$\sigma_b(A) = \{\lambda \in \mathbb{C} : \ A - \lambda \text{ is not Browder}\} \text{ for the Browder spectrum of} A$$
$$\sigma_{\ell e}(A) = \sigma_\ell(\pi(A)) \text{ for the left essential spectrum of } A$$
$$\sigma_{re}(A) = \sigma_r(\pi(A)) \text{ for the right essential spectrum of } A$$
$$\sigma_{\ell re}(A) = \sigma_{\ell e}(A) \cap \sigma_{re}(A)$$
$$\sigma_\circ(A) = \{\lambda \in \mathbb{C} : 0 < \alpha(A - \lambda)\} \text{ for the point spectrum of } A$$

In Hilbert context, it is well known that

$$\sigma_\ell(A) = \sigma_a(A) \text{ and } \sigma_r(A) = \sigma_\delta(A).$$

For convenient let us introduce the following additional notations:

$$\rho_{SF}(A) = \{\lambda \in \mathbb{C} : A - \lambda \in \rho_{SF}\}$$
$$\rho_{SF}^\circ(A) = \{\lambda \in \mathbb{C} : A - \lambda \in \rho_{SF}^\circ\}$$
$$\rho_{SF}^+(A) = \{\lambda \in \mathbb{C} : A - \lambda \in \rho_{SF} \text{ with } i(A - \lambda) > 0\}$$

$$\rho_{SF}^{-}(A) = \{\lambda \in \mathbb{C} : A - \lambda \in \rho_{SF} \text{ with } i(A-\lambda) < 0\}$$
$$\rho_{SF}^{\pm}(A) = \rho_{SF}^{+}(A) \cup \rho_{SF}^{-}(A)$$
$$\rho_{SF}^{n}(A) = \{\lambda \in \mathbb{C} : A - \lambda \in \rho_{SF} \text{ with } i(A-\lambda) = n\}$$
$$\text{for } -\infty \leq n \leq \infty$$
$$\Gamma_{\circ}(A) = \text{the union of all trivial components of}$$
$$\left(\sigma(A) \setminus \overline{[\rho_{SF}^{\pm}(A)]}\right) \cup \left(\bigcup_{-\infty<n<\infty} \overline{[\rho_{SF}^{n}(A)]} \setminus \rho_{SF}^{n}(A)\right)$$
$$\Gamma_{\circ e}(A) = \Gamma_{\circ}(A) \setminus \sigma^{R}(A), \text{ where } \sigma^{R}(A) \text{ denotes the set of}$$
$$\text{Riesz points of } A, \text{ i.e. } \sigma^{R}(A) = \sigma(A) \setminus \sigma_b(A)$$

If K is a subset of $\mathbb{C}$, we denote the set of all isolated points of K by $\operatorname{iso} K$.

Recall([1, 4, 8, 9, 10, 13, 18]) that an operator $A \in B(H)$ is said to be *p-hyponormal* if

$$(A^*A)^p - (AA^*)^p \geq 0 \text{ for } p \in (0,1].$$

If $p = 1$, A is said to be hyponormal and if $p = 1/2$, A is said to be semi-hyponormal ([17]). If A is p-hyponormal, then A is also q-hyponormal for every $0 < q \leq p$. An operator A is called *log-hyponormal* ([5, 6, 16]) if A is invertible and satisfies the following inequality

$$\log(A^*A) \geq \log(AA^*).$$

It is well known ([16]) that any invertible p-hyponormal operator is a log-hyponormal operator and the converse is not true. Let $\mathcal{H}(p)$ denote the class of all p-hyponormal operators $(0 < p < 1/2)$ and $\mathcal{L}$ denote the class of log-hyponormal operators. Let $A \in \mathcal{H}(p)$ or $\mathcal{L}$ have the polar decomposition $A = U|A|$. Then the *Aluthge transform* of A is defined by $\widetilde{A} = |A|^{1/2}U|A|^{1/2}$. The Aluthge transform $\widetilde{A}$, in conjunction with some related arguments of Xia[17], has proved to be a very useful tools in the study of p-hyponormal and log-hyponormal operators.

In this paper we prove that if A is a p-hyponormal or log-hyponormal operator, then there exists an associated hyponormal operator T. The existence of T is deduced from the Aluthge transform $\widetilde{A}$ of A . The relationship between A and T is a deep one: we exploit it here to study the spectral properties of p-hyponormal and log-hyponormal operators. In particular, it is shown that the operator A and T have the same spectral picture, which give a slight improvement of [18, Theorem 6]. Also, we apply these results to give brief proofs of some well known spectral properties of p-hyponormal and log-hyponormal operators, amongst them that the spectrum is a continuous function on these classes of operators.

2. Results

For an operator $A \in B(H)$, let $H_0(A-\lambda)$ denote the quasi-nilpotent part,

$$H_0(A-\lambda) = \left\{x \in H : \lim_{n\longrightarrow\infty} ||(A-\lambda)^n x||^{\frac{1}{n}} = 0\right\},$$

of A. Then $(A-\lambda)^{-n}(0) \subseteq H_0(A-\lambda)$ for all natural numbers n and $\lambda \in \mathbb{C}$. Since hyponormal operators $A \in B(H)$ satisfy $||(A-\lambda)x||^n \le ||(A-\lambda)^n x||$ for all natural numbers n and unit vectors $x \in H$, $H_0(A-\lambda) = (A-\lambda)^{-1}(0)$.

The following is the main result.

THEOREM 1. *If $A \in \mathcal{H}(p)$ or $\mathcal{L}$, then there exists a hyponormal operator T such that the following holds.*

(i) *There exists a quasi-affinity X and an injection operator Y such that $TX = XA$ and $AY = YX$. Here the operators X and Y are invertible in the case in which $A \in \mathcal{L}$; also, the operator X is invertible in the case in which $A \in \mathcal{H}(p)$ and 0 is not in the approximate point spectrum of the pure (=* completely non-normal*) part of A.*

(ii) $H_0(A-\lambda) = (A-\lambda)^{-1}(0) = H_0(T-\lambda) = (T-\lambda)^{-1}(0)$ *for all* $\lambda \in \mathbb{C}$.

(iii) $\sigma_s(A) = \sigma_s(T)$, *where σ_s stands for either of σ, $\sigma_\circ$, $\sigma_{\circ\circ}$, σ_a, σ_r, σ_δ, σ_b, σ_w, σ_e, $\sigma_{\ell e}$, σ_{re}, $\sigma_{\ell re}$, $\Gamma_\circ$ or $\Gamma_{\circ e}$. Also, $i(A-\lambda) = i(T-\lambda)$ for all $\lambda \in \mathbb{C}$.*

(iv) *If the function f is analytic in an open neighborhood of $\sigma(A)$, then $\{\lambda \in \mathbb{C} : f(A)-\lambda \in \rho_{SF}^\circ\} = \{\lambda \in \mathbb{C} : f(T)-\lambda \in \rho_{SF}^\circ\}$ and $\sigma_w(f(A)) = \sigma_w(f(T)) = \sigma(f(T)) - \sigma_{\circ\circ}(f(T))$.*

Proof. Let $A \in \mathcal{H}(p)$ or $\mathcal{L}$. We start by constructing the "associated hyponormal" operator T. If $A \in \mathcal{L}$ has the polar decomposition $A = U|A|$, then

$$\{A(s,t)A(s,t)^*\}^{\frac{\min(s,t)}{s+t}} \le |A|^{2\min(s,t)} \le \{A(s,t)^*A(s,t)\}^{\frac{\min(s,t)}{s+t}},$$

where $A(s,t) = |A|^s U |A|^t$ for some numbers $s,t > 0$ ([15, Theorem 4]). The operator $A(s,t) \in \mathcal{H}(\frac{\min(s,t)}{s+t})$. Choose $0 < s \le t$ such that $s+t = 1$. Then $A(s,t) = |A|^s U |A|^{1-s}$. Now, choose $s = t$ (so that A is similar to an $\mathcal{H}(1/2)$ operator); applying the Löwner inequality, then it follows A is similar to an $\mathcal{H}(p)$ operator, $0 < p < 1/2$, and our problem reduces to that of constructing the associated hyponormal operator T for a given $A \in \mathcal{H}(p)$. Given $A \in \mathcal{H}(p)$, decompose A into its normal and pure parts by $A = A_\circ \oplus A_1$(with respect to the decomposition $H = H_\circ \oplus H_1$, say). Then $A_1 \in \mathcal{H}(p)$. Let A_1 have the polar decomposition $A_1 = U_1|A_1|$, where (necessarily) U_1 is an isometry. Define the (first Aluthge) transform $\widetilde{A}_1$ of A_1 by $\widetilde{A}_1 = |A_1|^{1/2} U_1 |A_1|^{1/2}$. Then $\widetilde{A}_1 \in \mathcal{H}(p+1/2)$([1]), $\widetilde{A}_1 |A_1|^{1/2} = |A_1|^{1/2} A_1$ and $A_1 U_1 |A_1|^{1/2} = U_1 |A_1|^{1/2} \widetilde{A}_1$ (where $|A_1|^{1/2}$ is a quasi-affinity and $U_1|A_1|^{1/2}$ is injective). Define the (second Aluthge) transform T_1 of A_1 by $T_1 = (\widetilde{\widetilde{A}_1})$. Then T_1 is hyponormal ([1]), and there exists a quasi-affinity X_1 and an injective operator Y_1 such that $T_1 X_1 = X_1 A_1$ and $A_1 Y_1 = Y_1 T_1$. Let $T = A_\circ \oplus T_1$. Then T is hyponormal. Defining the quasi-affinity X by $X = I_{H_\circ} \oplus X_1$ and the injective operator Y by $Y = I_{H_\circ} \oplus Y_1$ it follows that $TX = XA$ and $AY = YT$. Here it is clear that the operators X and Y are invertible in the case in which $A \in \mathcal{L}$. Suppose now that $A \in \mathcal{H}(p)$ and $0 \notin \sigma_a(A_1)$. Then $0 \notin \sigma(|A_1|)$, the operator $|A_1|$ is invertible and the operator $\widetilde{A}_1$ is similar to A_1. Repeating this argument it now follows that the operator X_1 is invertible. This proves (i).

Towards (ii), we assume that $p = \frac{1}{2}$, $T = \tilde{A}$ is hyponormal, and prove that $H_0(A-\lambda) = (A-\lambda)^{-1}(0) = H_0(T-\lambda) = (T-\lambda)^{-1}(0)$: the proof for $p < \frac{1}{2}$ follows from a repeated application of the argument with $A \in \mathcal{H}(\frac{1}{2})$ and T replaced by $A \in \mathcal{H}(p)$ and $\tilde{A} \in \mathcal{H}(p+\frac{1}{2})$, respectively. As seen in part (i) above, $TX = XA$, where X is the quasi-affinity $X = I_{H_0} \oplus |A_1|^{\frac{1}{2}}$. Let $x \in H_0(A-\lambda)$. Then, as $n \longrightarrow \infty$,

$$\|(T-\lambda)^n Xx\|^{\frac{1}{2}} = \|X(A-\lambda)^n x\|^{\frac{1}{n}} \le \|X\|^{\frac{1}{n}} \, \|(A-\lambda)^n x\|^{\frac{1}{n}} \longrightarrow 0.$$

Hence $Xx \in H_0(T-\lambda) = (T-\lambda)^{-1}(0)$, which implies that $X(A-\lambda)x = 0$. Since X is injective, $x \in (A-\lambda)^{-1}(0) \Longrightarrow H_0(A-\lambda) = (A-\lambda)^{-1}(0)$. To prove that $H_0(A-\lambda) = H_0(T-\lambda)$ we argue as follows. If $x \in (A-\lambda)^{-1}(0)$, then $|A|^{\frac{1}{2}}x = |\lambda|^{\frac{1}{2}}x$. Set $y = |\lambda|^{\frac{-1}{2}}x$; then $y \in (A-\lambda)^{-1}(0)$ and $|A|^{\frac{1}{2}}(|\lambda|^{\frac{-1}{2}}x) = x$. Conversely, if $y \in (A-\lambda)^{-1}(0)$ and $|A|^{\frac{1}{2}}y = x$, then $|A|^{\frac{1}{2}}y = |\lambda|^{\frac{1}{2}}y \Longrightarrow y = |\lambda|^{\frac{-1}{2}}x \Longrightarrow x \in (A-\lambda)^{-1}(0)$. Conclusion:

$x \in (A-\lambda)^{-1}(0)$ if and only if there exists a $y \in (A-\lambda)^{-1}(0)$ such that $|A|^{\frac{1}{2}}y = x$.

Let $x \in H_0(A-\lambda) = (A-\lambda)^{-1}(0)$. Then there exists a $y \in (A-\lambda)^{-1}(0)$ such that $|A|^{\frac{1}{2}}y = x$. Since $y \in (A-\lambda)^{-1}(0) \Longrightarrow \lambda x = \lambda|A|^{\frac{1}{2}}y = |A|^{\frac{1}{2}}Ay = T|A|^{\frac{1}{2}}y = Tx$, $H_0(A-\lambda) \subseteq (T-\lambda)^{-1}(0) = H_0(T-\lambda)$. For the reverse inclusion, let $x \in H_0(T-\lambda)$. Then $\lambda U|A|^{\frac{1}{2}}x = A(U|A|^{\frac{1}{2}}x)$. Since λ is a normal eigenvalue of A (all eigenvalues of a p-hyponormal operator are normal), we have the following sequence of implications:

$$\begin{aligned} A^*(U|A|^{\frac{1}{2}}x) = \overline{\lambda}(U|A|^{\frac{1}{2}}x) &\Longrightarrow |A|^{\frac{3}{2}}x = \overline{\lambda}(U|A|^{\frac{1}{2}}x) \\ &\Longrightarrow |A|^2x = \overline{\lambda}Tx = |\lambda|^2x \\ &\Longrightarrow |A|^{\frac{1}{2}}x = |\lambda|^{\frac{1}{2}}x \\ &\Longrightarrow \lambda x = Tx = |A|^{\frac{1}{2}}(U|A|^{\frac{1}{2}}x) \\ &\qquad\quad = |\lambda|^{\frac{-1}{2}}|A|^{\frac{1}{2}}(U|A|x) = |\lambda|^{\frac{-1}{2}}|A|^{\frac{1}{2}}Ax \\ &\Longrightarrow |A|^{\frac{1}{2}}Ax = \lambda|\lambda|^{\frac{1}{2}}x = \lambda|A|^{\frac{1}{2}}x \\ &\Longrightarrow (A-\lambda)Ax = 0. \end{aligned}$$

Again, since λ is a normal eigenvalue of A, this implies that $(A^*-\overline{\lambda})Ax = 0 \Longrightarrow \overline{\lambda}Ax = |A|^2x = |\lambda|^2x \Longrightarrow Ax = \lambda x$. Hence $H_0(T-\lambda) \subseteq (A-\lambda)^{-1}(0) = H_0(A-\lambda)$.

The proof of (iii) of Theorem for the case in which σ_s stand for either of σ, σ_o, σ_{oo}, σ_a, σ_δ, σ_b, or σ_w either appears in or follows from the proof of [18, Theorem 5]. It is clear from the similarity of A and T in the case in which $A \in \mathcal{L}$ that $i(A-\lambda) = i(T-\lambda)$ for all $\lambda \in \mathbb{C}$. Furthermore, if $A \in \mathcal{H}(p)$, then $i(A-\lambda) = i(T-\lambda)$ in the case in which either $\lambda \neq 0$([18, Lemma 3]) or $0 = \lambda \notin \sigma_a(A_1)$ (because then A and T are similar). Finally, since $0 \in \sigma_a(A_1) = \sigma_a(T_1)$ implies that ranA_1 and ranT_1 are not closed, we conclude that $i(A-\lambda) = i(T-\lambda)$ for all $\lambda \in \mathbb{C}$. Consequently, $\Gamma_o(A) = \Gamma_o(T)$.

Now we show that $\sigma_{le}(A) = \sigma_{le}(T)$ and $\sigma_{re}(A) = \sigma_{re}(T)$. If we let S denote either of A and T, then part (ii) of the theorem implies that asc$(S-\lambda) \leq 1$ for all $\lambda \in \mathbb{C}$, and hence that $\alpha(S-\lambda) \leq \beta(S-\lambda)$ ([12, Proposition 38.5]). Recall that

$$\sigma_{le}(S) = \{\lambda \in \mathbb{C} : \text{either } \operatorname{ran}(S-\lambda) \text{ is not closed or } \alpha(S-\lambda) = \infty\}$$

and

$$\mathbb{C} \setminus \sigma_{re}(S) = \{\lambda \in \mathbb{C} : \operatorname{ran}(S-\lambda) \text{ is closed and } \beta(S-\lambda) < \infty\}.$$

Since $i(A-\lambda) = i(T-\lambda)$ and $\alpha(A-\lambda) = \alpha(T-\lambda)$ by part (ii) above, $\beta(A-\lambda) = \beta(T-\lambda)$. Thus the only way $\sigma_{le}(A)$ and $\sigma_{le}(T)$, as also $\mathbb{C}\setminus\sigma_{re}(A)$ and $\mathbb{C}\setminus\sigma_{re}(T)$, can fail to be equal is that $\alpha(A-\lambda) = \alpha(T-\lambda) < \infty$ and either $\operatorname{ran}(A-\lambda)$ is closed but $\operatorname{ran}(T-\lambda)$ is not or $\operatorname{ran}(T-\lambda)$ is closed but $\operatorname{ran}(A-\lambda)$ is not. (Recall that $\alpha(S-\lambda) \leq \beta(S-\lambda)$, so that $\beta(S-\lambda) < \infty \Longrightarrow \alpha(S-\lambda) < \infty$.) We consider the case $\operatorname{ran}(A-\lambda)$ is closed but $\operatorname{ran}(T-\lambda)$ is not: the proof for the other case is similar (see also the Remark below). If $\operatorname{ran}(A-\lambda)$ is closed and $0 \leq \alpha(A-\lambda) < \infty$, then either $A-\lambda$ is injective or λ is in the point spectrum of A. Since $A-\lambda$ is injective $\Longrightarrow A-\lambda$ is left invertible $\Longrightarrow T-\lambda$ is left invertible, and λ is in the point spectrum of $A \Longrightarrow \lambda$ is a normal eigenvalue of T, in either case we have a contradiction. Hence $\sigma_{le}(A) = \sigma_{le}(T)$ and $\sigma_{re}(A) = \sigma_{re}(T)$. Trivially, $\sigma_e(A) = \sigma_e(T)$ and $\sigma_{lre}(A) = \sigma_{lre}(T)$.

Recall that the isolated points of the spectrum of a hyponormal operator are eigenvalues of the operator and that the eigenvalues of a hyponormal operator are normal eigenvalues of the operator. Hence if $\lambda \in \sigma_{oo}(A) = \sigma_{oo}(T)$, then λ is a Riesz point of A (and T). This implies that $\Gamma_{oe}(A) = \Gamma_{oe}(T)$, and the proof of (iii) is complete.

To prove (iv), we start by noting that it is enough to consider polynomials f. Thus, given $\lambda \in \mathbb{C}$, let $f(T) - \lambda = a_0 \prod_{i=1}^n (T-\lambda_i)$ for some scalar a_0 and complex numbers λ_i such that $\lambda = a_0 \prod_{i=1}^n (\lambda_i)$. Then, using the properties of the "index function",

$$\begin{aligned} f(T) - \lambda \in \rho_{SF}^{\circ} &\Leftrightarrow a_0 \prod_{i=1}^n (T-\lambda_i) \in \rho_{SF}^{\circ} \\ &\Leftrightarrow T-\lambda_i \in \rho_{SF}^{\circ} \text{ for all } i = 1, 2, \cdots, n \\ &\Leftrightarrow \lambda_i \in \sigma_{oo}(T) = \sigma_{oo}(A) \text{ for all } i = 1, 2, \cdots, n \\ &\Leftrightarrow A-\lambda_i \in \rho_{SF}^{\circ} \text{ for all } i = 1, 2, \cdots, n \\ &\Leftrightarrow a_0 \prod_{i=1}^n (A-\lambda_i) \in \rho_{SF}^{\circ} \\ &\Leftrightarrow f(A) - \lambda \in \rho_{SF}^{\circ} \end{aligned}$$

and

$$\begin{aligned} f(T) - \lambda \in \rho_{SF}^{\circ} &\Leftrightarrow \lambda_i \in \sigma_{oo}(T)(= \sigma_{oo}(A)) \text{ for all } i = 1, 2, \cdots, n \\ &\Leftrightarrow \lambda_i \in \sigma_{oo}(f(T)) \\ &\Leftrightarrow \lambda_i \in \sigma_{oo}(f(A)) \text{ for all } i = 1, 2, \cdots, n \end{aligned}$$

Hence

$$\begin{aligned}\sigma_w(f(A)) &= \sigma(f(A)) - \{\lambda \in \mathbb{C} : f(A) - \lambda \in \rho_{SF}^{\circ}\} \\ &= f(\sigma(A)) - \sigma_{oo}(f(T)) \\ &= f(\sigma(T)) - \sigma_{oo}(f(T)) \\ &= \sigma(f(T)) - \sigma_{oo}(f(T)) \\ &= \sigma(f(T)) - \{\lambda \in \mathbb{C} : f(T) - \lambda \in \rho_{SF}^{\circ}\} \\ &= \sigma_w(f(T)).\end{aligned}$$

□

REMARK. An alternative proof of $\sigma_e(A) = \sigma_e(T)$ in part (iii) of Theorem 1 using the Calkin map π is obtained as follows. We may assume without loss of generality that $p = 2^{-n}$ for some natural number n. Since

$$\pi\left(|A|^2\right) = \pi\left(A^*A\right) = \pi(A^*)\pi(A) = \pi(A)^*\pi(A)$$

and

$$\pi\left(|A|^2\right) = \pi\left(\left\{|A|^{2p}\right\}^{\frac{1}{p}}\right) = \left(\pi\left(|A|^{2p}\right)\right)^{\frac{1}{p}},$$

it follows that

$$\pi\left(|A|^{2p}\right) = \{\pi(A^*)\pi(A)\}^p$$

and

$$\pi\left(|A|^{2p} - |A^*|^{2p}\right) = \{\pi(A^*)\pi(A)\}^p - \{\pi(A)\pi(A^*)\}^p.$$

Recall that if $\mathcal{A}$ is an algebra of operators and $a \in \mathcal{A}$, then a is positive if and only if the (algebra) *numerical range* $V(\mathcal{A}; a)$ of a is a subset of the set $\mathcal{R}_+$ of non-negative reals. Let $a = |A|^{2p} - |A^*|^{2p}$. Then the p-hyponormality of A implies that $V(B(H); a) \subseteq \mathcal{R}_+$. Since

$$V\left(C(H); \pi(a)\right) = \bigcap V\left(a + J : J \in K(H)\right) \subseteq V(B(H); a) \subseteq \mathcal{R}_+,$$

$\pi(a)$ is a positive operator [3, Proposition 10.7], and hence $\pi(A) \in \mathcal{H}(p)$. Define $\tilde{A}_1$, $\tilde{\tilde{A}}_1 = T_1$, $\tilde{A}$ and T as in the proof of Theorem 1(i). Then

$$\pi(A_1) = \pi(U)\pi(|A_1|) = \pi(U_1)\left(\pi(|A_1|^{\frac{1}{2}})\right)^2,$$

$$\pi(\tilde{A}_1) = \pi\left(|A_1|^{\frac{1}{2}}\right)\pi\left(U_1\right)\pi\left(|A_1|^{\frac{1}{2}}\right) = \widetilde{\pi(A_1)}$$

and $\pi(T_1) = \pi(\tilde{\tilde{A}}_1) = \widetilde{\widetilde{\pi(A_1)}}$, where $\widetilde{\pi(A_1)} \in \mathcal{H}(p + \frac{1}{2})$ and $\pi(T_1) \in \mathcal{H}(1)$. Clearly, $\sigma_e(A) = \sigma_e(T)$ if and only if $\sigma(\pi(A_1)) = \sigma(\pi(T_1))$.

Since $\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$ for all $a, b \in \mathcal{A}$ for every algebra $\mathcal{A}$ [3, Proposition 5.3], to prove $\sigma_e(A) = \sigma_e(T)$ we have to prove only that $0 \in \sigma(\pi(A_1))$ if and only if $0 \in \sigma(\pi(T_1))$. Furthermore, since $i(A_1 - \lambda) = i(T_1 - \lambda)$ for all λ, and both A_1 and T_1 are injective, it will suffice to prove that ranA_1 is not closed if and only if ranT_1 is not closed. Assume that $0 \in \sigma(\pi(A_1))$. Then ranA_1 is not closed. We assert that ranT_1 is not closed. For if ranT_1 is closed, then T_1 is left invertible, which (since $||T_1 x|| \leq |||\tilde{A}_1|^{\frac{1}{2}}||||\tilde{A}_1|^{\frac{1}{2}} x||$ for all $x \in H_1$) implies that $|\tilde{A}_1|$ is invertible. Consequently, $\tilde{A}_1$ is left invertible, which (in turn) implies that A_1 is left invertible. Hence ranA_1 is closed: a contradiction. A similar argument shows that ranT_1 is not closed implies that ranA_1 is not closed, and the proof is complete.

3. Applications

In this section we apply Theorem 1 to Weyl's theorem and the spectral continuity. In the following we assume (without explicitly saying so) that the corresponding results are known to hold for hyponormal operators. Let $A \in \mathcal{H}(p)$ or $\mathcal{L}$, and let T be the associated hyponormal operator. It is then clear from Theorem 1 that

$$\sigma_w(A) = \sigma_w(T) = \sigma(T) - \sigma_{oo}(T) = \sigma(A) - \sigma_{oo}(A),$$

i.e., A satisfies Weyl's theorem. Again, if the function f is analytic on an open neighborhood of $\sigma(A)$, then

$$\sigma_w(f(A)) = \sigma(f(A)) - \sigma_{oo}(f(A)) = f(\sigma_w(A))$$

and $f(A)$ satisfies Weyl's theorem. We have

Corollary 2.([4, 10]) *If $A \in \mathcal{H}(p)$ or $\mathcal{L}$, then A satisfies Weyl's theorem. Furthermore, if the function f is analytic on an open neighborhood of $\sigma(A)$, then $f(A)$ satisfies Weyl's theorem.*

Let $P_1(A) = \{\lambda \in \mathbb{C} : A - \lambda$ is semi-Fredholm with $i(A - \lambda) \neq 0\}$. Then since $i(A - \lambda) = i(T - \lambda)$ for all $\lambda \in \mathbb{C}$, we have $P_1(A) = P_1(T)$.

Recall from [7] that the function "σ" is continuous at the points $B \in B(H)$ if and only if, for each $\lambda \in \sigma(B) \setminus \overline{P_1(B)}$ and $\epsilon > 0$, the ϵ-neighborhood of λ contains a component of

$$\sigma^\circ(B) = \sigma_{oo}(B) \cup [\sigma_{re}(B) \cap \sigma_{\ell e}(B)].$$

Let $\lambda \in \sigma(A) \setminus \overline{P_1(A)} = \sigma(T) \setminus \overline{P_1(T)}$. Since hyponormal operators are points of continuity of σ [14], given $\epsilon > 0$ and $\lambda \in \sigma(T) \setminus \overline{P_1(T)}$, the ϵ-neighborhood of λ contains a component of $\sigma^\circ(T) = \sigma^\circ(A)$. We have proved:

COROLLARY 3.([6, 8, 11, 13]) *If $A \in \mathcal{H}(p)$ or $\mathcal{L}$, then A is a point of continuity of σ.*

More is true. The conclusion that A satisfies Weyl's theorem implies that $\sigma_b(A) = \sigma_w(A)$ and that $\sigma_{oo}(A)$ coincides with the set of normal eigenvalues, denoted $\sigma_n(A)$, of A. Hence

$$\begin{aligned}\sigma_e(A) \cap \overline{\sigma_n(A)} &= \sigma_e(A) \cap \overline{\sigma_{oo}(A)} \\ &\subseteq \sigma_w(A) \cap \overline{\sigma_{oo}(A)} \\ &= (\sigma(A) \setminus \sigma_{oo}(A)) \cap \overline{\sigma_{oo}(A)} \\ &\subset \Gamma_{oe}.\end{aligned}$$

Since (already) A is a point of continuity of σ, [2, Theorem 14.17] implies the following

COROLLARY 4. *If $A \in \mathcal{H}(p)$ or $\mathcal{L}$, then A is a point of continuity of $\sigma_e, \sigma_b, \sigma_w$.*

The operator $B \in B(H)$ is said to satisfy *a-Weyl's theorem* if

$$\sigma_{\ell e}(B) = \sigma_a(B) \setminus \sigma_{oo}^a(B).$$

In general, it is true that

$$\text{a-Weyl's theorem} \implies \text{Weyl's theorem.}$$

Let $A^* \in \mathcal{H}(p)$ or $\mathcal{L}$, and let T^* be the associated hyponormal operator. Then

$$\sigma_{oo}(A) = \sigma_{oo}^a(A) = \sigma_{oo}^a(T),$$

and

$$\begin{aligned}\sigma_{\ell e}(A) &= \sigma_{re}(A^*)^* = \sigma_{re}(T^*)^* = \sigma_{\ell e}(T) \\ &= \sigma_a(T) \setminus \sigma_{oo}^a(T) = \sigma_a(A) \setminus \sigma_{oo}^a(A).\end{aligned}$$

Thus we have

COROLLARY 5.([6, 8, 11]) *If $A^* \in \mathcal{H}(p)$ or $\mathcal{L}$, then A satisfies a-Weyl's theorem. Furthermore, A is a point of continuity of $\sigma_{\ell e}$.*

Proof. As already seen, A satisfies a-Weyl's theorem. Recall that hyponormal operators are points of continuity of $\sigma_{\ell e}$. Hence (see [2, Theorem 14.24])

(i) $\rho_{SF}^{-\infty}(T) = \text{interior}\overline{[\rho_{SF}^{-\infty}(T)]}$,

(ii) $\sigma_e(T) = \text{boundary}(\rho_{SF}^{\pm}(T)) \cup \rho_{SF}^{\pm\infty}(T) \cup \overline{\Gamma_{oe}(T)}$,

(iii) if $\lambda \in \sigma_{\ell re}(T)$ is an interior point of $\overline{[\rho_{SF}^{n}(T)]}$ for some non-zero integer n, then $\lambda \in \overline{\Gamma_{oe}(T)}$. Since

$$\rho_{SF}^{\pm\infty}(T) = \rho_{SF}^{\pm\infty}(A),\ \rho_{SF}^{\pm}(T) = \rho_{SF}^{\pm}(A) \text{ and } \Gamma_{oe}(T) = \Gamma_{oe}(A),$$

conditions (i), (ii) and (iii) are satisfied with T replaced by A. Hence by [2, Theorem 14.24], A is a point of continuity of $\sigma_{\ell e}$. □

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BHAGWATI P. DUGGAL, 8 REDWOOD GROVE, NORTHFIELDS AVENUE EALING, LONDON W5 4SZ, UNITED KINGDOM
E-mail: bpduggal@yahoo.co.uk

IN HO JEON, DEPARTMENT OF MATHEMATICS, EWHA WOMEN'S UNIVERSITY, SEOUL 120-750, KOREA
E-mail: jih@math.ewha.ac.kr
Recent Address: Department of Mathematics, Seoul National University, Seoul 151-747, Korea