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SOME PROPERTIES OF THE AP-DENJOY INTEGRAL

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ABSTRACT. In this paper, we define the ap-Denjoy integral and investigate some properties of the ap-Denjoy integral.

1. Introduction

For a measurable set E of real numbers, we denote by $|E|$ its Lebesgue measure. Let E be a measurable set and let c be a real number. The *density* of E at c is defined by

$$d_cE = \lim_{h\to 0^+} \frac{\big|(E \cap (c-h, c+h)\big|}{2h},$$

provided the limit exists. The point c is called a *point of density* of E if $d_cE = 1$ and a *point of dispersion* of E if $d_cE = 0$. The set E^d represents the set of all points $x \in E$ such that x is a point of density of E.

A function $F : [a,b] \to \mathbb{R}$ is said to be *approximately differentiable* at $c \in [a,b]$ if there exists a measurable set $E \subseteq [a,b]$ such that $c \in E^d$ and $\lim_{\substack{x\to c\\ x\in E}} \frac{F(x)-F(c)}{x-c}$ exists. The approximate derivative of F at c is denoted by $F'_{ap}(c)$.

An *approximate neighborhood* (or ap-nbd) of $x \in [a,b]$ is a measurable set $S_x \subseteq [a,b]$ containing x as a point of density. For every $x \in E \subseteq [a,b]$, choose an ap-nbd $S_x \subseteq [a,b]$ of x. Then we say that $S = \{S_x : x \in E\}$ is a *choice* on E. A tagged interval $(x, [c,d])$ is said to be *subordinate* to the choice $S = \{S_x\}$ if $c, d \in S_x$. Let $\mathcal{P} = \{(x_i, [c_i, d_i]) : 1 \le i \le n\}$ be a finite collection of non-overlapping tagged intervals. If $(x_i, [c_i, d_i])$ is subordinate to a choice S for each i, then we say that $\mathcal{P}$ is subordinate

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to S. If $\mathcal{P}$ is subordinate to S and $[a,b] = \bigcup_{i=1}^{n}[c_i, d_i]$, then we say that $\mathcal{P}$ is a tagged partition of $[a,b]$ that is subordinate to S.

2. The ap-Denjoy and ap-Henstock integrals

We introduce the notion of the approximate Lusin function. This function is used to define the ap-Denjoy integral.

For a function $F : [a,b] \to \mathbb{R}$, F can be treated as a function of intervals by defining $F([c,d]) = F(d) - F(c)$.

DEFINITION 2.1. Let $F : [a,b] \to \mathbb{R}$ be a function. The function F is an approximate Lusin function(or F is an AL function) on $[a,b]$ if for every measurable set $E \subseteq [a,b]$ of measure zero and for every $\varepsilon > 0$ there exists a choice S on E such that $|(\mathcal{P})\sum F(I)| < \varepsilon$ for every finite collection $\mathcal{P}$ of non-overlapping tagged intervals that is subordinate to S.

Recall that $F : [a,b] \to \mathbb{R}$ is AC_s on a measurable set $E \subseteq [a,b]$ if for each $\varepsilon > 0$ there exist a positive number η and a choice S on E such that $|(\mathcal{P})\sum F(I)| < \varepsilon$ for every finite collection $\mathcal{P}$ of non-overlapping tagged intervals that is subordinate to S and satisfies $(\mathcal{P})\sum |I| < \eta$. The function F is ACG_s on E if E can be expressed as a countable union of measurable sets on each of which F is AC_s.

LEMMA 2.2. *If $F : [a,b] \to \mathbb{R}$ is ACG_s on $[a,b]$, then F is an AL function on $[a,b]$.*

Proof. Suppose that $E \subseteq [a,b]$ is a measurable set of measure zero. Let $E = \bigcup_{n=1}^{\infty} E_n$, where $\{E_n\}$ is a sequence of disjoint measurable sets and F is AC_s on each E_n. Let $\varepsilon > 0$. For each n, there exist a choice $S^n = \{S_x^n : x \in E_n\}$ on E_n and a positive number η_n such that $|(\mathcal{P})\sum F(I)| < \epsilon/2^n$ whenever $\mathcal{P}$ is subordinate to S^n and $(\mathcal{P})\sum |I| < \eta_n$. For each n, choose an open set O_n such that $E_n \subseteq O_n$ and $|O_n| < \eta_n$. Let $S_x = S_x^n \cap O_n$ for each $x \in E_n$. Then $S = \{S_x : x \in E\}$ is a choice on E. Suppose that $\mathcal{P}$ is subordinate to S. Let $\mathcal{P}_n$ be a subset of $\mathcal{P}$ that has tags in E_n and note that $(\mathcal{P}_n)\sum |I| < |O_n| < \eta_n$. Hence

$$\left|(\mathcal{P})\sum F(I)\right| \leq \sum_{n=1}^{\infty} \left|(\mathcal{P}_n)\sum F(I)\right| < \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon. \qquad \square$$

DEFINITION 2.3. A function $f : [a,b] \to \mathbb{R}$ is *ap-Denjoy integrable* on $[a,b]$ if there exists an AL function F on $[a,b]$ such that F is approximately differentiable a.e. on $[a,b]$ and $F'_{ap} = f$ a.e. on $[a,b]$. The function f is ap-Denjoy integrable on a measurable set $E \subseteq [a,b]$ if $f\chi_E$ is ap-Denjoy integrable on $[a,b]$.

If we add the condition $F(a) = 0$, then the function F is unique. We will denote this function $F(x)$ by $(AD)\int_a^x f$.

It is easy to show that if $f : [a,b] \to \mathbb{R}$ is ap-Denjoy integrable on $[a,b]$, then f is ap-Denjoy integrable on every subinterval of $[a,b]$. This gives rise to an interval function F such that $F(I) = (AD)\int_I f$ for every subinterval $I \subseteq [a,b]$. The function F is called the primitive of f.

Recall that $F : [a,b] \to \mathbb{R}$ is AC_* on a measurable set $E \subseteq [a,b]$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum_{i=1}^n \omega(F,[c_i,d_i]) < \varepsilon$ whenever $\{[c_i,d_i] : 1 \leq i \leq n\}$ is a finite collection of non-overlapping intervals that have endpoints in E and satisfy $\sum_{i=1}^n (d_i - c_i) < \delta$, where $\omega(F,[c_i,d_i]) = \sup\{|F(y) - F(x)| : c_i \leq x < y \leq d_i\}$. The function F is ACG_* on E if $F|_E$ is continuous on E, $E = \bigcup_{n=1}^\infty E_n$ and F is AC_* on each E_n. It is easy to show that is F is ACG_* on $[a,b]$, then F is ACG_s on $[a,b]$. A function $f : [a,b] \to \mathbb{R}$ is Denjoy integrable on $[a,b]$ if there exists an ACG_* function $F : [a,b] \to \mathbb{R}$ such that $F' = f$ almost everywhere on $[a,b]$.

The following theorem shows that the ap-Denjoy integral is an extension of the Denjoy integral.

THEOREM 2.4. *If $f : [a,b] \to \mathbb{R}$ is Denjoy integrable on $[a,b]$, then f is ap-Denjoy integrable on $[a,b]$.*

Proof. Suppose that $f : [a,b] \to \mathbb{R}$ is Denjoy integrable on $[a,b]$. Then there exists an ACG_* function $F : [a,b] \to \mathbb{R}$ such that $F' = f$ almost everywhere on $[a,b]$. Since F is ACG_s on $[a,b]$, by Lemma 2.2 F is an AL function on $[a,b]$ and $F'_{ap} = F' = f$ almost everywhere on $[a,b]$. Hence f is ap-Denjoy integrable on $[a,b]$ □

Recall that a function $f : [a,b] \to \mathbb{R}$ is ap-Henstock integrable on $[a,b]$ if there exists a real number A with the following property; for each $\epsilon > 0$ there exists a choice S on $[a,b]$ such that $\left|(\mathcal{P})\sum f(x)|I| - A\right| < \epsilon$ whenever $\mathcal{P} = \{(x,I) : x \in [a,b]\}$ is a tagged partition of $[a,b]$ that is subordinate to S. A function $f : [a,b] \to \mathbb{R}$ is Khintchine integrable on $[a,b]$ if there exists an ACG function $F : [a,b] \to \mathbb{R}$ such that $F'_{ap} = f$ a.e. $[a,b]$.

There exists a function that is ap-Denjoy integrable on $[a,b]$, but not Denjoy integrable on $[a,b]$.

EXAMPLE 2.5. Let $\{(a_n, b_n)\}$ be a sequence of disjoint open intervals in (a, b) with the following properties:

(1) $b_1 < b$ and $b_{n+1} < b_n$ for all n;
(2) $\{a_n\}$ converges to a;
(3) a is a point of dispersion of $O = \bigcup_{n=1}^{\infty}(a_n, b_n)$.

Define $F : [a, b] \to \mathbb{R}$ by $F(x) = 0$ for all $x \in [a, b] - O$ and

$$F(x) = \sin^2\left(\frac{x - a_n}{b_n - a_n}\pi\right)$$

for $x \in (a_n, b_n)$. Then it is easy to show that F is approximately differentiable on $[a, b]$. It is well-known [4] that the approximate derivative F'_{ap} is ap-Henstock integrable but not Khintchine integrable on $[a, b]$. By [4, Lemma 16.17 and Theorem 16.18], F is an AL function and hence F'_{ap} is ap-Denjoy integrable on $[a, b]$. But F'_{ap} is not Denjoy integrable on $[a, b]$, since every Denjoy integrable function is Khintchine integrable.

THEOREM 2.6. *Let $f : [a, b] \to \mathbb{R}$ be ap-Denjoy integrable on $[a, b]$ and let $F(x) = (AD) \int_a^x f$ for each $x \in [a, b]$. Then*

(a) *the function F is approximately differentiable* a.e. *on $[a, b]$ and $F'_{ap} = f$* a.e. *on $[a, b]$; and*
(b) *the functions F and f are measurable.*

Proof. (a) follows from the definition of the ap-Denjoy integral. Since F is approximately continuous a.e. on $[a, b]$, F is measurable by [4, Theorem 14.7]. It follows from [4, Theorem 14.12] that f is measurable. □

THEOREM 2.7. *Let $F : [a, b] \to \mathbb{R}$ be an AL function on $[a, b]$. If F is approximately differentiable* a.e. *on $[a, b]$, then F'_{ap} is ap-Denjoy integrable on $[a, b]$ and $(AD) \int_a^x F'_{ap} = F(x) - F(a)$ for each $x \in [a, b]$.*

Proof. Suppose that F is an AL function on $[a, b]$ and F is approximately differentiable a.e. on $[a, b]$. Then for every constant C, $F + C$ is also an AL function on $[a, b]$, approximately differentiable a.e. on $[a, b]$ and $(F + C)'_{ap} = F'_{ap}$ a.e. on $[a, b]$. Hence F'_{ap} is ap-Denjoy integrable on $[a, b]$ and

$$F(x) + C = (AD) \int_a^x F'_{ap} \quad \text{for each } x \in [a, b].$$

Since $F(a) + C = 0$, $C = -F(a)$ and

$$(AD) \int_a^x F'_{ap} = F(x) - F(a) \quad \text{for each } x \in [a, b].$$

□

We can easily show that if f is ap-Denjoy integrable on each of intervals $[a,c]$ and $[c,b]$, then f is ap-Denjoy integrable on $[a,b]$ and

$$(AD)\int_a^b f = (AD)\int_a^c f + (AD)\int_c^b f.$$

THEOREM 2.8. *Suppose that $f : [a,b] \to \mathbb{R}$ is ap-Denjoy integrable on each subinterval $[c,d] \subseteq (a,b)$. If $(AD)\int_c^d f$ converges to a finite limit as $c \to a^+$ and $d \to b^-$, then f is ap-Denjoy integrable on $[a,b]$ and $(AD)\int_a^b f = \lim_{\substack{c\to a^+\\ d\to b^-}} (AD)\int_c^d f$.*

Proof. Choose a point $p \in (a,b)$ and fix it. First, we will prove that if f is ap-Denjoy integrable on $[p,d]$ for each $d \in (p,b)$ and $(AD)\int_p^d f$ converges to a finite limit as $d \to b^-$, then f is ap-Denjoy integrable on $[p,b]$ and $(AD)\int_p^b f = \lim_{d\to b^-} (AD)\int_p^d f$.

Let $L = \lim_{d\to b^-} (AD)\int_p^d f$, let $a_0 = p$ and $\{a_k\}$ be an increasing sequence in (p,b) that converges to b. Define a function $F : [p,b] \to \mathbb{R}$ by

$$F(x) = F_i(x) \quad \text{if } x \in [a_{i-1}, a_i] \quad \text{for each } i = 1,2,3,\cdots$$

and $F(b) = L$, where F_i is the primitive of f on $[a_{i-1}, a_i]$ and $F_i(a_{i-1}) = 0$ for each i. Since each F_i is an AL function on $[a_{i-1}, a_i]$ such that F_i is approximately differentiable a.e. on $[a_{i-1}, a_i]$ and $(F_i)'_{ap} = f$ a.e. on $[a_{i-1}, a_i]$, the function F is an AL function on $[p,b]$ such that F is approximately differentiable a.e. on $[p,b]$ and $F'_{ap} = f$ a.e. on $[p,b]$. Hence, f is ap-Denjoy integrable on $[p,b]$ and

$$(AD)\int_p^b f = F(b) = L = \lim_{d\to b^-} (AD)\int_p^d f.$$

Similarly, we can prove that if f is ap-Denjoy integrable on $[c,p]$ for each $c \in (a,p)$ and $(AD)\int_c^p f$ converges to a finite limit as $c \to a^+$, then f is ap-Denjoy integrable on $[a,p]$ and $(AD)\int_a^p f = \lim_{c\to a^+} (AD)\int_c^p f$.

If $(AD)\int_c^d f$ converges to a finite limit as $c \to a^+$ and $d \to b^-$, then for any $p \in (a,b)$ $(AD)\int_c^p f$ converges to a finite limit as $c \to a^+$ and $(AD)\int_p^d f$ converges to a finite limit as $d \to b^-$. By the proof of the previous parts, f is ap-Denjoy integrable on $[a,p] \cup [p,b] = [a,b]$ and

$$\begin{aligned}(AD)\int_a^b f &= (AD)\int_a^p f + (AD)\int_p^b f \\ &= \lim_{c\to a^+}(AD)\int_c^p f + \lim_{d\to b^-}(AD)\int_p^d f \\ &= \lim_{\substack{c\to a^+ \\ d\to b^-}}(AD)\int_c^d f. \end{aligned}$$ □

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