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WHEN IS THE CLASSIFYING SPACE FOR ELLIPTIC FIBRATIONS RANK ONE?

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Abstract. We give a necessary and sufficient condition of a rationally elliptic space X such that the Dold-Lashof classifying space $\mathrm{Baut}_1 X$ for fibrations with the fiber X is rank one. It is only when X has the rational homotopy type of a sphere or the total space of a spherical fibration over a product of spheres.

1. Introduction

Let X be a simply connected CW complex of finite type and $\mathrm{Baut}_1 X$ the identity component of the space of self-homotopy equivalences of X. The Dold-Lashof classifying space, $\mathrm{Baut}_1 X$, is the classifying space for orientable fibrations with fiber the homotopy type of X ([1]). Recall that X is said to be (rationally) elliptic if the dimensions of rational cohomology $H^*(X;Q)$ and homotopy $\pi_*(X)\otimes Q$ are finite. We denote $X \simeq_0 Y$ if X is rationally homotopic to Y. Let $M(X)$ be the Sullivan minimal model of X ([2], [7]). By explicit calculations of derivations of $M(X)$, we see

Theorem 1.1. *For elliptic spaces X, rank $\pi_*(\mathrm{Baut}_1 X) = 1$ if and only if $X \simeq_0 S^m$, the m-dimensinal sphere, or $M(X) \cong (\Lambda(x_1, \cdots, x_n, y_1, \cdots, y_n, v), d)$, where $dx_i = dy_i = 0$, $dv = \sum_{i=1}^n x_i y_i$ with $\deg x_i = \deg y_i$ for $i = 1, .., n$ and they are oddly generated.*

Note that $\mathrm{Baut}_1 X \simeq_0 K(Q, 2n)$, $K(Q, 4n)$ and $K(Q, \deg v + 1)$ if $X \simeq_0 S^{2n-1}$, S^{2n} and $M(X) \cong (\Lambda(x_1, \cdots, y_n, v), d)$ of above, respectively. Here $K(Q, n)$ is the Eilenberg-Maclane space and a spatial realization of $(\Lambda(x_1, \cdots, y_n, v), d)$ is the total space of a fibration

$$S^{\deg v} = S^{4m+1} \to E \to (S^{2m+1})^{\times 2n},$$

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where $\deg x_i = \deg y_i = 2m+1$. See [4, Theorem 2.3] for a property of such spaces.

2. Preliminary

The simply connected minimal model $M = M(X)$ is given by a free Q-commutative ($xy = (-1)^{\deg x \cdot \deg y} yx$) differential graded algebra $(\Lambda V, d)$ with a graded Q-vector space $V = \bigoplus_{i>1} V^i$ and a minimal differential d, i.e., $d(V^i) \subset (\Lambda^+ V \cdot \Lambda^+ V)^{i+1}$. In this paper, we assume the normality condition: $\ker(d|_V) = \ker[d|_V]$. For example, if $dx = 0$, $dv = x^2$ and $dw = x^3$ in $M = (\Lambda(x,v,w),d)$ with $\deg x$ even, we take the normal model $M' = (\Lambda(x,v,w),d)$ with $dx = 0$, $dv = x^2$ and $dw = 0$, which is isomorphic to M.

Let $\mathrm{Der}_i M$ be the set of Q-derivations of M decreasing the degree by i. They are linear self-maps σ of M satisfying $\sigma(M^j) \subset M^{j-i}$ and $\sigma(xy) = \sigma(x)y + (-1)^{i \deg x} x\sigma(y)$. The boundary operator $\partial : \mathrm{Der}_i M \to \mathrm{Der}_{i-1} M$ is defined by $\partial(\sigma) = d \circ \sigma - (-1)^i \sigma \circ d$. Then $\partial \circ \partial = 0$. We denote $\bigoplus_{i>0} \mathrm{Der}_i M$ by $\mathrm{Der}M$. It is known that $\pi_{*+1}(\mathrm{Baut}_1 X) \otimes Q \cong H_*(\mathrm{Der}M(X), \partial)$ ([3, 7]).

In rational homotopy theory, the category of rational homotopy types of simply connected spaces and the category of Q-commutative differential graded algebras A^* with $A^1 = 0$ are equivalent [7]. Refer [2] for a general introduction and notations. Especially note that $V^i \cong \mathrm{Hom}(\pi_i(Y), Q)$ and $H^*(M(X)) \cong H^*(X;Q)$. So our title is translated to "*When* $\dim H_*(\mathrm{Der}M) = 1$ *for* $M = (\Lambda(v_1, \cdots, v_n), d)$ *with* $\dim H^*(M) < \infty$? ".

In the followings, $\langle * \rangle$ is the ideal of ΛV generated by $*$, and put the derivation which send an element v of V to an element f of ΛV and the other generators to zero as (v, f). For example, for $M = (\Lambda(x,y,z),d)$ with $dx = dy = 0$ and $dz = xy$, we have from definitions $\partial(y,1) = (-1)^{\deg y(1+\deg x)+1}(z,x)$ and $\partial(z,1) = 0$. If $\deg x < \deg y$ and they are odd, (y,x) is a non exact ∂-cycle too.

3. Proof

LEMMA 3.1. *Let* $M = (\Lambda V, d)$ *be the minimal model with* $\dim V < \infty$ *and* $\dim H^*(M) < \infty$. *Then for an element* a *of* V *with* $\deg a$ *even, there is* $u \in V$ *such that* $du = a^n + h \neq 0$ *for some* $h \in \Lambda V$.

Proof. Let $(\Lambda V, d) \to (\Lambda V/\langle V^{<\deg a}\rangle, \overline{d}) = (\Lambda V^{\geq \deg a}, \overline{d})$ be the natural projection. Then a is a $\overline{d}$-cocycle. Since $\dim H^*(\Lambda V^{\geq \deg a}, \overline{d}) < \infty$ [5, p.183 Corollary], there is a minimal integer m such that $[a^m] = 0$ in it. Then there is an element $u \in V$ such that $\overline{d}u = a^n + g \neq 0$ for some $g \in \Lambda V^{\geq a}$ $(n \leq m)$. Note $h = g + g'$ for some $g' \in \langle V^{<\deg a}\rangle$. □

Proof of Theorem 1.1. We consider the condition of $M = M(X) = (\Lambda V, d)$ such that

(1) $$\dim H_*(\mathrm{Der} M, \partial) = 1.$$

Let $V_0 = \{v \in V \mid dv = 0\}$, $s = \min V = \min\{i \mid V^i \neq 0\}$ and $t = \max V = \max\{i \mid V^i \neq 0\}$, which is finite since X is elliptic. Take an element v from V^t. Then the derivation $(v, 1)$ is a non-exact ∂-cycle. From (1), $V^t = Q\{v\}$. We fix this v. Since $\mathrm{Baut}_1 X$ is simply connected, (1) is equivalent to

$$H_*(\mathrm{Der} M, \partial) = Q\{(v,1)\}.$$

Note $\deg v = t$ is odd since $\dim \Lambda v = \dim H^*(\Lambda V/\langle V^{<t}\rangle, \overline{d}) < \infty$ from [5, p.183 Corollary]. We give the proof by dividing s into two cases.

• WHEN s IS ODD.

If $\dim V_0 = 1$, then $M(X) = (\Lambda x, 0)$ for an element x of $\deg x = 2n+1$, i.e., $X \simeq_0 S^{2n+1}$. If $\dim V_0 > 1$ and $d = 0$, then $\dim H_*(\mathrm{Der} M) > 1$. For both $(x, 1)$ and $(y, 1)$ are non exact ∂-cycles, which are not homologous, for some linearly independent elements x and y of V_0.

Let $\dim V_0 > 1$ and $d \neq 0$. Take an element x_1 form ${V_0}^s$. Since the ∂-cycle (v, x_1) must be ∂-exact from (1), there is an element y_1 of V $(y_1 \notin Q\{x_1\})$ such that

(2) $$dv = x_1 y_1 + f$$

with $f \in \Lambda^+ V \cdot \Lambda^+ V$ and

(3) $$\partial(y_1, 1) = -(v, x_1).$$

Suppose that $\deg y_1 > s$. Then there is the ∂-cycle (y_1, x_1) from (3). Since it must be exact from (1), there is an element z of V such that $dy_1 = x_1 z + g$ with $g \in \Lambda V$ and

(4) $$\partial(z, 1) = -(y_1, x_1).$$

Note $s \leq \deg z < \deg y_1$. If $\deg z = s$, $z \in V_0$. Then (v, z) is a non-exact cycle since (4) does not contain the term $(v, *)$, and it contradicts (1). Even if $\deg z > s$, there is the ∂-cycle (z, x_1) from (4). By iterating this argument, we reach an element w of ${V_0}^s$ such that (v, w) is a non-exact

∂-cycle. Thus the assumption is false, i.e., $\deg y_1 = s$ and $y_1 \in V_0$. From the argument of degree,

$$f = \sum_{i=2}^{n} x_i y_i \in \Lambda V_0$$

for a set $S = \{x_1, \cdots, x_n, y_1, \cdots, y_n\} \subset {V_0}^{\mathrm{odd}}$, where $\deg x_i = \deg y_i = s$ and $Q\{x_i\} \neq Q\{y_i\}$ for $i = 1, .., n$. Since

$$(5) \quad \partial \left(\sum_{i=1}^{n} \lambda_i (x_i, 1) + \sum_{i=1}^{n} \mu_i (y_i, 1) \right) = \left(v, - \sum_{i=1}^{n} \lambda_i y_i - \sum_{i=1}^{n} \mu_i x_i \right)$$

for $\{\lambda_i, \mu_i\}_i \subset Q$, S must be linearly independent in V_0 from (1). For if $\sum_i \lambda_i(x_i, 1) + \sum_i \mu_i(y_i, 1)$ for some $\{\lambda_i, \mu_i\}_i$ is a ∂-cycle, it can not be exact. If $\dim {V_0}^{\mathrm{odd}} > 2n$, there is an element $z \in {V_0}^{\mathrm{odd}} - Q\{S\}$ such that (v, z) is a non exact ∂-cycle from (5). So ${V_0}^{\mathrm{odd}} = Q\{S\}$. If there is a non zero element a in ${V_0}^{\mathrm{even}}$, $\deg a > s$. Since $t + 1 = 2s < 2 \deg a$, $[a^n] \neq 0$ for any n. From $\dim H^*(M) < \infty$, ${V_0}^{\mathrm{even}} = 0$. Hence we have $V_0 = Q\{S\}$ and $M(X) \cong (\Lambda(x_1, \cdots, x_n, y_1, \cdots, y_n, v), d)$.

• WHEN s IS EVEN.

Take an element x from ${V_0}^s$. Since the ∂-cycle (v, x) must be exact from (1), there is an element a of V such that

$$\partial(a, 1) = -(v, x)$$

and $dv = xa + f$ for some $f \in \Lambda^+ V \cdot \Lambda^+ V$. Then $\deg a$ is even, and there is an element u of V and an integer $m > 1$ such that

$$du = a^m + h \neq 0$$

for some $h \in \Lambda V$ from Lemma 3.1. If $\deg a > s$, we have $t < \deg u$. It contradicts $\max V$. Therefore $\deg a = s$, i.e., $a \in V_0$. Since $V = V_0 \oplus Q\{v\}$ from $t = 2s - 1$, we have $\dim {V_0}^{\mathrm{even}} = 1$. So $a = \lambda x$ for some $\lambda \in Q - 0$ and $f = 0$.

If there is a non zero element y in ${V_0}^{\mathrm{odd}}$, the derivation (v, y) is a non exact ∂-cycle since $t + 1 = 2s < s + \deg y$. Thus ${V_0}^{\mathrm{odd}} = 0$. Hence we have $M(X) \cong (\Lambda(x, v), d)$ with $dx = 0$ and $dv = x^2$ with $\deg x = s = 2n$, i.e., $X \simeq_0 S^{2n}$ □

QUESTION : When is $H^*(\mathrm{Baut}_1 X; Q)$ free?

When rank $\pi_*(\mathrm{Baut}_1 X) \leq 2$, $H^*(\mathrm{Baut}_1 X; Q)$ is a free graded algebra. See [6] for certain cases.

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