Bull. Korean Math. Soc. **42** (2005), No. 3, pp. 509–520

REMARKS ON THE MINIMIZER OF A p-GINZBURG-LANDAU TYPE

YUTIAN LEI

ABSTRACT. The author studies the asymptotic behavior of the radial minimizer for a variant of the p-Ginzburg-Landau type functional, in the case of p larger than the dimension, when the parameter tends to zero. The $C^{1,\alpha}$ convergence of the radial minimizer is proved. And the estimation of the convergent rate of the minimizer is given.

1. Introduction

Let $G \subset R^n (n \geq 2)$ be a bounded and simply connected domain with smooth boundary ∂G. $g(x) : \partial G \to S^1$ is smooth map satisfying $d = \deg(g, \partial G)$. When $n = 2$, many papers studied the asymptotic behavior of minimizer u_ε of the Ginzburg-Landau functional

$$E_\varepsilon^1(u) = \frac{1}{2}\int_G |\nabla u|^2 dx + \frac{1}{4\varepsilon^2}\int_G (1 - |u|^2)^2 dx$$

on $H_g^1(G, R^2)$ as $\varepsilon \to 0$. In particular, some subsequence u_{ε_k} of the minimizer u_ε satisfies

$$\lim_{k\to\infty} u_{\varepsilon_k} = u_* \quad \text{in} \quad C_{loc}^{1,\alpha}(\overline{G} \setminus A), \tag{1.1}$$

where $\alpha \in (0, 1)$, u_* is a harmonic map (see [1]). Here A is the set of the singularities of u_*. The papers [2] and [5] presented the properties of the radial minimizer of $E_\varepsilon^1(u)$ in the function class

$$V = \left\{ u(x) = f(r)\frac{x}{|x|} \in H^1(B, R^2)\,;\, f(1) = 1,\, r = |x| \right\},$$

Received March 4, 2004. Revised June 21, 2005.
2000 Mathematics Subject Classification: 35J65, 49K20.
Key words and phrases: p-Ginzburg-Landau functional, radial minimizer, asymptotic behavior.

where $B = \{x \in R^2; |x| < 1\}$. The asymptotic behavior of the minimizer in $H_g^1(G, R^2)$ of the Ginzburg-Landau type functional

$$E_\varepsilon^2(u) = \frac{1}{2}\int_G |\nabla u|^2 dx + \frac{1}{4\varepsilon^2}\int_G |u|^2(1-|u|^2)^2 dx,$$

with the different penalization, was discussed extensively in [3]. The same result as (1.1) was also derived. Afterwards, it is researched asymptotic properties of the radial minimizer of

$$E_\varepsilon(u, B) = \frac{1}{p}\int_B |\nabla u|^p dx + \frac{1}{4\varepsilon^p}\int_B |u|^2(1-|u|^2)^2 dx, \quad (p > n)$$

in $W = \{u(x) = f(r)\frac{x}{|x|} \in W^{1,p}(B, R^n); f(1) = 1, r = |x|\}$, where $B = \{x \in R^n; |x| < 1\}$. The following properties have been proved (see Theorems 3.6 and 4.3 in [4]):

(1.2) The radial minimizer u_ε is unique as long as ε is sufficiently small,

and the convergence which is weaker than (1.1)

$$u_\varepsilon \to \frac{x}{|x|} \quad \text{in} \quad W_{loc}^{1,p}(\overline{B} \setminus \{0\}) \quad \text{as} \quad \varepsilon \to 0. \tag{1.3}$$

In this paper, we will prove that the radial minimizer of $E_\varepsilon(u, B)$ also satisfies the convergent property as (1.1). To do this, the $C^{1,\alpha}$ uniform estimation of the minimizer u_ε should be obtained. Indeed, it is difficult since the Euler-Lagrange equation, which the minimizer satisfies, is degenerate when $p > 2$. There may not be any classical solution to the equation. Hereby, we consider the regularized functional

$$E_\varepsilon^\tau(u, B) = \frac{1}{p}\int_B (|\nabla u|^2 + \tau)^{p/2} dx + \frac{1}{4\varepsilon^p}\int_B |u|^2(1-|u|^2)^2 dx, \quad (\tau \in (0,1)).$$

It is easy to see that the minimizer $u_\varepsilon^\tau(x) = f_\varepsilon^\tau(r)\frac{x}{|x|}$ exists in W. By the argument of the weak low semi-continuity we can deduce that

$$\lim_{\tau \to 0} u_\varepsilon^\tau = \tilde{u}_\varepsilon \quad \text{in} \quad W^{1,p}(B, R^n), \tag{1.4}$$

where $\tilde{u}_\varepsilon$ is a radial minimizer of $E_\varepsilon(u, B)$ in W. Noticing (1.2), we know that as ε is sufficiently small, the limit $\tilde{u}_\varepsilon$ must be the unique radial minimizer u_ε. Hence, we may derive the $C^{1,\alpha}$ convergence of the radial minimizer via establishing the $C^{1,\alpha}$ estimation of u_ε^τ (see Theorem 2.2). In addition, we also concern with the convergent rate of $|u_\varepsilon| \to 1$ in $W^{1,p}(B \setminus \{0\})$ when $\varepsilon \to 0$. In fact, if $T > 0$, we can obtain firstly that $E_\varepsilon(u_\varepsilon, B \setminus B_T(0)) - \frac{1}{p}\int_{B \setminus B_T(0)} |\nabla \frac{x}{|x|}|^p dx \le C\varepsilon^{[p]-p+1}$ (cf. Theorem 3.1).

Next, by improving the exponent $[p]-p+1$ of ε step by step, we can see at last the convergent rate in $W^{1,p}([T,1])$:

$$\int_T^1 r^{n-1}[(f_\varepsilon')^p+\frac{1}{\varepsilon^p}(1-f_\varepsilon^2)^2]dr \le C\varepsilon^p,$$

where $f_\varepsilon(r)=|u_\varepsilon(x)|$ (cf. Theorem 3.2).

2. $C^{1,\alpha}$ convergence

Since u_ε^τ is a minimizer, it is not difficult to see that $f_\varepsilon^\tau=|u_\varepsilon^\tau|$ solves

$$(2.1)\quad -\left(rA^{(p-2)/2}f_r\right)_r+\frac{n-1}{r}A^{(p-2)/2}f=\frac{1}{2\varepsilon^p}rf\left(4f^2-3f^4-1\right),$$

and $|f_\varepsilon^\tau|\le 1$ in $[0,1]$ by the maximum principle, where $A=f_r^2+\frac{(n-1)f^2}{r^2}+\tau$. By the same argument of (3.8) and (4.2) in [4], we also see that for any $T>0$, there exists $C>0$ which is independent of ε and τ, such that

$$(2.2)\qquad |f_\varepsilon^\tau|\ge 29/30 \quad\text{in}\quad [T,1],$$

$$(2.3)\qquad \int_T^1 A^{p/2}dr\le C.$$

PROPOSITION 2.1. *Denote $u_\varepsilon^\tau=u=f(r)\frac{x}{|x|}$. Then for any compact subset $K\subset(0,1]$, there exists a positive constant C which does not depend on ε and τ, such that $\|f\|_{C^{1,\beta}(K,R)}\le C,\quad \forall\beta\in(0,1/2)$.*

Proof. Take $T>0$ so small that $K_{3T}\subset\subset K\subset\subset K_{2T}\subset K_T=(T,1-T)$. Let $\zeta\in C_0^\infty(K_T,[0,1])$ satisfy that $\zeta=0$ on $[0,1]\setminus K_T$, $\zeta=1$ on K_{2T}, and $|\zeta_r|\le C(T)$ on $(0,1)$. Multiplying (2.1) with r^{-1} and differentiating, then multiplying by $f_r\zeta^2$ and integrating on $(0,1)$, we have

$$\begin{aligned}-\int_0^1\left(A^{(p-2)/2}f_r\right)_{rr}\left(f_r\zeta^2\right)dr-\int_0^1\left(r^{-1}A^{(p-2)/2}f_r\right)_r\left(f_r\zeta^2\right)dr&\\+(n-1)\int_0^1\left(r^{-2}A^{(p-2)/2}f\right)_r\left(f_r\zeta^2\right)dr&\\=\frac{1}{2\varepsilon^p}\int_0^1\left[f\left(4f^2-3f^4-1\right)\right]_r\left(f_r\zeta^2\right)dr.&\end{aligned}$$

Integrating by parts and noting

$$-\frac{1}{2\varepsilon^p}\int_{K_T}f^2(f_r)^2(12f^2-8)\zeta^2dr\le 0,$$

we get

$$\int_{K_T}\left(A^{\frac{p-2}{2}}f_r\right)_r(f_r\zeta^2)_r dr+\int_{K_T}A^{\frac{p-2}{2}}(f_r\zeta^2)_r\left[\frac{f_r}{r}-\frac{(n-1)f}{r^2}\right]dr \le \frac{1}{2\varepsilon^p}\int_{K_T}\left(4f^2-3f^4-1\right)f_r^2\zeta^2 dr.$$

Denote

$$I=\int_{K_T}\zeta^2\left(A^{(p-2)/2}f_{rr}^2+(p-2)A^{(p-4)/2}f_r^2f_{rr}^2\right)dr.$$

Noting

$$A_r=2\left[f_rf_{rr}+(n-1)(ff_rr^{-2}-r^{-3}f^2)\right],$$

and using Young inequality, we see that for any $\delta\in(0,1)$,

$$I\le\delta I+C(\delta,T)\int_{K_T}A^{p/2}\zeta_r^2dr+\frac{1}{2\varepsilon^p}\left|\int_{K_T}f_r^2(4f^2-3f^4-1)\zeta^2dr\right|. \tag{2.4}$$

From (2.1) and (2.2) and by Young inequality, it follows

$$\frac{1}{2\varepsilon^p}\left|\int_{K_T}(4f^2-3f^4-1)f_r^2\zeta^2dr\right|\le\delta I+C(\delta,T)\int_{K_T}A^{(p+2)/2}\zeta^2dr$$

with $\delta\in(0,1)$. Substituting this into (2.4) and choosing δ sufficiently small, we have

$$I\le C\int_{K_T}A^{p/2}\zeta_r^2dr+C\int_{K_T}A^{(p+2)/2}\zeta^2dr. \tag{2.5}$$

To estimate the second term of the right hand side, we take $\phi=\zeta^{2/q}f_r^{(p+2)/q}$ in the embedding inequality

$$\|\phi\|_{L^q}\le C\|\phi_r\|_{L^1}^{1-1/q}\|\phi\|_{L^1}^{1/q},\quad q\in(1+\frac{2}{p},2). \tag{2.6}$$

Applying Young inequality we see that for any $\delta\in(0,1)$,

$$\int_{K_T}f_r^{p+2}\zeta^2dr\le C\left(\int_{K_T}\zeta^{2/q}|f_r|^{(p+2)/q}dr\right)\times\left(\int_{K_T}\zeta^{2/q-1}|\zeta_r||f_r|^{(p+2)/q}+\delta I+C(\delta)\int_{K_T}A^{\frac{p+2}{q}-\frac{p}{2}}\zeta^{4/q-2}dr\right)^{q-1}. \tag{2.7}$$

Noting $q \in (1+\frac{2}{p}, 2)$, we can use the Holder inequality to estimate the right hand side of (2.7). Thus, from (2.3) it leads to $\int_{K_T} f_r^{p+2}\zeta^2 dr \leq \delta I + C(\delta)$ for any $\delta \in (0,1)$. Substituting this into (2.5) and choosing δ sufficiently small, we obtain $\int_{K_T} A^{(p-2)/2} f_{rr}^2 \zeta^2 dr \leq C$. Combining this with (2.3), we get $\|A^{p/4}\zeta\|_{H^1(K_T)} \leq C$. Noticing that $\zeta = 1$ on K, we see that $\|A^{p/4}\|_{H^1(K)} \leq C$. Applying the embedding inequality, we know that for any $\beta \leq 1/2$, $\|A^{p/4}\|_{C^\beta(K)} \leq C$. The inner estimation is set up.

In the following, we consider the estimation near the boundary point $r=1$. Denote $g(r) = f(r+1) - 1$. Set $\tilde{g}(r) = g(r)$ as $-1 \leq r \leq 0$, $\tilde{g}(r) = -g(-r)$ as $0 < r \leq 1$. If denote $f(r) = \tilde{g}(r-1) + 1$ in $[0,2]$, then $f(r)$ solves (2.1) in $[0,2]$. Take $R < \frac{1}{4}$. Assume that $\zeta \in C^\infty(0,1]$ satisfies $\zeta = 1$ when $r \geq 1 - R$; $\zeta = 0$ when $r \leq 2R$. Differentiating (2.1) and multiplying by $f_r\zeta^2$, then integrating on $[R,1]$, we have

$$\begin{aligned}-\int_R^1 \left(A^{(p-2)/2} f_r\right)_{rr} (f_r\zeta^2)\, dr - \int_R^1 \left(r^{-1} A^{(p-2)/2} f_r\right)_r (f_r\zeta^2) dr \\ + (n-1)\int_R^1 \left(r^{-2} A^{(p-2)/2} f\right)_r (f_r\zeta^2) dr \\ = \frac{1}{\varepsilon^p}\int_R^1 \left[f(1-f^2)(3f^2-1)\right]_r (f_r\zeta^2) dr.\end{aligned}$$

Integrating by parts leads to

$$\begin{aligned}&\int_R^1 \left(A^{(p-2)/2} f_r\right)_r (f_r\zeta^2)_r\, dr \\ &+ \int_R^1 A^{(p-2)/2}(f_r\zeta^2)_r \left[r^{-1} f_r - (n-1) r^{-2} f\right] dr \\ \leq\, & \frac{1}{\varepsilon^p}\int_R^1 (1-f^2)(3f^2-1) f_r^2 \zeta^2 dr \\ &- \frac{1}{\varepsilon^p}\int_R^1 f^2(12f^2-8)(f_r)^2\zeta^2 dr + |I(1) - I(R)|,\end{aligned}$$

where $I(r) = (A^{(p-2)/2} f_r)_r + \frac{1}{r} A^{(p-2)/2} f_r - (n-1)\frac{1}{r^2} A^{(p-2)/2} f] f_r \zeta^2$. From (2.1) it follows that $I(r) = \frac{1}{r\varepsilon^p} f(f^2-1)(3f^2-1) f_r\zeta^2$. Noting $f(1) = 1$ and $\zeta(R) = 0$, we obtain $I(1) = I(R) = 0$. Substituting this into the inequality above, and by the same argument as the inner estimation, we also derive (2.5). Now, take $\phi = \zeta^{2/q} f_r^{(p+2)/q}$ in the embedding inequality

$$\|\phi\|_{L^q} \le C(\|\phi_r\|_{L^1} + \|\phi\|_{L^1})^{1-1/q}\|\phi\|_{L^1}^{1/q}, \quad q \in (1+\frac{2}{p}, 2)$$

instead of (2.6) (in fact, (2.6) is not valid since $\phi \neq 0$ near $r = 1$). Thus, (2.7) can be still derived. The rest proof is same as the proof of the inner estimation.

THEOREM 2.2. *Let $u_\varepsilon = f_\varepsilon(r)\frac{x}{|x|}$ be a radial minimizer of $E_\varepsilon(u, B)$. Then for any compact subset $K \subset \overline{B}\backslash\{0\}$, $\lim_{\varepsilon\to 0} u_\varepsilon = \frac{x}{|x|}$ in $C^{1,\alpha}(K, R^n)$ for all $\alpha \in (0, 1/2)$.*

Proof. For any compact subset $K \subset \overline{B} \setminus \{0\}$, by using Proposition 2.1 we know that for some $\beta \in (0, 1/2)$,

$$\|u_\varepsilon^\tau\|_{C^{1,\beta}(K)} \le C = C(K) \tag{2.8}$$

with $C > 0$ independent of ε, τ. From this and the embedding theorem, we see that for some $\beta_1 < \beta$, there exist $w_\varepsilon^* \in C^{1,\beta_1}(K, R^n)$ and a subsequence τ_k of τ, such that as $k \to \infty$,

$$u_\varepsilon^{\tau_k} \to w_\varepsilon^* \quad \text{in} \quad C^{1,\beta_1}(K, R^n). \tag{2.9}$$

Combining this with (1.4) and (1.2), we have $w_\varepsilon^* = u_\varepsilon$.

Applying (2.8) and the embedding theorem again, we know that for some $\beta_2 < \beta$, there exist $w^* \in C^{1,\beta_2}(K, R^n)$ and a subsequence τ_m of τ_k, such that as $m \to \infty$,

$$u_{\varepsilon_m}^{\tau_m} \to w^* \quad \text{in} \quad C^{1,\beta_2}(K, R^n). \tag{2.10}$$

Set $\alpha = \min(\beta_1, \beta_2)$. Thus, when $m \to \infty$, using (2.9) and (2.10) we obtain

$$\begin{aligned} &\|u_{\varepsilon_m} - w^*\|_{C^{1,\alpha}(K,R^n)} \\ &\le \|u_{\varepsilon_m} - u_{\varepsilon_m}^{\tau_m}\|_{C^{1,\alpha}(K,R^n)} + \|u_{\varepsilon_m}^{\tau_m} - w^*\|_{C^{1,\alpha}(K,R^n)} \\ &\le o(1), \end{aligned} \tag{2.11}$$

which, together with (1.3), implies $w^* = \frac{x}{|x|}$. Noting the uniqueness of the limit $\frac{x}{|x|}$, we deduce that the convergence (2.11) holds not only for the subsequence, but also for the whole u_ε.

3. Analysis of the convergent rate

From (4.2) in [4], It was led to, for any compact subset of $K \subset (0,1]$, the convergent rate

$$\frac{1}{4\varepsilon^p}\int_K \left(1-f_\varepsilon^2\right)^2 r^{n-1}dr \le C. \tag{3.1}$$

In this section, we shall present the better rate.

THEOREM 3.1. *Let $u_\varepsilon(x) = f_\varepsilon(r)\frac{x}{|x|}$ be a radial minimizer of $E_\varepsilon(u,B)$. Then for any $T > 0$, there exists a constant $C > 0$ which is independent of ε, such that as $\varepsilon \to 0$,*

$$\int_T^1 |f_\varepsilon'|^p r^{n-1}dr + \frac{1}{\varepsilon^p}\int_T^1 f_\varepsilon^2(1-f_\varepsilon^2)^2 r^{n-1}dr \le C\varepsilon^{[p]-p+1}. \tag{3.2}$$

$$\begin{aligned}\frac{1}{p}\int_{B\setminus B_T(0)} |\nabla u_\varepsilon|^p + \frac{1}{4\varepsilon^p}\int_{B\setminus B_T(0)} |u_\varepsilon|^2(1-|u_\varepsilon|^2)^2 \\ \to \frac{1}{p}\int_{B\setminus B_T(0)} |\nabla \frac{x}{|x|}|^p.\end{aligned} \tag{3.3}$$

Proof. From [4,Theorem 4.2] it follows that

$$E_\varepsilon(f_\varepsilon;T) \le \frac{1}{p}\int_T^1 (n-1)^{p/2} r^{n-p-1}dr + C\varepsilon^{[p]-p+1}. \tag{3.4}$$

Here $E_\varepsilon(f;T) = \frac{1}{p}\int_T^1 |f_r|^p r^{n-1}dr + \frac{1}{4\varepsilon^p}\int_T^1 f^2(1-f^2)^2 r^{n-1}dr$. On the other hand, Jensen's inequality with $p > 2$ implies that

$$\begin{aligned}E_\varepsilon(f_\varepsilon;T) \ge \frac{1}{p}\int_T^1 |f_\varepsilon'|^p r^{n-1}dr + \frac{1}{p}\int_T^1 \left((n-1)\frac{f_\varepsilon^2}{r^2}\right)^{p/2} r^{n-1}dr \\ + \frac{1}{4\varepsilon^p}\int_T^1 f_\varepsilon^2\left(1-f_\varepsilon^2\right)^2 r^{n-1}dr.\end{aligned} \tag{3.5}$$

Combining this with (3.4) yields

$$\begin{aligned}&\frac{1}{p}\int_T^1 \left((n-1)\frac{f_\varepsilon^2}{r^2}\right)^{p/2} r^{n-1}dr \\ &\le E_\varepsilon(f_\varepsilon;T) \\ &\le C\varepsilon^{[p]-p+1} + \frac{1}{p}\int_T^1 (n-1)^{p/2} r^{n-p-1}dr.\end{aligned} \tag{3.6}$$

By using (3.1) and Holder inequality we deduce

$$\int_T^1 (n-1)^{p/2} r^{n-p-1}(1-f_\varepsilon^p)dr \le C\varepsilon^{p/2}.$$

Substituting this into (3.6) leads to

$$E_\varepsilon(f_\varepsilon;T) - \frac{1}{p}\int_T^1 \left((n-1)r^{-2}\right)^{p/2} r^{n-1}dr \le C\varepsilon^{[p]-p+1}. \tag{3.7}$$

Noticing $\int_{B\setminus B_T(0)} |\nabla \frac{x}{|x|}|^p dx = |S^{n-1}| \int_T^1((n-1)r^{-2})^{p/2} r^{n-1}dr$, from (3.5) and (3.7), we can obtain (3.2) and (3.3).

THEOREM 3.2. *Let $u_\varepsilon(x) = f_\varepsilon(r)\frac{x}{|x|}$ be a radial minimizer of $E_\varepsilon(u,B)$. Then for any $T>0$, there exists $C>0$ which is independent of ε, such that $\int_T^1 r^{n-1}[(f_\varepsilon')^p + \frac{1}{\varepsilon^p}(1-f_\varepsilon^2)^2]dr \le C\varepsilon^p$.*

Proof. From Jensen's inequality and (2.2) it follows

$$\begin{aligned} E_\varepsilon(f_\varepsilon;T) \ge & \frac{1}{p}\int_T^1 (f_\varepsilon')^p r^{n-1}dr + \frac{1}{8\varepsilon^p}\int_T^1 (1-f_\varepsilon^2)^2 r^{n-1}dr \\ & + \frac{1}{p}\int_T^1 \frac{(n-1)^{p/2}}{r^p} f_\varepsilon^p r^{n-1}dr. \end{aligned}$$

Combining this with (3.4) and using (3.1), we have

$$\frac{1}{p}\int_T^1 (f_\varepsilon')^p r^{n-1}dr + \frac{1}{8\varepsilon^p}\int_T^1 (1-f_\varepsilon^2)^2 r^{n-1}dr \le C\varepsilon^{[p]-p+1}. \tag{3.8}$$

Noting (3.4) and (3.8), and applying the integral mean value theorem, we see that there exists $T_1 \in [T, 2T]$, such that

$$\left[\frac{1}{\varepsilon^p}\left(1-f_\varepsilon^2\right)^2\right]_{r=T_1} \le C_1\varepsilon^{[p]-p+1}. \tag{3.9}$$

Clearly, we may find a minimizer ρ_1 in $W_{f_\varepsilon}^{1,p}((T_1,1), R^+\cup\{0\})$ of the functional

$$E(\rho,T_1) = \frac{1}{p}\int_{T_1}^1 (\rho_r^2+1)^{p/2}dr + \frac{1}{2\varepsilon^p}\int_{T_1}^1 (1-\rho)^2 dr.$$

PROPOSITION 3.3. *(3.9) implies that $E(\rho_1,T_1) \le C\varepsilon^{F[1]}$. Here $F[j] = \frac{[p]+1-p}{2^j} + \frac{(2^j-1)p}{2^j}$, $j=0,1,\cdots$.*

Proof. The minimizer ρ_1 solves the problem

$$-\varepsilon^p (v^{(p-2)/2}\rho_r)_r = 1-\rho \quad \text{on} \quad [T_1, 1], \tag{3.10}$$

$$\rho(T_1) = f_\varepsilon(T_1), \quad \rho(1) = f_\varepsilon(1) = 1, \tag{3.11}$$

where $v = \rho_r^2 + 1$. Obviously, $\rho \le 1$. Noting that ρ_1 is a minimizer, we deduce from (3.8) in [4] and (3.4) that

$$E(\rho_1, T_1) \le E(f_\varepsilon, T_1) \le C E_\varepsilon(f_\varepsilon; T_1) \le C. \tag{3.12}$$

Take $\zeta \in C^\infty(0,1]$, $\zeta = 1$ on $(0, T_1]$, $\zeta = 0$ near $r = 1$, and $|\zeta_r| \le C(T_1)$. Multiplying (3.10) with $\zeta\rho_r$ and integrating over $[T_1, 1]$, we have

$$\begin{aligned} v^{(p-2)/2}\rho_r^2\Big|_{r=T_1} &+ \int_{T_1}^1 v^{(p-2)/2}\rho_r(\zeta_r\rho_r + \zeta\rho_{rr})dr \\ &= \frac{1}{\varepsilon^p}\int_{T_1}^1 (1-\rho)\zeta\rho_r dr. \end{aligned} \tag{3.13}$$

At first, by using (3.12) we obtain
(3.14)

$$\begin{aligned} &\left|\int_{T_1}^1 v^{(p-2)/2}\rho_r(\zeta_r\rho_r + \zeta\rho_{rr})dr\right| \\ &\le \int_{T_1}^1 v^{(p-2)/2}|\zeta_r|(\rho_r)^2 dr + \frac{1}{p}\left|\int_{T_1}^1 \left[\left(v^{(p-2)/2}\zeta\right)_r - v^{(p-2)/2}\zeta_r\right] dr\right| \\ &\le C + \frac{1}{p} v^{p/2}\Big|_{r=T_1}. \end{aligned}$$

Next, by applying (3.12), (3.11), and (3.9), we derive

$$\begin{aligned} &\frac{1}{\varepsilon^p}\left|\int_{T_1}^1 (1-\rho)\zeta\rho_r dr\right| \\ &= \frac{1}{2\varepsilon^p}\left|\int_{T_1}^1 \left[((1-\rho)^2\zeta)_r - (1-\rho)^2\zeta_r\right] dr\right| \\ &\le \frac{1}{2\varepsilon^p}(1-\rho)^2\Big|_{r=T_1} + \frac{C}{2\varepsilon^p}\int_{T_1}^1 (1-\rho)^2 dr \\ &\le C. \end{aligned} \tag{3.15}$$

Combining (3.13)–(3.15), we get $v^{(p-2)/2}\rho_r^2|_{r=T_1} \le C + \frac{1}{p}v^{p/2}|_{r=T_1}$. Substituting this into $v^{p/2}|_{r=T_1} = v^{(p-2)/2}(\rho_r^2+1)|_{r=T_1}$, and using Young inequality, we see that for any $\delta \in (0,1/2)$, $v^{p/2}|_{r=T_1} \le C(\delta) + (\frac{1}{p} + \delta)v^{p/2}|_{r=T_1}$. Choosing δ sufficiently small yields

$$v^{p/2}\Big|_{r=T_1} \le C. \tag{3.16}$$

Multiplying (3.10) with $(\rho - 1)$ and integrating over $[T_1, 1]$, we have

$$\int_{T_1}^1 \left[v^{(p-2)/2}\rho_r(\rho-1)\right]_r dr = \int_{T_1}^1 v^{(p-2)/2}\rho_r^2 dr + \frac{1}{\varepsilon^p}\int_{T_1}^1 (\rho-1)^2 dr.$$

Hence, by applying (3.16), (3.11), and (3.9), we obtain

$$\begin{aligned} E(\rho_1, T_1) &\le C\left|\int_{T_1}^1 \left[v^{(p-2)/2}\rho_r(\rho-1)\right]_r dr\right| \\ &= Cv^{(p-2)/2}|\rho_r|\,|\rho-1|\Big|_{r=T_1} \\ &\le C\varepsilon^{F(1)}. \end{aligned}$$

Proposition is proved.

PROPOSITION 3.4. *Proposition 3.3 implies that*

$$E_\varepsilon(f_\varepsilon; T_1) \le C\varepsilon^{F[1]} + \frac{1}{p}\int_{T_1}^1 \frac{(n-1)^{p/2}}{r^{p-n+1}} dr.$$

Proof. Set $w_\varepsilon = f_\varepsilon$ if $r \in [0, T_1]$, $w_\varepsilon = \rho_1$ if $r \in [T_1, 1]$. Noticing that u_ε is a minimizer, we have $E_\varepsilon(u_\varepsilon, B) \le E_\varepsilon(w_\varepsilon \frac{x}{|x|}, B)$. Hence

$$\begin{aligned} E_\varepsilon(f_\varepsilon, T_1) &\le \frac{1}{p}\int_{T_1}^1 \left(\rho_r^2 + \frac{n-1}{r^2}\rho^2\right)^{p/2} r^{n-1} dr \\ &\quad + \frac{1}{4\varepsilon^p}\int_{T_1}^1 \rho^2\left(1-\rho^2\right)^2 r^{n-1} dr \\ &\le \frac{1}{p}\int_{T_1}^1 \left(\frac{n-1}{r^2}\rho^2\right)^{p/2} r^{n-1} dr + CE(\rho_1, T_1). \end{aligned}$$

Proposition 3.4 is seen by using Proposition 3.3.

Complete the proof of Theorem 3.2. Using Proposition 3.4 and (3.1), we can deduce that

$$\int_{T_1}^1 (f_\varepsilon')^p r^{n-1} dr + \frac{1}{8\varepsilon^p}\int_{T_1}^1 (1-f_\varepsilon^2)^2 r^{n-1} dr \le C\varepsilon^{F[1]} + C\varepsilon^{p/2} \le C_2\varepsilon^{F[1]}$$

by the same derivation of (3.8). Comparing with (3.8), we find the rate is better than (3.8), since the exponent of ε is improved from $F[0]$ to $F[1]$.

Set $T_m \in [T_{m-1}, 2T]$. By the same argument above (whose idea is improving the exponent of ε from $F[k]$ to $F[k+1]$), we know that there exists a sufficiently large integer m satisfying $\frac{p}{2}+1 \le F[m]$, such that

$$\begin{aligned}\int_{T_m}^1 (f_\varepsilon')^p r dr + &\frac{1}{8\varepsilon^p}\int_{T_m}^1 (1-f_\varepsilon^2)^2 r^{n-1} dr \\ &\le C\varepsilon^{F[m]} + C\varepsilon^{p/2} \\ &\le C\varepsilon^{p/2}.\end{aligned} \tag{3.17}$$

Similar to the derivations of (3.9), we see that there exists $T_{m+1} \in [T_m, 2T]$ such that

$$\left[\frac{1}{\varepsilon^p}\left(1-f_\varepsilon^2\right)^2\right]_{r=T_{m+1}} \le C\varepsilon^{p/2}. \tag{3.18}$$

Obviously, the minimizer $\rho_2 \in W^{1,p}_{f_\varepsilon}((T_1,1),R^+)$ of

$$E(\rho, T_{m+1}) = \frac{1}{p}\int_{T_{m+1}}^1 (\rho_r^2+1)^{p/2} dr + \frac{1}{2\varepsilon^p}\int_{T_{m+1}}^1 (1-\rho)^2 dr$$

exists. By the analogous proof of Proposition 3.3, form (3.18) we can also obtain that

$$E(\rho_2, T_{m+1}) \le v^{\frac{p-2}{2}} \rho_{4r}(1-\rho_1)\Big|_{r=T_{m+1}} \le C(1-\rho_2(T_{m+1})) \le C\varepsilon^{G[1]},$$

where $G[j] = \frac{p/2}{2^j} + \frac{(2^j-1)p}{2^j}, j = m+1, m+2, \cdots$. So, by the same proof of Proposition 3.4 we also conclude that

$$E_\varepsilon(f_\varepsilon; T_{m+1}) \le C\varepsilon^{G[1]} + \frac{1}{p}\int_{T_{m+1}}^1 \frac{(n-1)^{p/2}}{r^{p-1}} dr.$$

Similar to the derivation of (3.8), using (3.17) we have

$$\int_{T_{m+1}}^1 (f_\varepsilon')^p r^{n-1} dr + \frac{1}{8\varepsilon^p}\int_{T_{m+1}}^1 (1-f_\varepsilon^2)^2 r^{n-1} dr \le C\varepsilon^{G[1]}.$$

By the same argument above (whose idea is improving the exponent of ε from $G[k]$ to $G[k+1]$), we know that for any $k \in N$,

$$\int_{T_{m+k}}^1 (f_\varepsilon')^p r^{n-1} dr + \frac{1}{8\varepsilon^p}\int_{T_{m+k}}^1 (1-f_\varepsilon^2)^2 r^{n-1} dr \le C\varepsilon^{\frac{p/2}{2^k}+\frac{(2^k-1)p}{2^k}}.$$

Letting $k \to \infty$, we can see the conclusion of Theorem.

References

[1] F. Bethuel, H. Brezis, and F. Helein, *Ginzburg-Landau Vortices,* Birkhauser, 1994.

[2] R. M. Herve and M. Herve, *Etude qualitative des solutions reelles d'une equation differentielle liee a l'equation de Ginzburg-Landau,* Ann. Inst. H. Poincare Anal. Non Lineaire **11** (1994), 427–440.

[3] L. Lassoued and C. Lefter, *On a variant of the Ginzburg-Landau energy,* Nonlinear Differ. Equ. Appl. **5** (1998), no. 43, 39–51.

[4] Y. Lei, *Radial minimizer of a variant of p-Ginzburg-Landau functional,* Electron. J. Differential Equations **2003** (2003), no. 35, 1–12.

[5] P. Mironescu, *On the stability of radial solution of the Ginzburg-Landau equation,* J. Funct. Anal. **130** (1995), 334–344.

DEPARTMENT OF MATHEMATICS, NANJING NORMAL UNIVERSITY, NANJING, JIANGSU 210097, P.R.CHINA
E-mail: lythxl@163.com