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RANK PRESERVER OF BOOLEAN MATRICES

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ABSTRACT. A Boolean matrix with rank 1 is factored as a left factor and a right factor. The perimeter of a rank-1 Boolean matrix is defined as the number of nonzero entries in the left factor and the right factor of the given matrix. We obtain new characterizations of rank preservers, in terms of perimeter, of Boolean matrices.

1. Introduction

There are many papers on Boolean matrices, and linear preserver problems have been the subject of research by many authors ([1]–[6]).

The *Boolean algebra* consists of the set $\mathbb{B} = \{0, 1\}$ equipped with two binary operations, addition and multiplication. The operations are defined as usual except that $1 + 1 = 1$.

For two distinct rank-1 matrices A and B, a rank-1 matrix C is called a *separating matrix* of A and B if the rank of $A + C$ is 2 but the rank of $B + C$ is 1 or vice versa. In this case, rank-1 matrices A and B are said to be *separable*. Then it is natural to ask which pairs of distinct rank-1 matrices are separable. In [1], the fact that every pair of distinct rank-1 matrices (see Section 2) is separable were studied and used for the research on Boolean rank-preserving operators.

We are interested in obtaining characterizations, in terms of perimeter, of rank-preserving linear operator of Boolean matrices.

These are motivated by the separable matrices studied in [1], which was used to characterize the linear operators that preserve Boolean rank. In section 2, we give definitions and preliminaries for our purpose. In Section 3, we obtain new characterizations of Boolean rank preservers using the perimeter of Boolean rank-1 matrices.

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2. Preliminaries

Let $\mathbb{M}_{m,n}(\mathbb{B})$ denote the set of all $m \times n$ matrices with entries in the Boolean algebra $\mathbb{B}$. The usual definitions for adding and multiplying matrices apply to Boolean matrices as well. Throughout this paper, we shall adopt the convention that $1 < m \leq n$.

If an $m \times n$ Boolean matrix A is not zero, then its *Boolean rank*, $b(A)$, is the least k for which there exist $m \times k$ and $k \times n$ Boolean matrices B and C with $A = BC$. The Boolean rank of the zero matrix is 0.

It is well known that $b(A)$ is the least k such that A is the sum of k matrices of Boolean rank 1 ([5]).

Let lowercase, bold face letters represent vectors and all vectors $\mathbf{a}$ are column vectors ($\mathbf{a}^t$ is a row vector) for $\mathbf{a} \in \mathbb{B}^m\, [= \mathbb{M}_{m,1}(\mathbb{B})\,]$.

For our purposes, we can define *a Boolean vector space* to be any subset of $\mathbb{B}^m$ containing 0 which is closed under addition.

If $\mathbf{a}$ and $\mathbf{b}$ are vectors in $\mathbb{B}^m$, we say $\mathbf{a}$ *absorbs* $\mathbf{b}$, written $\mathbf{b} \leq \mathbf{a}$, if $b_i = 0$ whenever $a_i = 0$, for all $1 \leq i \leq m$.

If $\mathbb{V}, \mathbb{W}$ are vector spaces with $\mathbb{V} \subseteq \mathbb{W}$, then $\mathbb{V}$ is called a *subspace* of $\mathbb{W}$. We identify $\mathbb{M}_{m,n}(\mathbb{B})$ with $\mathbb{B}^{mn}$ in the usual way when discuss it as a Boolean vector space and consider its subspaces.

Let $\mathbb{V}$ be a Boolean vector space. If S is a subset of $\mathbb{V}$, then $\langle S \rangle$ denotes the intersection of all subspaces of $\mathbb{V}$ containing S. This is a subspace of $\mathbb{V}$ and is called *the subspace generated by* S. If $S = \{s_1, s_2, \cdots, s_p\}$, then $\langle S \rangle = \{\sum_{i=1}^p a_i s_i : a_i \in \mathbb{B}\}$, the set of *linear combinations* of S. Note that $\langle \Phi \rangle = \{0\}$. Define the *dimension* of $\mathbb{V}$, written $\dim(\mathbb{V})$, to be the minimum of the cardinalities of all subsets S of $\mathbb{V}$ generating $\mathbb{V}$. We call a generating set of cardinality equal to $\dim(\mathbb{V})$ a *basis* of $\mathbb{V}$. It is well-known that every Boolean vector space has only one basis. A subset of $\mathbb{V}$ is called *independent* if none of its members is a linear combination of the others. Then every basis is independent.

Let $E_{i,j}$ be the $m \times n$ matrix whose (i,j)-th entry is 1 and whose other entries are all 0, and we call $E_{i,j}$ a *cell*. Let $\mathbb{E} = \{E_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ be the set of cells in $\mathbb{M}_{m,n}(\mathbb{B})$.

If A and B are matrices in $\mathbb{M}_{m,n}(\mathbb{B})$, we say A *dominates* B (written $B \leq A$ or $A \geq B$) if $a_{i,j} = 0$ implies $b_{i,j} = 0$ for all i, j. Equivalently, $B \leq A$ if and only if $A + B = A$.

It is easy to verify that the Boolean rank of A is 1 if and only if there exist nonzero (Boolean) vectors $\mathbf{a} \in \mathbb{M}_{m,1}(\mathbb{B})$ and $\mathbf{x} \in \mathbb{M}_{n,1}(\mathbb{B})$ such that $A = \mathbf{a}\mathbf{x}^t$. And these vectors $\mathbf{a}$ and $\mathbf{x}$ are uniquely determined by A. We call $\mathbf{a}$ the *left factor* and $\mathbf{x}$ the *right factor* of A. Therefore there are exactly $(2^m-1)(2^n-1)$ rank-1 $m \times n$ Boolean matrices.

For any vector $\mathbf{u} \in \mathbb{M}_{m,1}(\mathbb{B})$, let $|\mathbf{u}|$ be the number of nonzero entries in $\mathbf{u}$, and when $A = \mathbf{a}\mathbf{x}^t$ is not zero, define the *perimeter* of A, $p(A)$, as $|\mathbf{a}| + |\mathbf{x}|$. Since the factorization of A as $\mathbf{a}\mathbf{x}^t$ is unique, the perimeter of A is also unique.

3. Boolean rank preserver

If V, W are Boolean vector spaces, a mapping $T : V \to W$ which preserves sums and O is said to be a *Boolean linear transformation.* If $V = W$, the word *operator* is used instead of "transformation".

A linear transformation $T : V \to W$ is *invertible* if and only if T is injective and $T(V) = W$.

Beasley and Pullman[1] obtained characterizations of invertible Boolean linear operator which preserves Boolean rank.

LEMMA 3.1. [1] *If T is a Boolean linear operator on V, then the following statements are equivalent:*

(a) *T is invertible;*
(b) *T is injective;*
(c) *T is surjective;*
(d) *T permutes $\mathbb{E}$.*

An $n \times n$ Boolean matrix A is said to be *invertible* if for some X, $AX = XA = I_n$, where I_n is the $n \times n$ identity matrix. This matrix X is necessarily unique when it exists. It is well known that the permutation matrices are the only invertible Boolean matrices ([3]).

Let T be a linear operator on $\mathbb{M}_{m,n}(\mathbb{B})$. Then

(i) T is a (U, V)-operator if there exist invertible matrices U and V such that $T(A) = UAV$ for all A in $\mathbb{M}_{m,n}(\mathbb{B})$, or $m = n$ and $T(A) = UA^tV$ for all A in $\mathbb{M}_{m,n}(\mathbb{B})$.
(ii) T preserves Boolean rank r if $b(T(A)) = r$ whenever $b(A) = r$ for all $A \in \mathbb{M}_{m,n}(\mathbb{B})$.
(iii) T preserves perimeter k of Boolean rank-1 matrices if $p(T(A)) = k$ whenever $p(A) = k$ for all $A \in \mathbb{M}_{m,n}(\mathbb{B})$ with $b(A) = 1$.

THEOREM 3.2. [1] *If T is a linear operator on $\mathbb{M}_{m,n}(\mathbb{B})$, then the following statements are equivalent:*

(a) *T is invertible and preserves the rank of all rank-1 matrices;*
(b) *T preserves the ranks of all rank-1 matrices and rank-2 matrices and preserves the dimension of all rank-1 spaces;*
(c) *T is a (U, V)-operator.*

In the following we have new characterizations of the linear operators that preserve the rank of Boolean matrices.

LEMMA 3.3. *If T is a (U, V)-operator on $\mathbb{M}_{m,n}(\mathbb{B})$, then T preserves the perimeter of Boolean rank-1 matrices.*

Proof. Let A be an $m \times n$ Boolean rank-1 matrix. Then $A = \mathbf{a}\mathbf{x}^t$ has perimeter $p(A) = |\mathbf{a}| + |\mathbf{x}|$. Thus $T(A) = UAV = U(\mathbf{a}\mathbf{x}^t)V = (U\mathbf{a})(V^t\mathbf{x})^t$ has perimeter $p(T(A)) = |U\mathbf{a}| + |V^t\mathbf{x}| = |\mathbf{a}| + |\mathbf{x}| = p(A)$ since U and V^t are permutation matrices in $\mathbb{M}_{m,m}(\mathbb{B})$ and $\mathbb{M}_{n,n}(\mathbb{B})$ respectively.

If $m = n$ and $T(A) = UA^tV$, then we can show that $p(T(A)) = |\mathbf{a}| + |\mathbf{x}|$ by a similar method as above. Hence (U, V)-operator preserves the perimeter. □

LEMMA 3.4. *If T preserves Boolean rank 1 and the set of perimeter 2 and the set of perimeter k ($k \geq 3$) of Boolean rank 1 matrices on $\mathbb{M}_{m,n}(\mathbb{B})$ with $n \geq m \geq 2$, then we have:*

(a) *T maps a cell into a cell.*
(b) *T maps a row into a row or a column if $m = n$.*
(c) *T maps two distinct cells in a row onto two distinct cells in a row (or a column if $m = n$).*
(d) *T maps two distinct cells in a column onto two distinct cells in a column (or a row if $m = n$).*
(e) *If T maps a row of a matrix A into a row, then T maps each row of A into a row of $T(A)$. Similarly, if T maps a column of a matrix A into a column, then T maps each column of A into a column of $T(A)$.*

Proof. (a) Since T preserves perimeter 2 of Boolean rank-1 matrices, T maps each cell into a cell on $\mathbb{M}_{m,n}(\mathbb{B})$ since cells are the only matrices with perimeter 2.

(b) Let $T(E_{i,1}) = E_{p,q}$, $T(E_{i,h}) = E_{r,s}$ for $1 < h \leq n$. If $p \neq r$ and $q \neq s$, then $b(E_{i,1} + E_{i,h}) = 1$ but $b(T(E_{i,1} + E_{i,h})) = 2$. This contradicts the assumption. Hence $p = r$ or $q = s$. This means that the i-th row is mapped into a row or a column if $m = n$ under T.

(c) Suppose $T(E_{i,j}) = T(E_{i,h}) = E_{p,q}$ for some distinct j and h. Then T maps the i-th row into the p-th row and both the j-th and h-th column into the q-th column by (b). Thus for any Boolean rank-1 matrix A with perimeter k (≥ 3) which dominates $E_{i,j}$ and $E_{i,h}$, we can show that $T(A)$ has perimeter at most $k-1$, a contradiction. Thus T maps distinct cells in a row into distinct cells in either a row or a column by (b).

(d) It is similar to (c).

(e) If not, then there exist two rows, say, i-th row and j-th row such that T maps the i-th row into an r-th row but j-th row into an s-th column for some r and s. Consider a matrix $D = E_{i,p}+E_{i,q}+E_{j,p}+E_{j,q}$ with Boolean rank 1. Then

$$T(D) = T(E_{i,p}+E_{i,q})+T(E_{j,p}+E_{j,q}) = (E_{r,p'}+E_{r,q'})+(E_{p'',s}+E_{q'',s})$$

for some $p' \neq q'$ and $p'' \neq q''$ by (c). Therefore $b(T(D)) \neq 1$ and T does not preserve rank 1, a contradiction. Hence T maps each row into a row. Similarly, T maps each column into a column. □

EXAMPLE 3.5. Consider a linear map T on $\mathbb{M}_{m,n}(\mathbb{B})$ with $m \geq 2$ and $n \geq 3$ such that

$$T(A) = B = (b_{i,j}),$$

where $A = (a_{i,j})$, $b_{i,j} = 0$ with $i \geq 2$ and $b_{1,j} = \sum_{i=1}^{m} a_{i,r}$ with $r \equiv i+(j-1) \pmod{n}$ and $1 \leq r \leq n$. Then T maps each row and each column into the first row. And T preserves Boolean rank and perimeters $2, 3$ and $n+1$ of Boolean rank-1 matrices. But T does not preserve perimeters k ($k \geq 4$ and $k \neq n+1$) of Boolean rank-1 matrices: For, if $4 \leq k \leq n$, then we can choose a $2 \times (k-2)$ submatrix with perimeter k which is mapped to distinct k position in the first row of B under T. Then this $1 \times k$ submatrix has perimeter $k+1$. Therefore T does not preserve perimeter k of Boolean rank-1 matrices. □

THEOREM 3.6. *If T is a linear operator on $\mathbb{M}_{m,n}(\mathbb{B})$ with $m \geq 2$ and $n \geq 4$, then the following statements are equivalent:*

(a) *T is invertible and preserves the rank of all Boolean rank-1 matrices;*
(b) *T preserves the ranks of all Boolean rank-1 matrices and rank-2 matrices and preserves the dimension of all rank-1 spaces;*
(c) *T is a (U,V)-operator;*
(d) *T preserves the rank and the perimeter of all Boolean rank-1 matrices;*
(e) *T preserves the rank and the perimeters 2 and 3 of all Boolean rank-1 matrices;*

(f) *T permutes $\mathbb{E}$ and preserves the rank of all Boolean rank-1 matrices.*

Proof. The equivalence of (a), (b) and (c) comes from Theorem 3.2 and the implications of (c) $\Rightarrow$ (d) $\Rightarrow$ (e) come from Lemma 3.3. We now show that (e) implies (f).

Since T maps a cell into a cell by Lemma 3.4 (a), we have to show that $T(E_{i,j}) \neq T(E_{p,q})$ for all distinct pairs (i,j) and (p,q) of indices. But we have shown it by Lemma 3.4 (c) and (d) for $i = p$ or $j = q$. For the case $i \neq p$ and $j \neq p$, assume that $T(E_{i,j}) = T(E_{p,q}) = E_{k,l}$. Without loss of generality, we may assume that T maps the i-th row into the r-th row. Then T maps each row into a row by Lemma 4.4.

If $k = n + k' \geq n + 2$, consider the $k' \times n$ matrix

$$D = \sum_{j'=1}^{n} E_{i,j'} + \sum_{q'=1}^{n} E_{p,q'} + \sum_{g=1}^{k'-2} \sum_{h=1}^{n} E_{g,h}$$

with Boolean rank 1 and perimeter $n + k' = k$. Then T maps the i-th and p-th row of D into the r-th row by Lemma 3.4. Then the perimeter of $T(D)$ is less than $n + k' = k$, a contradiction. Thus $T(E_{i,j}) \neq T(E_{p,q})$ for all distinct (i,j) and (p,q).

If $4 \leq k \leq n$, we will show that we can choose a $2 \times (k-2)$ submatrix from the i-th and p-th row whose image under T is either $1 \times k$ submatrix in the r-th row (and hence its perimeter is $k+1$) or $1 \times k - 2$ submatrix in the r-th row (and hence its perimeter is $k - 1$) as follows : Since T maps each row into a row and $T(E_{i,j}) = T(E_{p,q}) = E_{r,l}$, T maps the i-th row and the p-th row into the r-th row. But T maps distinct cells in each row (or column) to distinct cells by Lemma 3.4.

If $T(E_{i,q}) = T(E_{p,j})$ then the image of $A = E_{i,j} + E_{i,q} + E_{p,j} + E_{p,q}$ under T is a 1×2 submatrix in the r-th row. Hence $p(A) = 4$ but $p(T(A)) = 3$. Then T does not preserve perimeter 4.

If $T(E_{i,q}) \neq T(E_{p,j})$, then we can choose a 2×2 submatrix $B = E_{i,j} + E_{i,h} + E_{p,j} + E_{p,h}$ with $h \neq q$ whose image under T is a 1×4 submatrix in the r-th row. Hence $p(B) = 4$ but $p(T(B)) = 5$. Then T does not preserve perimeter 4.

Applying this method to the case of perimeter k, we can choose a $2 \times (k-2)$ submatrix of 1's whose image under T is either (i) a $1 \times k - 2$ submatrix A' of 1's in the r-th row or (ii) $1 \times k$ submatrix B' of 1's in the r-th row. For the case (i), we have $p(A') = k$ but $p(T(A')) = k - 1$. For the case (ii), we have $p(B') = k$ but $p(T(B')) = k + 1$. These two cases show that T does not preserve the perimeter k of a Boolean rank-1 matrix, a contradiction.

Therefore we have shown that $T(E_{i,j}) \neq T(E_{p,q})$ for any distinct pairs (i,j) and (p,q) of indices. Hence T permutes $\mathbb{E}$.

Finally, the implication of (f) $\Rightarrow$ (a) comes from Lemma 3.1. $\square$

Thus we have characterizations of the linear operators that preserve the perimeter of Boolean rank-1 matrices.

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