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q-ANALOGUE OF EULER-BARNES' NUMBERS AND POLYNOMIALS

LEECHAE JANG AND TAEKYUN KIM

ABSTRACT. Recently, Kim[2,6] has introduced an interesting Euler-Barnes' numbers and polynomials. In this paper, we construct the q-analogue of Euler-Barnes' numbers and polynomials, and investigate their properties.

1. Introduction

Let $w, a_1, a_2, \cdots, a_r$ be complex numbers such that $a_i (\neq 0)$ for each $i = 1, 2, \cdots, r$. Then the Euler-Barnes' polynomials of w with parameters $a_1, a_2, \cdots, a_r$ are defined as

$$\frac{(1-u)^r}{\prod_{j=1}^{r}(e^{a_j t}-u)}\, e^{wt} = \sum_{n=0}^{\infty} H_n^{(r)}(w, u \mid a_1, a_2, \cdots, a_r)\frac{t^n}{n!},$$

for $u \in \mathbb{C}$ with $|u| > 1$, cf.[6]. In the special case $w = 0$, the above polynomials are called the r-th Euler-Barnes' numbers. We write

$$H_n^{(r)}(u \mid a_1, a_2, \cdots, a_r) = H_n^{(r)}(0, u \mid a_1, a_2, \cdots, a_r).$$

Throughout this paper, the symbols $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ will respectively denote the ring of rational integers, the ring of p-adic integers, the field of p-adic numbers, the complex number field and the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$, cf. [2–5]. If $q \in \mathbb{C}$, one normally assumes

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$|q| < 1$. If $q \in \mathbb{C}_p$, one normally assumes $|1-q|_p \le p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$. In this paper we use the notation:

$$[x] = [x:q] = \frac{1-q^x}{1-q}, \quad \text{cf. [1,2,8]}.$$

The ordinary Euler numbers E_m are defined by the generating function in the complex number field as

$$\frac{2}{e^t+1} = \sum_{m=0}^{\infty} E_m \frac{t^m}{m!}, \quad (|t| < \pi), \quad \text{cf. [9]}.$$

Let u be an algebraic in complex number field. Then Frobenius-Euler numbers are defined as

$$\frac{1-u}{e^t-u} = \sum_{n=0}^{\infty} H_n(u) \frac{t^n}{n!}, \quad (|t| < \pi), \quad \text{cf. [9,10]}.$$

Note that $H_n(-1) = E_n$. Also, Carlitz defined the q-analogue of Frobenius-Euler numbers and polynomials as follows:

$$H_0(u:q) = 1, \ (qH+1)^k - uH_k(u:q) = 0 \text{ if } k \ge 1,$$

where u is a complex number with $|u| > 1$:

$$H_k(u, x:q) = (q^x H + [x])^k \text{ if } k \ge 0, \quad \text{cf. [2,11]},$$

with the usual convention about replacing $H^k(u:q)$ by $H_k(u:q)$. For any positive integer N, $z \in \mathbb{C}_p$,

$$\mu_z(a + p^N \mathbb{Z}_p) = \frac{z^a}{[p^N : z]}$$

can be extended to distribution on $\mathbb{Z}_p$, cf. [1,2,7,13]. Let $UD(\mathbb{Z}_p)$ be denoted by the set of uniformly differentiable functions on $\mathbb{Z}_p$. Then this distribution admits the following integral for $f \in UD(\mathbb{Z}_p)$:

$$I_z(f) = \int_{\mathbb{Z}_p} f(x) d\mu_z(x) = \lim_{N \to \infty} \frac{1}{[p^N : z]} \sum_{x=0}^{p^N-1} f(x) z^x, \text{ cf. [1,2,12]}.$$

The purpose of this paper is to construct the q-analogue of Euler-Barnes' numbers and investigate their properties.

2. q-analogue of multiple Euler numbers and polynomials

Let d be a fixed integer and let p be a fixed prime number. We set

$$X = \varprojlim_N (\mathbb{Z}/dp^N\mathbb{Z}),$$
$$X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p,$$
$$a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},$$

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^N$.

Let $u \in \mathbb{C}_p$ with $|1-u^f|_p \ge 1$ for each positive integer f and let $a_1, a_2, \cdots, a_r$ be non-zero p-adic integers. For $w \in \mathbb{Z}_p$, we consider the q-analogue of Euler-Barnes' polynomials by using p-adic invariant integrals as follows: For $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{1-p}}$, define

$$H_n^{(r)}(w, u, q \mid a_1, a_2, \cdots, a_r)$$
$$= \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r \text{ times}} \left[w + \sum_{j=1}^r a_j x_j : q\right]^n d\mu_u(x_1) \cdots d\mu_u(x_r). \tag{1}$$

By (1), we note that

$$\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r \text{ times}} \left[w + \sum_{j=1}^r a_j x_j : q\right]^n d\mu_u(x_1) \cdots d\mu_u(x_r)$$
$$= \lim_{N\to\infty} \frac{1}{[p^N : u]^r} \sum_{x_1,\cdots,x_r=0}^{p^N-1} \left[w + \sum_{j=1}^r a_j x_j : q\right]^n u^{\sum_{j=1}^r x_j}$$
$$= \lim_{N\to\infty} \left(\frac{1-u}{1-u^{p^N}}\right)^r$$
$$\times \sum_{x_1,\cdots,x_r=0}^{p^N-1} \left(\sum_{l=0}^n \binom{n}{l} \left(\frac{1}{1-q}\right)^n (-1)^l q^{l(w+\sum_{j=1}^r a_j x_j)} u^{\sum_{j=1}^r x_j}\right)$$
$$= \frac{(1-u)^r}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{lw} \left(\frac{1}{\prod_{j=1}^r (1-q^{la_j}u)}\right),$$

where $\binom{n}{l}$ is binomial coefficient. Therefore we obtain the following:

THEOREM 1. *For $n \geq 0$, we have*

$$\begin{aligned} & H_n^{(r)}(w,u,q \mid a_1,\cdots,a_r) \\ &= \frac{(1-u)^r}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lw}\left(\frac{1}{\prod_{j=1}^{r}(1-q^{la_j}u)}\right). \end{aligned}$$

Moreover,

$$\lim_{q\to 1} H_n^{(r)}(w,u,q \mid a_1,\cdots,a_r) = H_n^{(r)}(w,u^{-1} \mid a_1,\cdots,a_r).$$

REMARK. (1) In the special case $w=0$, we write

$$H_n^{(r)}(u,q \mid a_1,\cdots,a_r) = H_n^{(r)}(0,u,q \mid a_1,\cdots,a_r).$$

(2) Note that $\lim_{q\to 1} H_n^{(1)}(u,q \mid 1) = H_n(u^{-1})$, cf. [8,9].

Let $G_q^{(r)}(t,u \mid a_1,a_2,\cdots,a_r)$ be the generating function of $H_n^{(r)}(u,q \mid a_1,\cdots,a_r)$:

$$G_q^{(r)}(t,u \mid a_1,\cdots,a_r) = \sum_{k=0}^{\infty} H_k^{(r)}(u,q \mid a_1,\cdots,a_r)\frac{t^k}{k!},$$

for $q \in \mathbb{C}_p$ with $|1-q|_p < 1$, $u \in \mathbb{C}_p$ with $|1-u^f|_p \geq 1$. Then we have

$$\begin{aligned} & G_q^{(r)}(t,u \mid a_1,\cdots,a_r) \\ &= \sum_{k=0}^{\infty} H_k^{(r)}(u,q \mid a_1,\cdots,a_r)\frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty}\frac{(1-u)^r}{(1-q)^k}\sum_{i=0}^{k}\binom{k}{i}(-1)^i\left(\prod_{l=1}^{r}\frac{1}{1-q^{ia_l}u}\right)\frac{t^k}{k!} \\ &= (1-u)^r e^{\frac{t}{1-q}}\sum_{j=0}^{\infty}\left(\prod_{l=1}^{r}\frac{1}{1-q^{ja_l}u}\right)\left(\frac{1}{1-q}\right)^j\frac{t^j}{j!}. \end{aligned}$$

Therefore we obtain the following:

THEOREM 2. *For $q \in \mathbb{C}_p$ with $|1-q|_p < 1$, $u \in \mathbb{C}_p$ with $|1-u^f|_p \geq 1$, we have*

$$G_q^{(r)}(t,u|a_1,\cdots,a_r) = e^{\frac{t}{1-q}}(1-u)^r\sum_{j=0}^{\infty}\left(\prod_{l=1}^{r}\frac{1}{1-q^{ja_l}u}\right)\left(\frac{1}{1-q}\right)^j\frac{t^j}{j!}.$$

COROLLARY 3. *For $q \in \mathbb{C}_p$ with $|1-q|_p < 1$, $u \in \mathbb{C}_p$ with $|1-u|_p \geq 1$, we have*

$$\begin{aligned} &G_q^{(r)}(x,t,u \mid a_1,\cdots,a_r) \\ &= \sum_{n=0}^{\infty} H_n^{(r)}(x,u,q \mid a_1,\cdots,a_r)\frac{t^n}{n!} \\ &= e^{\frac{t}{1-q}}(1-u)^r \sum_{j=0}^{\infty} \left(\prod_{l=1}^{r} \frac{1}{1-q^{ja_l}u}\right)\left(\frac{1}{1-q}\right)^j q^{jx}\frac{t^j}{j!}. \end{aligned}$$

Note that

$$\lim_{q\to 1} G_q^{(r)}(x,t,u \mid a_1,\cdots,a_r) = \frac{(1-u^{-1})^r}{\prod_{l=1}^{r}(e^{a_j t}-u^{-1})}\, e^{xt}.$$

By (1), the Euler-Barnes' polynomials of x can be rewritten as

$$H_n^{(r)}(w,u,q \mid a_1,\cdots,a_r) = \sum_{k=0}^{n}\binom{n}{k}[w:q]^{n-k}q^{wk}H_k^{(r)}(u,q \mid a_1,\cdots,a_r).$$

From the above Eq.(1), we have the distribution relation for the q-analogue of Euler-Barnes'polynomials as follows:

THEOREM 4. *For $f \in \mathbb{N}$, we have*

$$\begin{aligned} &\frac{1}{(u-1)^r}H_n^{(r)}(w,u,q \mid a_1,\cdots,a_r) \\ (2)\qquad &= [f:q]^n \sum_{i_1,\cdots,i_r=0}^{f-1} \frac{u^{\sum_{j=1}^{r} i_j}}{(u^f-1)^r} \\ &\qquad \times H_n^{(r)}\left(\frac{w+\sum_{j=1}^{r} a_j i_j}{f}, u^f, q^f \mid a_1,\cdots,a_r\right). \end{aligned}$$

For $k \geq 0$, $f \in \mathbb{N}$, we set

$$(3)\quad E_{u:a_1,q}^{(k)}(x+fp^k\mathbb{Z}_p) = \frac{[fp^N:q]^k u^x}{1-u^{fp^N}}H_k^{(1)}\left(\frac{a_1 x}{fp^N}, u^{fp^N}, q^{fp^N} \mid a_1\right),$$

and this can be extended to a distribution on X. We show that $E_{u:a_1,q}^{(k)}$ is a distribution on X. For this, it suffices to check that

$$\sum_{i=0}^{p-1} E_{u:a_1,q}^{(k)}(x+ifp^N+fp^{N+1}\mathbb{Z}_p) = E_{u:a_1,q}^{(k)}(x+fp^k\mathbb{Z}_p).$$

By (2), we easily see that

$$\sum_{i=0}^{p-1} \frac{[p:q^{fp^N}]^k}{1-(u^{fp^N})^p}(u^{fp^N})^i H_k^{(1)}\left(\frac{\frac{a_1x}{fp^N}+ia_1}{p}, (u^{fp^N})^p, (q^{fp^N})^p \mid a_1\right)$$
$$= \frac{1}{1-u^{fp^N}} H_k^{(1)}\left(\frac{a_1x}{fp^N}, u^{fp^N}, q^{fp^N} \mid a_1\right).$$

Therefore, we have

$$\begin{aligned}
&\sum_{i=0}^{p-1} E_{u:a_1,q}^{(k)}(x+ifp^N+fp^{N+1}\mathbb{Z}_p)\\
&= \sum_{i=0}^{p-1} \frac{[fp^{N+1}:q]^k u^{(x+ifp^N)}}{1-u^{fp^{N+1}}}\\
&\quad\times H_k^{(1)}\left(\frac{a_1(x+ifp^N)}{fp^{N+1}}, u^{fp^{N+1}}, q^{fp^{N+1}} \mid a_1\right)\\
&= u^x \sum_{i=0}^{p-1} \frac{[fp^N:q]^k [p:q^{fp^N}]^k}{1-(u^{fp^N})^p}(u^{fp^N})^i \qquad (4)\\
&\quad\times H_k^{(1)}\left(\frac{\frac{a_1x}{fp^N}+ia_1}{p}, (u^{fp^N})^p, (q^{fp^N})^p \mid a_1\right)\\
&= \frac{u^x[fp^N:q]^k}{1-u^{fp^N}} H_k^{(1)}\left(\frac{a_1x}{fp^N}, u^{fp^N}, q^{fp^N} \mid a_1\right)\\
&= E_{u:a_1,q}^{(k)}(x+fp^k\mathbb{Z}_p).
\end{aligned}$$

Next we show that $|E_{u:a_1,q}^{(k)}|_p \le 1$. Indeed,

$$\begin{aligned}
&E_{u:a_1,q}^{(k)}(x+fp^N\mathbb{Z}_p)\\
&= \sum_{i=0}^{k} \binom{k}{i}\left(\frac{u^x}{1-u^{fp^N}}\right)[a_1x:q]^{k-i}[fp^N:q]^i q^{a_1xi} \qquad (5)\\
&\quad\times H_i^{(1)}\left(u^{fp^N}, q^{fp^N} \mid a_1\right).
\end{aligned}$$

By induction on i, we see that

$$\left|\frac{u^x}{1-u^{fp^N}} H_i^{(1)}\left(u^{fp^N}, q^{fp^N} \mid a_1\right)\right|_p \le 1, \text{ for all } i,$$

where we use the assumption $|1-u^f|_p \geq 1$, it follows that we have

$$\left| E^{(k)}_{u:a_1,q}\left(x + ifp^N + fp^N\mathbb{Z}_p\right)\right|_p \leq 1. \tag{6}$$

Thus $E^{(k)}_{u:a_1,q}$ is a measure on X. This measure yields an integral for each non-negative integers k as follows:

PROPOSITION 5. *For $k \geq 0$, we have*

$$\int_X dE^{(k)}_{u:a_1,q}(x) = \int_{\mathbb{Z}_p} dE^{(k)}_{u:a_1,q} = \frac{1}{1-u} H^{(1)}_k(u, q \mid a_1).$$

It is easy to see that

$$H_0(u, q \mid a_1) = 1.$$

We may now mention the following formula which is easy to prove by (5) and (6):

$$E^{(k)}_{u:a_1,q}(x + fp^N\mathbb{Z}_p) = [a_1 x : q]^k \frac{u^k}{1-u^{fp^N}} + [fp^N : q] \times (p\text{-integral}).$$

Hence, we obtain the following :

$$\begin{aligned}\int_X dE^{(k)}_{u:a_1,q}(x) &= \frac{1}{1-u}\int_X [a_1 x : q]^k d\mu_u(x)\\ &= \frac{1}{1-u} H^{(1)}_k(u, q \mid a_1).\end{aligned}$$

From the above definition, we have the following:

THEOREM 6. *Let $a_1, a_2, \cdots, a_r$ be p-adic integers. Then we obtain:*

$$\begin{aligned}&\left(\frac{1}{1-u}\right)^r H^{(r)}_{k,\chi}(u, q \mid a_1, \cdots, a_r)\\ &= \frac{1}{(1-u^d)^r}[d : q]^k \sum_{i_1,\cdots,i_r=0}^{d-1} u^{\sum_{j=1}^r i_j}\left(\prod_{j=1}^r \chi(i_j)\right)\\ &\quad\times H^{(r)}_k\left(\frac{\sum_{j=1}^r a_j i_j}{d}, u^d, q^d \mid a_1, \cdots, a_r\right).\end{aligned} \tag{7}$$

Note that

$$\int_X \chi(x)dE^{(k)}_{u:a_1,q}(x) = \frac{1}{1-u}H^{(1)}_{k,\chi}(u,q|a_1). \tag{8}$$

Let ω be denoted as the Teichmuller character mod p (if $p = 2$, mod 4). For $x \in X^*$, we set

$$\langle x : q\rangle = \frac{[x:q]}{w(x)}.$$

Note that $|\langle x : q\rangle - 1|_p < p^{-\frac{1}{p-1}}$, $\langle x : q\rangle^s$ is defined as $\exp(s \log_p \langle x : q\rangle)$ for $|s|_p \le 1$. For $s \in \mathbb{Z}_p$, define

$$L_{p,q:a_1}(u \mid s, \chi) = \int_{X^*} \langle a_1 x : q\rangle^{-s}\chi(x)d\mu_u(x).$$

Then we have

$$\begin{aligned}&\frac{1}{1-u}L_{p,q:a_1}(u : -k, x)\\ &= \frac{1}{1-u}H^{(1)}_{k,\chi}(u,q \mid a_1) - \frac{\chi(p)[p:q]^k}{1-u^p}H^{(1)}_{k,\chi}(u^p, q^p \mid a_1).\end{aligned}$$

Indeed,we see

$$\begin{aligned}&\int_{X^*}\langle a_1 x : q\rangle^k \chi\omega^k(x)d\mu_u(x)\\ &= \int_X \chi(x)[a_1x:q]^k d\mu_u(x) - \chi(p)[p:q]^k\frac{1-u}{1-u^p}\int_X [a_1x : q^p]^k d\mu_{u^p}(x).\end{aligned}$$

Since $|\langle a_1x : q\rangle - 1|_p < p^{-\frac{1}{p-1}}$ for $x \in X^*$, we obtain

$$\langle a_1 x : q\rangle^{p^n} \equiv 1 \pmod{p^n}.$$

For $k \equiv k' \pmod{(p-1)p^n}$, we have

$$L_{p,q:a_1}(u : -k, \chi\omega^k) \equiv L_{p,q:a_1}(u : -k', \chi\omega^{k'}) \pmod{p^n}.$$

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Leechae Jang, Department of Mathematics and Computer Science, Konkuk University, Choongju 380-701, Korea
E-mail: leechae.jang@kku.ac.kr

Taekyun Kim, Institute of Science Education, Kongju National University, Kongju 314-701, Korea
E-mail: tkim@kongju.ac.kr