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ON THE GLOBAL MINIMAL DRINFELD MODULE EQUATIONS

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ABSTRACT. We show that any Drinfeld module of rank 2 on $\mathbb{F}_q[T]$ over a global function field K has a global minimal Drinfeld module equation if and only if $h(\mathcal{O}_K) = 1$.

1. Introduction

Let k be a global function field over a finite constant field $\mathbb{F}_q$ where q is a power of a prime number p. Drinfeld[1] introduced the notion of elliptic modules, which are now known as Drinfeld modules, on k in analogy with classical elliptic curves. Drinfeld modules of rank 2 have many interesting properties analogous to those of elliptic curves.

Let L be a number field and E an elliptic curve defined over L. As is well known [3], if L has class number 1, then there exists a global minimal Weierstrass equation for E. The converse to this statement was proved by Silverman[2]. In this article, we will prove that the equivalence holds for the Drinfeld module case.

We fix the following notations:

$$\begin{aligned} A &= \mathbb{F}_q[T],\ k = \mathbb{F}_q(T),\ \infty = \left(\frac{1}{T}\right) \\ k_\infty &= \text{the completion of } k \text{ at } \infty \\ C &= \text{the completion of algebraic closure of } k_\infty \\ K &= \text{a separable extension of } k \text{ in } C \\ M_K &= \text{a complete set of inequivalent absolute values of } K \\ \mathcal{O}_K &= \text{the integral closure in } K \text{ of } A. \end{aligned}$$

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2. Drinfeld module equations

Let $K\{\tau_p\}$ be the noncommutative ring with a commutation relation,

$$\tau_p x = x^p \tau_p \quad \text{for} \quad x \in K.$$

Any injective ring homomorphism

$$\phi : A \to K\{\tau_p\}$$

has its value in $K\{\tau\}$ where $\tau = {\tau_p}^n$ with $q = p^n$. For each $u \in K\{\tau\}$, denote $\deg_\tau(u)$ the degree of u as a polynomial in τ.

By a Drinfeld module of rank r over K, we mean an injective ring homomorphism

$$\phi : A \to K\{\tau\}$$

such that, for $a \in A$

(i) $\deg_\tau \phi(a) = r \cdot \deg a$,
(ii) the constant term of $\phi(a)$ is equal to a.

We write ϕ_a for $\phi(a)$.

Let ϕ and ψ be two Drinfeld modules over K. By a *homomorphism*

$$u : \phi \to \psi,$$

we mean an element $u \in K\{\tau_p\}$ such that

$$u \cdot \phi_a = \psi_a \cdot u$$

for all $a \in A$. A homomorphism u is called an *isomorphism* if u is invertible in $K\{\tau_p\}$, i.e., a nonzero constant in K.

In this article, by a Drinfeld module over K, we always mean a Drinfeld module of rank 2. Thus a Drinfeld module ϕ is completely determined by

$$\phi_T(X) = TX + g(\phi)X^q + \Delta(\phi)X^{q^2},$$

where $g(\phi), \Delta(\phi) \in K$. The value $\Delta(\phi)$ is called a *discriminant* of ϕ.

Let S be the subset of M_K whose elements are induced by the places of K lying above ∞. For any $v \in M_K - S$, we denote by K_v(resp. $\mathcal{O}_{K,v}$) the completion of K(resp. $\mathcal{O}_K$) at v. For $x \in K_v$, $v(x)$ denotes the normalized valuation of x at v.

DEFINITION 2.1. Let $v \in M_K - S$ and let ϕ be a Drinfeld module over K. For a Drinfeld module ψ which is isomorphic to ϕ over K, the equation $\psi_T(X) = TX + gX^q + \Delta X^{q^2}$ is called a *minimal equation* for ϕ over K at v if

(1) $g, \Delta \in \mathcal{O}_{K,v}$,
(2) $v(\Delta)$ is minimal subject to (1).

An equation as above is called a *global minimal equation* for ϕ over K if it is simultaneously minimal for every $v \in M_K - S$.

Let $\Delta_{\phi,v}$ be the discriminant of a minimal equation for ϕ over K at v. We define the *discriminant ideal* $\mathcal{D}_{\phi/K}$ by the formula:

$$\mathcal{D}_{\phi/K} = \prod_{v \notin S} \mathfrak{p}_v^{v(\Delta_{\phi,v})},$$

where $\mathfrak{p}_v$ is the prime ideal of $\mathcal{O}_K$ associated to v.

From the definition, it follows that ϕ over K has a global minimal equation over K if and only if there is a Drinfeld module ψ which is isomorphic to ϕ over K such that $(\Delta(\psi)) = \mathcal{D}_{\phi/K}$.

Let $J = J_K(S)$ be the set of K-ideles without w-component for all $w \in S$. Throughout this paper, we assume that none of v is contained in S. For each v, let U_v be the group of units of $\mathcal{O}_{K,v}$ and let $U = U_K(S) = \prod_{v \notin S} U_v$.

DEFINITION 2.2. Let ϕ be a Drinfeld module over K. The *minimal discriminant* of ϕ is the element $\Delta(\phi/K) \in J/U^{q^2-1}$ with the property that, for all v, the local component $\Delta(\phi/K)_v \in K_v^\times/U_v^{q^2-1}$ contains the discriminant of any minimal equation for ϕ over K at v.

REMARK 2.3. The discriminant ideal is precisely the ideal corresponding to $\Delta(\phi/K)$ under the natural map

$$J/U^{q^2-1} \to J/U.$$

For each $u \in J$, we shall denote the image of u under the above map by (u). Then $\mathcal{D}_{\phi/K} = (\Delta(\phi/K))$. Thus $\Delta(\phi/K)$ is a somewhat finer invariant than the minimal discriminant ideal.

PROPOSITION 2.4. *Let ϕ be a Drinfeld module over K. Then the minimal discriminant $\Delta(\phi/K)$ is contained in $K^\times J^{q^2-1}/U^{q^2-1}$.*

Proof. Take any Drinfeld module equation for ϕ over K, say with discriminant Δ. Then for each v, there exists $u_v \in K_v$ which gives a discriminant $\Delta(\phi/K)_v = u_v^{q^2-1}\Delta$. Letting $u \in J$ be the ideal with local components u_v, we see that $\Delta(\phi/K) \equiv \Delta u^{q^2-1} \mod U^{q^2-1}$. □

An immediate corollary of Proposition 2.4 is the fact that the discriminant ideal is a (q^2-1)-th power in the ideal class group $Cl(\mathcal{O}_K)$.

Let Δ be as in the proof of Proposition 2.4. We define the ideal δ_Δ by the formula:

$$\delta_\Delta = \prod \mathfrak{p}_v{}^{v(u_v)},$$

where u_v's are also as in the proof of Proposition 2.4. Then

$$\mathcal{D}_{\phi/K} = (\Delta)\delta_{\Delta}{}^{q^2-1}.$$

The class of the ideal δ_Δ in $Cl(\mathcal{O}_K)$ is independent of the choice of a defining equation of ϕ over K. We denote this class by $\delta_{\phi/K}$. Then $\delta_{\phi/K}$ is a ideal class whose (q^2-1)-th power is the ideal class of $\mathcal{D}_{\phi/K}$. Here we propose to call $\delta_{\phi/K}$ the *Drinfeld class.*

There is another way of finding an ideal class whose (q^2-1)-th power is the ideal class of $\mathcal{D}_{\phi/K}$. Consider the following composition of maps

$$\mu : \frac{K^\times J^{q^2-1}}{U^{q^2-1}} \twoheadrightarrow \frac{K^\times J^{q^2-1}}{K^\times U^{q^2-1}} \approx \frac{J^{q^2-1}}{K^{\times q^2-1}U^{q^2-1}} \overset{\sim}{\longleftarrow} \frac{J}{K^\times U}.$$

For the latter, the right-hand map is given by raising to the (q^2-1)-th power, while the two groups in the center are isomorphic because

$$K^\times \cap J^{q^2-1} = K^{\times q^2-1}.$$

But the above two methods give the the same result.

LEMMA 2.5. $\mu(\Delta(\phi/K)) = \delta_{\phi/K}$.

Proof. It is trivial from the construction. □

The importance of the Drinfeld class is that it determines whether the Drinfeld module has a global minimal equation.

THEOREM 2.6. *Let ϕ be a Drinfeld module over K. Then the following are equivalent:*

(a) $\Delta(\phi/K) \in K^\times U^{q^2-1}/U^{q^2-1}$;
(b) $\delta_{\phi/K} = 1$;
(c) *ϕ has a global minimal equation over K.*

Proof. (a)⇔(b). Since $\delta_{\phi/K} = \mu(\Delta(\phi/K))$, this is clear from the fact that $K^\times U^{q^2-1}/U^{q^2-1}$ is the kernel of μ.

(c)⇒(b). For each v, we can take $\Delta(\phi/K)_v = \Delta$ where Δ is the discriminant of a global minimal equation for ϕ over K. Thus $\delta_{\phi/K} = \mu(\Delta(\phi/K)) = 1$.

(b)⇒(c). Consider a Drinfeld module equation $\phi_T(X) = TX + g(\phi)X^q + \Delta(\phi)X^{q^2-1}$. Then $\delta_{\Delta(\phi)} = \prod \mathfrak{p}_v{}^{v(u_v)}$, where u_v's are as in the proof of Proposition 2.4. Let $\delta_{\Delta(\phi)} = (u)$ for some $u \in K$ i.e., $v(u_v) = v(u)$ for all v. Put $\psi = u^{-1}\phi u$, then $\Delta(\psi) = u^{q^2-1}\Delta(\phi)$ and so $v(\Delta(\psi)) = v(u^{q^2-1}\Delta(\phi)) = v((u/u_v)^{q^2-1}\Delta(\phi/K)_v) = v(\Delta(\phi/K)_v)$ for all v. Then $\psi_T(X)$ is a global minimal equation for ϕ over K. □

COROLLARY 2.7. *If $h(\mathcal{O}_K) = 1$, then any Drinfeld module ϕ over K has a global minimal Drinfeld module equation over K.*

The aim of the following theorem is to verify the converse of the Corollary 2.7 which is the main result in this work.

THEOREM 2.8. *Let $\mathfrak{a}$ be an ideal class of $\mathcal{O}_K$. Then there exists a Drinfeld module ϕ over K with the Drinfeld class $\delta_{\phi/K} = \mathfrak{a}$.*

Proof. Choose prime ideals $\mathfrak{p}_{v_1}, \mathfrak{p}_{v_2}, \mathfrak{p}_{v_3} \in \mathfrak{a}$ with v_1, v_2, v_3 all distinct. Then ${\mathfrak{p}_{v_2}}^{-1}$, ${\mathfrak{p}_{v_3}}^{-1} \in \mathfrak{a}^{-1}$ and then $\mathfrak{p}_{v_1}{\mathfrak{p}_{v_2}}^{-1} = (a)$, $\mathfrak{p}_{v_1}{\mathfrak{p}_{v_3}}^{-1} = (b)$ for some $a, b \in K$. Let $\pi \in K$ with $v_1(\pi) = 1$. Set $g = a^{-1}$, $\Delta = b^{-1}$ and $u = (u_v) \in J$, where $u_v = 1$ if $v \neq v_1$ and $u_{v_1} = \pi$. Consider a Drinfeld module ϕ determined by $\phi_T(X) = TX + gX^q + \Delta X^{q^2-1}$. Then for any v with $v \neq v_i$, $i = 1, 2, 3$, $v(g) = v(\Delta) = 0$, i.e., $\phi_T(X)$ is minimal at v. Also $v_2(g) = v_2(a^{-1}) = 1$ and $v_2(\Delta) = v_2(b^{-1}) = 0$, i.e., $\phi_T(X)$ is minimal at v_2. Similarly, $\phi_T(X)$ is minimal at v_3. Put $\psi = \pi^{-1}\phi\pi$. Then $\psi_T = T + \pi^{q-1}g\tau + \pi^{q^2-1}\Delta\tau^2$. Then $v_1(\pi^{q-1}g) = v_1(\pi^{q-1}) + v_1(g) = q-2$ and $v_1(\pi^{q^2-1}\Delta) = v_1(\pi^{q^2-1}) + v_1(\Delta) = q^2 - 2$. Since $0 \leq q-2 < q-1$, $0 \leq q^2-2 < q^2-1$, $\psi_T(X)$ is minimal at v_1. Therefore $\Delta(\phi/K) \equiv \Delta u^{q^2-1}$ mod U^{q^2-1} and then $\mu(\Delta(\phi/K)) = \overline{(u)} = \bar{\mathfrak{p}}_{v_1} = \mathfrak{a}$. □

COROLLARY 2.9. *If $h(\mathcal{O}_K) > 1$, then there exists a Drinfeld module ϕ over K which does not have a global minimal Drinfeld module equation over K.*

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