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PRIME RADICALS OF SKEW LAURENT POLYNOMIAL RINGS

JUNCHEOL HAN

ABSTRACT. Let R be a ring with an automorphism σ. An ideal I of R is *σ-ideal* of R if $\sigma(I) = I$. A proper ideal P of R is *σ-prime ideal* of R if P is a σ-ideal of R and for σ-ideals I and J of R, $IJ \subseteq P$ implies that $I \subseteq P$ or $J \subseteq P$. A proper ideal Q of R is *σ-semiprime ideal* of Q if Q is a σ-ideal and for a σ-ideal I of R, $I^2 \subseteq Q$ implies that $I \subseteq Q$. The σ-prime radical is defined by the intersection of all σ-prime ideals of R and is denoted by $P_\sigma(R)$. In this paper, the following results are obtained: (1) For a principal ideal domain R, $P_\sigma(R)$ is the smallest σ-semiprime ideal of R; (2) For any ring R with an automorphism σ and for a skew Laurent polynomial ring $R[x, x^{-1}; \sigma]$, the prime radical of $R[x, x^{-1}; \sigma]$ is equal to $P_\sigma(R)[x, x^{-1}; \sigma]$.

1. Introduction and some definitions

Throughout this paper, R will denote an associative ring with identity, σ will be an automorphism of R. A left (resp. right, two-sided) ideal I of R is called a left (resp. right, two-sided) *σ-ideal* if $\sigma(I) = I$. An ideal P of R is called *σ-prime ideal* if P ($\neq R$) is a σ-ideal and for σ-ideals I, J of R, $IJ \subseteq P$ implies that $I \subseteq P$ or $J \subseteq P$. An ideal Q of R is called *σ-semiprime ideal* if for any σ-ideal I, $I^2 \subseteq Q$ implies that $I \subseteq Q$. R is called a *σ-prime* (resp. *σ-semiprime*) *ring* if (0) is a σ-prime (resp. σ-semiprime) ideal. For more things about these terminologies, refer to [3], [5], and [6]. Note that every σ-prime ideal of R is σ-semiprime ideal, and every prime (resp. semiprime) ring is σ-prime (resp. σ-semiprime).

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Recall that the prime radical (in other words, lower nil radical) of R (denoted by $P(R)$) is the intersection of all prime ideals of R. We can define σ-prime radical (in other words, σ-lower nil radical) of R (denoted by $P_\sigma(R)$) by the intersection of all σ-prime ideals of R. In Section 2, we will investigate some properties of $P_\sigma(R)$, in particular, we will show that $P_\sigma(R)$ is the smallest σ-semiprime ideal of principal ideal domain R.

Recall that the skew polynomial ring $R[x;\sigma]$ is a ring of polynomials in x with coefficients in R and subject to the relation $xa = \sigma(a)x$, for all $a \in R$. The skew Laurent polynomial ring $R[x, x^{-1};\sigma]$ is a localization of $R[x;\sigma]$ with respect to the set of powers of x and so $R[x, x^{-1};\sigma]$ consists of $\sum_{i=-\infty}^{\infty} a_i x^i$ with only finitely many nonzero terms (these are called the skew Laurent polynomials). In [6], A. Moussavi has found some results on semiprimitivity of $P(R[x;\sigma])$ for a left Noetherian ring with the assending chain condition on the right annihilators and a ring monomorphism σ of R and he proved that Jacobson radical of $P(R[x;\sigma])$ is equal $N(R)[x;\sigma]$ ($N(R)$ is the nilpotent radical of R) if such a ring R is semiprime or σ-prime. In [2], D. A. Jordan has obtained conditions which are sufficient for $R[x, x^{-1};\sigma]$ primitive. In [3], D. A. Jordan also has obtained some results on the primitivity of $R[x, x^{-1};\sigma]$ for a commutative Noetherian ring with an automorphism σ. In [5], A. Leroy and J. Matczuk have found the necessary and sufficient conditions for the primitivity of $R[x, x^{-1};\sigma]$ for a Noetherian P.I. ring with an automorphism σ. In Section 3, we will show that the prime radical of a skew Laurent polynomial ring $R[x, x^{-1};\sigma]$ is equal to the $P_\sigma(R)[x, x^{-1};\sigma]$.

EXAMPLE 1.1. Let $\mathbb{Z}$ be the ring of integers. Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$ be the upper 2×2 triangular matrix ring over $\mathbb{Z}$. Let $\sigma : R \to R$ be a map defined by $\sigma\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}$ for all $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in R$. Then σ is an automorphism of R and $I = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{pmatrix}$ is a σ-ideal of R.

EXAMPLE 1.2. Let F be any field and $R = F[x]$ be the polynomial ring over F. Let $\sigma : R \to R$ be a map defined by $\sigma(f(x)) = f(-x)$ for all $f(x) \in R$. Then σ is an automorphism of R and xR is a σ-prime ideal of R.

EXAMPLE 1.3. Let $\mathbb{Z}$ be the ring of integers and let $R = \mathbb{Z} \times \mathbb{Z}$. Consider a map $\sigma : R \to R$ defined by $\sigma((a, b)) = (b, a)$ for all $(a, b) \in R$.

Then σ is an automorphism of R. For an ideal $I = \mathbb{Z} \times \{0\}$ of R, I is not a σ-ideal of R since $\sigma(I) = \{0\} \times \mathbb{Z} \neq I$.

2. σ-prime radical of a ring R

Since the prime radical of R is the smallest semiprime ideal of R, we can also have the following question:

Question. For an automorphism σ of a ring R, is $P_\sigma(R)$ the smallest σ-semiprime ideal of R?

It is clear that for an automorphism σ of a ring R, $P_\sigma(R)$ is a σ-semiprime ideal of R. In this section, we will show that for any principal ideal domain R, the question is affirmative.

The definitions and the resluts in this section are obtained by the similar arguments on prime radical of ring R in [4]. A nonempty subset S of a ring R is called a *σ-m-system* if, for any $a, b \in S$ such that $\sigma(a) \in (a)$, $\sigma(b) \in (b)$, there exists $r \in R$ such that $arb \in S$.

PROPOSITION 2.1. *Let R be a principal ideal domain with an automorphism σ. If $P \subsetneq R$ is any σ-ideal of R, then the following are equivalent:*

(1) *P is σ-prime;*
(2) *For any $a, b \in R$ such that $\sigma(a) \in (a)$, $\sigma(b) \in (b)$, $(a)\cdot(b) \subseteq P$ implies that $a \in P$ or $b \in P$;*
(3) *For any $a, b \in R$ such that $\sigma(a) \in (a)$, $\sigma(b) \in (b)$, $aRb \subseteq P$ implies that $a \in P$ or $b \in P$.*

Proof. (1) $\Rightarrow$ (2). Clear.
(2) $\Rightarrow$ (3). If $aRb \subseteq P$ such that $\sigma(a) \in (a)$, $\sigma(b) \in (b)$, then $(a)\cdot(b) = RaRRbR \subseteq RPR = P$. By (2), $a \in P$ or $b \in P$.
(3) $\Rightarrow$ (1). Clear. □

COROLLARY 2.2. *Let R be a principal ideal domain with an automorphism σ. Then P is a σ-prime ideal of R if and only if $R \setminus P$ is a σ-m-system.*

Proof. It follows from the definition of σ-m-system and Proposition 2.1. □

For a σ-ideal I in a ring R with an automorphism σ, let $P_\sigma(R : I) = \{r \in R :$ every σ-m-system containing r meets $I\}$. Then we have the following theorem.

THEOREM 2.3. *Let R be a principal ideal domain with an automorphism σ. Then for any σ-ideal I of a ring R, $P_\sigma(R : I)$ equals to the intersection of all the σ-prime ideals containing I. In particular, $P_\sigma(R : I)$ is a σ-ideal of R.*

Proof. Let $a \in P_\sigma(R : I)$ and P be any σ-prime ideal of R containing I. Then $R \setminus P$ is a σ-m-system $R \setminus P$ by Corollary 2.2. This σ-m-system cannot contain a, for otherwise $(R \setminus P) \cap I \neq \emptyset$, a contradiction. Therefore, we have $a \in P$. Conversely, assume that $a \notin P_\sigma(R : I)$. Then by definition, there exists a σ-m-system S containing a which is disjoint from I. Note that there exists a σ-prime-ideal P which is maximal in the set of all σ-ideals of R disjoint from S and containing I. Indeed, consider the set Γ_σ of all σ-ideals of R disjoint from S and containing I. Then Γ_σ is nonempty since $I \in \Gamma_\sigma$. Since $\Gamma_\sigma \neq \emptyset$, every σ-ideal in Γ_σ is properly contained in R. Let Γ_σ be partially ordered by inclusion. By Zorn's Lemma there is a σ-ideal P of R which is maximal in Γ_σ. Let U, V be σ-ideals of R such that $UV \subseteq P$. If $U \not\subseteq P$ and $V \not\subseteq P$, then each of the σ-ideals $P + U$ and $P + V$ properly contains P and hence must meet S. Consequently, for some $p_i \in P$, $u \in U$ and $v \in V$, $p_1 + u = s_1 \in S$ and $p_2 + v = s_2 \in S$. Since S is a σ-m-system, there exists an element $r \in R$ such that $s_1 r s_2 \in S$. Thus $s_1 r s_2 = p_1 r p_2 + p_1 r v + u r p_2 + u r v \in P + UV \subseteq P$, a contradiction since $s_1 r s_2 \in S \cap P = \emptyset$. Therefore $U \subseteq P$ or $V \subseteq P$, and so P is prime. Hence we have $a \notin P$, as desired. □

A nonempty subset S of a ring R is called an *σ-n-system* if, for any $a \in S$ such that (a) is σ-ideal of R there exists $r \in R$ such that $ara \in S$.

PROPOSITION 2.4. *Let R be a principal ideal domain with an automorphism σ. For any σ-ideal Q of R, the following are equivalent:*

(1) *Q is σ-semiprime;*
(2) *For any $a \in R$ such that (a) is σ-ideal of R, $(a)^2 \subseteq Q$ implies that $a \in Q$;*
(3) *For any $a \in R$ such that (a) is σ-ideal of R, $aRa \subseteq Q$ implies that $a \in Q$.*

Proof. It is similar to the proof as given in the Proposition 2.1. □

COROLLARY 2.5. *Let R be a principal ideal domain with an automorphism σ. Then P is a σ-semiprime ideal of R if and only if $R \setminus P$ is a σ-n-system.*

Proof. It follows from the definition of *σ-n-system* and Proposition 2.4. □

THEOREM 2.6. *Let R be a principal ideal domain with an automorphism σ. For any σ-ideal Q of R, the following are equivalent:*

(1) *Q is a σ-semiprime ideal;*
(2) *Q is an intersection of σ-prime ideals;*
(3) *$Q = P_\sigma(R : Q)$.*

Proof. (3) $\Rightarrow$ (2). It follows from Theorem 2.3. since any σ-prime ideal is σ-semiprime.

(2) $\Rightarrow$ (1). It follows from the observation that every σ-prime ideal is σ-semiprime and the intersection of any σ-semiprime ideals is σ-semiprime.

(1) $\Rightarrow$ (3). Suppose that Q is a σ-semiprime ideal. By definition of σ-n-system, $Q \subseteq P_\sigma(R : Q)$. We want to show that $P_\sigma(R : Q) \subseteq Q$. Let $a \notin Q$ and let $N = R \setminus Q$. Then N is a σ-n-system containing a by Corollary 2.5. Then there exists a σ-m-system $M \subseteq N$ such that $a \in M$. Indeed, Consider a subset $M = \{a_1, a_2, a_3, \dots\}$ defined inductivley as follows: $a_1 = a$, $a_{i+1} = a_i r_i a_i \in N$ for some $r_i \in R$, where $i = 1, 2, \cdots$. We will show that M is a σ-m-system. Let a_i, $a_j \in M$ be arbitrary. If $i \leqslant j$, then $a_{j+1} \in a_j R a_j \subseteq a_i R a_j$, which means $a_{j+1} \in M$. If $j \leqslant i$, then similarly $a_{i+1} \in M$. Hence there is a σ-m-system $M \subseteq N$ such that $a \in M$. Since M is disjoint from Q, $a \notin P_\sigma(R : Q)$. □

COROLLARY 2.7. *Let R be a principal ideal domain with an automorphism σ. Then $P_\sigma(R : I)$ is the smallest σ-semiprime ideal of R which contains I.*

Proof. If follows from the Theorem 2.6. □

For a ring R with an automophism σ, $P_\sigma(R : (0))$ (simply denoted by $P_\sigma(R)$) is called the *σ-prime radical* of R. We can note that $P_\sigma(R)$ is the intersection of all σ-prime ideals of R by Theorem 2.3 and clearly, it is a σ-semiprime ideal of R and in particular, for any principal ideal domain R, it is the smallest σ-semiprime ideal of R by Corollary 2.7.

PROPOSITION 2.8. *Let R be a principal ideal domain with an automorphism σ. Then the following are equivalent:*

(1) *R is a σ-semiprime ring;*
(2) *$P_\sigma(R) = (0)$;*
(3) *R has no nonzero nilpotent σ-ideal.*

Proof. (1) $\Leftrightarrow$ (2) and (3) $\Rightarrow$ (1) are clear. It remains to show the implication (1) $\Rightarrow$ (3). Let R be a σ-semiprime ring and I be a nilpotent σ-ideal. Then $I^n = (0)$ and $I^{n-1} \neq (0)$ for some positive integer n. Suppose that R is a σ-semiprime ring. If $n \geqslant 2$, then $(I^{n-1})^2 = I^{2n-2} \subseteq I^{2n} = (0)$ implies $I^{n-1} = (0)$ since R is σ-semiprime, a contradiction. Thus $n = 1$ and so $I = (0)$. □

3. Prime radicals of skew Laurent polynomial rings

For any σ-ideal I of a ring R with an automorphism σ, we can have a reduced automorphism $\bar{\sigma}$ on R/I defined by $\bar{\sigma}(a + I) = \sigma(a) + I$ for all $a+I \in R/I$. Then we can note that $\bar{\sigma}$ is an automophism of R/I. Hence we can consider a skew Laurent polynomial ring $(R/I)[x, x^{-1}; \bar{\sigma}]$ with multiplication subject to the relation $x\bar{a} = \bar{\sigma}(\bar{a})x$ for all $\bar{a} = a+I \in R/I$.

LEMMA 3.1. *Let R be a ring with an automorphism σ and let K, I be ideals of R such that $R \supseteq K \supseteq I$. Then K is a σ-ideal of R if and only if K/I is a $\bar{\sigma}$-ideal of R/I.*

Proof. It follows from the definition of reduced automorphism $\bar{\sigma}$. □

LEMMA 3.2. *Let R be a ring with an automorphism σ and let I be an ideal of R. Then I is a σ-semiprime ideal of R if and only if R/I is a $\bar{\sigma}$-semiprime ring.*

Proof. ($\Rightarrow$) Suppose that I is a σ-semiprime ideal of R. If K/I is any $\bar{\sigma}$-ideal of R/I such that $(K/I)^2 = (\bar{0})$, the zero ideal of R/I. Then $K^2 = I$. By Lemma 3.1, K is σ-ideal of R. Since I is a σ-semiprime ideal, $K = I$ and so $K/I = (\bar{0})$, which means that R/I is a $\bar{\sigma}$-semiprime ring.

($\Leftarrow$) Suppose that R/I is a $\bar{\sigma}$-semiprime ring. If Q is any σ-ideal of R such that $Q^2 \subseteq I$, then $(\bar{0}) = Q^2/I = (Q/I)^2$. Since R/I is a $\bar{\sigma}$-semiprime ring, $Q/I = (\bar{0})$, so $Q = I$. Hence I is a σ-semiprime ideal of R. □

LEMMA 3.3. *Let R be a ring with an automorphism σ and let I be a σ-ideal of R. Then for such a reduced automorphism $\bar{\sigma}$ on R/I, we have $R[x, x^{-1}; \sigma]/I[x, x^{-1}; \sigma] \simeq (R/I)[x, x^{-1}; \bar{\sigma}]$.*

Proof. Define $\theta : R[x, x^{-1}; \sigma] \longrightarrow (R/I)[x, x^{-1}; \bar{\sigma}]$ by $\theta(f(x)) = \sum_{i=m}^{n} \bar{\sigma}(\bar{a_i})x^i$ for all $f(x) = \sum_{i=m}^{n} a_i x^i \in R[x, x^{-1}; \sigma]$. It is straitforward to show that θ is an epimorphism and the kernel of θ is equal to $I[x, x^{-1}; \sigma]$. Hence we have the result by the First Fundamental Homomorphism Theorem. □

LEMMA 3.4. *Let R be a ring with an automorphism σ. Then R is σ-semiprime if and only if $A = R[x, x^{-1}; \sigma]$ is semiprime.*

Proof. ($\Rightarrow$). Suppose that R is σ-semiprime. Let J be an ideal of A such that $J^2 = (0)$. Consider an ideal of R, J_0, the set of all leading coefficients of every $f(x) \in J$. Then J_0 is a σ-ideal of R. Indeed, for any $f \in J$, by letting $f = a_n x^n + \{\text{terms of lower degrees}\}$ where $a_n \in J_0$ and by considering $g = xf \in J$(resp. $h = x^{-1} f \in J$) we have $g = \sigma(a_n) x^{n+1} + \{\text{terms of lower degrees}\}$ (resp. $h = \sigma^{-1}(a_n) x^{n-1} + \{\text{terms of lower degrees}\}$, and so $\sigma(a_n) \in J_0$ (resp. $\sigma^{-1}(a_n) \in J_0$). Since $J_0^2 = (0)$, and so $J_0 = (0)$ by the assumption. Continuing in this way, every coefficients of $f(x)$ is equal to 0 for all $f(x) \in J$. Thus $J = (0)$, and so A is semiprime.

($\Leftarrow$). Suppose that A is semiprime. Let I be a nonzero σ-ideal of R. Then IA is a nonzero σ-ideal of A. Since A is semiprime, $(IA)^2 = I^2 A \neq (0)$, and then $I^2 \neq (0)$. Hence R is σ-semiprime. □

THEOREM 3.5. *Let R be a ring with an automorphism σ. Then the prime radical of $R[x, x^{-1}; \sigma]$ is equal to $P_\sigma(R)[x, x^{-1}; \sigma]$, i.e.,*

$$P(R[x, x^{-1}; \sigma]) = P_\sigma(R)[x, x^{-1}; \sigma].$$

Proof. Let $I = P_\sigma(R)$. Then I is the smallest σ-semiprime ideal of R by Corollary 2.7 and then R/I is $\bar{\sigma}$-semiprime by Lemma 3.2. Thus $(R/I)[x,\ x^{-1};\ \bar{\sigma}]$ is semiprime by Lemma 3.4 and so $I[x,\ x^{-1};\ \sigma]$is a semiprime ideal of $R[x,\ x^{-1};\ \sigma]$ by Lemma 3.3. Hence we have $I[x,\ x^{-1};\ \sigma] \supseteq P(R[x, x^{-1}; \sigma])$. To show the converse inclusion $I[x,\ x^{-1};\ \sigma] \subseteq P(R[x, x^{-1}; \sigma])$, let P be any prime ideal of $R[x,\ x^{-1};\ \sigma]$. Then $P \cap R$ is a σ-prime ideal of R by Proposition 1 in [2]. Since $P \cap R$ is a σ-prime ideal of R, $I \subseteq P \cap R \subseteq P$, which implies that $I[x,\ x^{-1};\ \sigma] \subseteq P$, and so $I[x,\ x^{-1};\ \sigma]\subseteq P(R[x, x^{-1}; \sigma])$. □

REMARK. We will have a question: For a ring with an automorphism σ, what is the prime radical of skew polynomial ring $R[x; \sigma]$? We might have some partial answer to this question. We can note that the above Lemma 3.3 holds for skew polynomial ring $R[x; \sigma]$, i.e., $R[x;\ \sigma]/I[x;\ \sigma] \simeq (R/I)[x;\ \bar{\sigma}]$. In [1], A. W. Goldie and G. O. Michler have shown that for a Noetherian ring R, I is σ-ideal of R if and only if $\sigma(I) \subseteq I$. By using this result, we can also note that for a Noetherian ring R, the above Lemma 3.4 holds for skew polynomial ring $R[x;\sigma]$, i.e., R is σ-semiprime if and only if $A = R[x;\sigma]$ is semiprime. Hence by the similar argument in the proof of Theorem 3.5 we have that for a Noetherian ring R with an automorphism σ, the prime radical of $R[x;\sigma]$ is equal to $P_\sigma(R)[x;\sigma]$.

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Department of Mathematics Education, Pusan National University, Pusan 609-735, Korea
E-mail: jchan@pusan.ac.kr