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CERTAIN SUMMATION FORMULAS DUE TO RAMANUJAN AND THEIR GENERALIZATIONS

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Abstract. The authors aim at deriving four generalized summation formulas, which, upon specializing their parameters, give many summation identities including, especially, the four very interesting summation formulas due to Ramanujan. The results are derived with the help of generalized Dixon's theorem obtained earlier by Lavoie, Grondin, Rathie, and Arora.

1. Introduction and the main results

Recently Lavoie et al.[4] have obtained a large number of interesting summation formulas contiguous to Dixon's theorem (cf. [5, p.250, Eq.(4)] in the following form:

(1.1)

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} a, & b, & c \\ 1+i+a-b, & 1+i+j+a-c \end{matrix} \;\middle|\; 1\right) \\
&= \frac{2^{-2c+i+j}\,\Gamma(1+i+a-b)\,\Gamma(1+i+j+a-c)\,\Gamma(b-i)\,\Gamma(c-i-j)}{\Gamma(a-2c+i+j+1)\,\Gamma(a-b-c+i+j+1)\,\Gamma(b)\,\Gamma(c)} \\
&\cdot\left[A_{i,j}\,\frac{\Gamma\left(\frac{a+1}{2}-c+\left[\frac{i+j+1}{2}\right]\right)\Gamma\left(\frac{a}{2}-b-c+1+i+\left[\frac{j+1}{2}\right]\right)}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{a}{2}-b+1+\left[\frac{i}{2}\right]\right)}\right. \\
&\qquad\left.+\,B_{i,j}\,\frac{\Gamma\left(\frac{a}{2}-c+1+\left[\frac{i+j}{2}\right]\right)\Gamma\left(\frac{a}{2}-b-c+\frac{3}{2}+i+\left[\frac{j}{2}\right]\right)}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{a}{2}-b+\frac{1}{2}+\left[\frac{i+1}{2}\right]\right)}\right]
\end{aligned}
$$

$(i,\, j = 0,\, 1,\, 2,\, 3;\; \Re(a-2b-2c) > -2-2j-j),$

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where $[x]$ denotes, as usual, the greatest integer less than or equal to x throughout this paper. The coefficients $A_{i,j}$ and $B_{i,j}$ appear at the tables in the last section.

The aim of this paper is to give four general summation formulas and then their some special cases. The main results are as follows:

(1.2)

$$\begin{aligned} &1+\left(\frac{1}{2}\right)^2\cdot\frac{1}{4}\cdot\frac{1}{(1+i)(\frac{5}{4}+i+j)\,1!} \\ &+\left(\frac{1\cdot 3}{2\cdot 2}\right)^2\cdot\left(\frac{1\cdot 5}{4\cdot 4}\right)\cdot\frac{1}{(1+i)(2+i)(\frac{5}{4}+i+j)(\frac{9}{4}+i+j)\,2!}+\cdots \\ &=\frac{2^{i+j-\frac{1}{2}}\,\Gamma(1+i)\,\Gamma\left(\frac{5}{4}+i+j\right)\Gamma\left(\frac{1}{2}-i\right)\Gamma\left(\frac{1}{4}-i-j\right)}{\Gamma(i+j+1)\,\Gamma\left(\frac{3}{4}+i+j\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)} \\ &\cdot\left[\alpha_{i,j}\,\frac{\Gamma\left(\frac{1}{2}+[\frac{1}{2}(i+j+1)]\right)\Gamma\left(\frac{1}{2}+i+[\frac{1}{2}(j+1)]\right)}{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{3}{4}+[\frac{1}{2}i]\right)}\right. \\ &\left.\quad+\beta_{i,j}\,\frac{\Gamma\left(1+[\frac{1}{2}(i+j)]\right)\Gamma\left(1+i++[\frac{1}{2}j]\right)}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{4}+[\frac{1}{2}(i+1)]\right)}\right] \end{aligned}$$

$(i=0,\ j=0,1,2,3;\ i=1,\ j=0,1,2;\ i=2,\ j=0,1,2;\ i=3,\ j=0)$,

where the coefficients $\alpha_{i,j}$ and $\beta_{i,j}$ can be obtained by simply putting $a=b=1/2$ and $c=1/4$ in $A_{i,j}$ and $B_{i,j}$ given at the tables of the last section.

(1.3)

$$\begin{aligned} &1+\frac{1}{2}\cdot\left(\frac{1}{4}\right)^2\cdot\frac{1}{(\frac{5}{4}+i)\,(\frac{5}{4}+i+j)\,1!} \\ &+\left(\frac{1\cdot 3}{2\cdot 2}\right)\cdot\left(\frac{1\cdot 5}{4\cdot 4}\right)^2\cdot\frac{1}{(\frac{5}{4}+i)\,(\frac{5}{4}+i+j)\,(\frac{9}{4}+i)\,(\frac{9}{4}+i+j)\,2!}+\cdots \\ &=\frac{2^{i+j-\frac{1}{2}}\,\Gamma\left(\frac{5}{4}+i\right)\Gamma\left(\frac{5}{4}+i+j\right)\Gamma\left(\frac{1}{4}-i\right)\Gamma\left(\frac{1}{4}-i-j\right)}{\{\Gamma(i+j+1)\}^2\,\{\Gamma\left(\frac{1}{4}\right)\}^2} \\ &\cdot\left[\gamma_{i,j}\,\frac{\Gamma\left(\frac{1}{2}+[\frac{1}{2}(i+j+1)]\right)\Gamma\left(\frac{3}{4}+i+[\frac{1}{2}(j+1)]\right)}{\Gamma\left(\frac{3}{4}\right)\Gamma\left(1+[\frac{1}{2}i]\right)}\right. \\ &\left.\quad+\delta_{i,j}\,\frac{\Gamma\left(1+[\frac{1}{2}(i+j)]\right)\Gamma\left(\frac{5}{4}+i+[\frac{1}{2}j]\right)}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}+[\frac{1}{2}(i+1)]\right)}\right] \end{aligned}$$

$(i=0,\ j=0,1,2,3;\ i=1,\ j=0,1,2;\ i=2,\ j=0,1,2;\ i=3,\ j=0)$, where the coefficients $\gamma_{i,j}$ and $\delta_{i,j}$ can be obtained by simply putting $a=1/2$ and $b=c=1/4$ in $A_{i,j}$ and $B_{i,j}$ given at the tables of the last section.

$$
\begin{aligned}
&1+\left(\frac{1}{2}\right)^3\cdot\frac{1}{(1+i)\,(1+i+j)\,1!}\\
&\quad+\left(\frac{1\cdot 3}{2\cdot 2}\right)^3\cdot\frac{1}{(1+i)\,(2+i)\,(1+i+j)\,(2+i+j)\,2!}+\cdots\\
&=\frac{2^{i+j-1}\,\Gamma(1+i)\,\Gamma(1+i+j)\,\Gamma\left(\frac{1}{2}-i\right)\,\Gamma\left(\frac{1}{2}-i-j\right)}{\left\{\Gamma\left(\frac{1}{2}+i+j\right)\right\}^2\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2}\\
&\quad\cdot\left[\epsilon_{i,j}\,\frac{\Gamma\left(\frac{1}{4}+\left[\frac{1}{2}(i+j+1)\right]\right)\,\Gamma\left(\frac{1}{4}+i+\left[\frac{1}{2}(j+1)\right]\right)}{\Gamma\left(\frac{3}{4}\right)\,\Gamma\left(\frac{3}{4}+\left[\frac{1}{2}i\right]\right)}\right.\\
&\qquad\left.+\,\zeta_{i,j}\,\frac{\Gamma\left(\frac{3}{4}+\left[\frac{1}{2}(i+j)\right]\right)\,\Gamma\left(\frac{3}{4}+i+\left[\frac{1}{2}j\right]\right)}{\Gamma\left(\frac{1}{4}\right)\,\Gamma\left(\frac{1}{4}+\left[\frac{1}{2}(i+1)\right]\right)}\right]
\end{aligned}
\tag{1.4}
$$

$(i=0,\ j=0,1,2,3; i=1,\ j=0,1,2;\ i=2,\ j=0,1,2;\ i=3,\ j=0)$, where the coefficients $\epsilon_{i,j}$ and $\zeta_{i,j}$ can be obtained by simply putting $a=b=c=1/2$ in $A_{i,j}$ and $B_{i,j}$ given at the tables of the last section.

(1.5)

$$
\begin{aligned}
&1-\left(\frac{1}{2}\right)^2\cdot\frac{1}{(1+i)\,1!}+\left(\frac{1\cdot 3}{2\cdot 2}\right)^2\cdot\frac{1}{(1+i)\,(2+i)\,2!}-\cdots\\
&=\frac{\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2\,\Gamma(1+i)}{\sqrt{2}\,\Gamma\left(\frac{1}{2}+\frac{1}{2}(i+|i|)\right)}\\
&\quad\cdot\left[\frac{\eta_i}{\Gamma\left(\frac{3}{4}+\frac{1}{2}i\right)\,\Gamma\left(\frac{3}{4}+\frac{1}{2}i-\left[\frac{1}{2}(i+1)\right]\right)}+\frac{\theta_i}{\Gamma\left(\frac{1}{4}+\frac{1}{2}i\right)\,\Gamma\left(\frac{1}{4}+\frac{1}{2}i-\left[\frac{1}{2}i\right]\right)}\right]
\end{aligned}
$$

$(i=0,1,2,3,4,5)$, where the coefficients η_i and θ_i can be obtained by simply putting $a=b=1/2$ in C_i and D_i given at the tables of the last section.

Note that the special case of (1.1) when $i=j=0$ reduces immediately to the classical Dixon's theorem (cf. [5, p.250, Eq.(4)]. Some interesting summation formulas due to Ramanujan (see [2, p.24, Entry 7, Examples 16–19]) are easily derivable as some special cases of our main results.

2. Derivation of the summation formulas (1.2) to (1.5)

In order to prove (1.2), if we take $a = b = 1/2$, $c = 1/4$ in (1.1), we see that the resulting left-hand side is

$$\sum_{k=0}^{\infty} \frac{\left\{\left(\frac{1}{2}\right)_k\right\}^2 \left(\frac{1}{4}\right)_k}{(1+i)_k \left(\frac{5}{4}+i+j\right)_k k!},$$

which is just the left-hand side of (1.2). Also it is seen that the resulting right-hand side of (1.1) in taking $a = b = 1/2$, $c = 1/4$ corresponds to the right-hand side of (1.2).

In the exactly same manner, the formulas (1.3)–(1.5) can be obtained.

3. Special cases

Now we will record some of the special cases of (1.2) to (1.5). Setting $(i, j) = (0,0)$, $(0,1)$, $(1,0)$ in (1.2), (1.3), (1.4), and $i = 0$, 1, 2 in (1.5), we obtain

$$1 + \frac{1}{5}\left(\frac{1}{2}\right)^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \cdot \frac{1\cdot 5}{5\cdot 9} + \cdots = \frac{\pi^2}{4\left\{\Gamma\left(\frac{3}{4}\right)\right\}^4} \tag{3.1}$$

$$1 + \frac{1}{9}\left(\frac{1}{2}\right)^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \cdot \frac{1\cdot 5}{9\cdot 13} + \cdots = \frac{5\,\pi^2}{18\left\{\Gamma\left(\frac{3}{4}\right)\right\}^4} - \frac{5}{9\pi} \tag{3.2}$$

$$1 + \frac{1}{9\cdot 2}\left(\frac{1}{2}\right)^2 + \frac{1}{3}\cdot\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{1\cdot 5}{9\cdot 13} + \cdots = \frac{5\,\pi^2}{9\left\{\Gamma\left(\frac{3}{4}\right)\right\}^4} - \frac{40}{9\pi} \tag{3.3}$$

$$1 + \frac{1}{5^2}\,\frac{1}{2} + \left(\frac{1\cdot 5}{5\cdot 9}\right)^2 \frac{1\cdot 3}{2\cdot 4} + \cdots = \frac{\pi^{5/2}}{8\sqrt{2}\left\{\Gamma\left(\frac{3}{4}\right)\right\}^2} \tag{3.4}$$

$$\begin{aligned} &1 + \frac{1}{5\cdot 9}\,\frac{1}{2} + \frac{1\cdot 5}{5\cdot 9}\,\frac{1\cdot 3}{2\cdot 4}\,\frac{1\cdot 5}{9\cdot 13} + \cdots \\ &\qquad = \frac{5\,\pi^{5/2}}{32\sqrt{2}\left\{\Gamma\left(\frac{3}{4}\right)\right\}^2} - \frac{5\,\pi^{3/2}}{48\sqrt{2}\left\{\Gamma\left(\frac{3}{4}\right)\right\}^2} \end{aligned} \tag{3.5}$$

$$1+\frac{1}{9^2}\,\frac{1}{2}+\left(\frac{1\cdot 5}{9\cdot 13}\right)^2\frac{1\cdot 3}{2\cdot 4}+\cdots = \frac{25\,\pi^{5/2}}{96\sqrt{2}\,\left\{\Gamma\left(\frac{3}{4}\right)\right\}^2}-\frac{125\,\pi^{3/2}}{288\sqrt{2}\,\left\{\Gamma\left(\frac{3}{4}\right)\right\}^2} \tag{3.6}$$

$$1+\left(\frac{1}{2}\right)^3+\left(\frac{1\cdot 3}{2\cdot 4}\right)^3+\cdots=\frac{\pi}{\left\{\Gamma\left(\frac{3}{4}\right)\right\}^4} \tag{3.7}$$

$$1+\frac{1}{2}\left(\frac{1}{2}\right)^3+\frac{1}{3}\left(\frac{1\cdot 3}{2\cdot 4}\right)^3+\cdots=\frac{\pi}{\left\{\Gamma\left(\frac{3}{4}\right)\right\}^4}-\frac{4\left\{\Gamma\left(\frac{3}{4}\right)\right\}^4}{\pi^3} \tag{3.8}$$

$$1+\frac{1}{2^2}\left(\frac{1}{2}\right)^3+\frac{1}{3^2}\left(\frac{1\cdot 3}{2\cdot 4}\right)^3+\cdots=\frac{2\pi}{\left\{\Gamma\left(\frac{3}{4}\right)\right\}^4}-\frac{24\left\{\Gamma\left(\frac{3}{4}\right)\right\}^4}{\pi^3} \tag{3.9}$$

$$1-\left(\frac{1}{2}\right)^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2-\cdots=\frac{\sqrt{\pi}}{\sqrt{2}\left\{\Gamma\left(\frac{3}{4}\right)\right\}^2} \tag{3.10}$$

$$1-\frac{1}{2}\left(\frac{1}{2}\right)^2+\frac{1}{3}\left(\frac{1\cdot 3}{2\cdot 4}\right)^2-\cdots=\frac{\sqrt{2\pi}}{\left\{\Gamma\left(\frac{3}{4}\right)\right\}^2}-\frac{2\sqrt{2}\left\{\Gamma\left(\frac{3}{4}\right)\right\}^2}{\pi^{3/2}} \tag{3.11}$$

$$1-\frac{1}{3}\left(\frac{1}{2}\right)^2+\frac{1}{6}\left(\frac{1\cdot 3}{2\cdot 4}\right)^2-\cdots=\frac{16\sqrt{2\pi}}{9\left\{\Gamma\left(\frac{3}{4}\right)\right\}^2}-\frac{16\sqrt{2}\left\{\Gamma\left(\frac{3}{4}\right)\right\}^2}{3\,\pi^{3/2}}. \tag{3.12}$$

It is remarked that (3.1), (3.4), (3.7), and (3.10) are known results due to Ramanujan, which are recorded in [2, p.24, Examples 16–19] as noted in Section 1.

4. Tables of special values of the coefficients $A_{i,j}$ and $B_{i,j}$, and Tables of C_i and D_i

Special values of $A_{i,j}$

$$A_{0,0} = A_{1,0} = -A_{0,1} = 1;\ A_{1,1} = c - a - 1;$$
$$A_{0,2} = \frac{1}{2}\left\{(a-b-c+1)^2 + (c-1)(c-3) - b^2 + a\right\};$$
$$A_{0,3} = 3ab + c(a-b-c+4) - (a+1)(a+2) - (a-1)(b-1);$$
$$A_{1,2} = a(a-1) + (b+c-3)(c-2a-1);$$
$$A_{2,0} = \frac{1}{2}(a-1)(a-4) - (b^2-5a+1) - (a-b+1)(b+c);$$
$$A_{2,1} = (b-1)(b-2) - (a-b+1)(a-b-c+3);$$
$$A_{2,2} = \frac{1}{2}(a-c+2)(a-2b-c+5)\left\{(a-c+2)(a-2b+2) - a(c-3)\right\} - (b-1)(b-2)(c-2)(c-3);$$
$$A_{3,0} = 5a - b^2 + (a+1)^2 - (2a-b+1)(b+c).$$

Special values of $B_{i,j}$

$$B_{0,0} = 0;\ B_{0,1} = -B_{1,0} = 1;$$
$$B_{0,2} = B_{2,0} = -2;\ B_{1,1} = a - 2b - c + 3;$$
$$B_{0,3} = (a+2)(a+4) - b(2a+5) - 3c(a-b-c+4) + 3;$$
$$B_{1,2} = -\left\{(a-b-c+2)(a-b-c+3) - (b-1)(b-c+1)\right\};$$
$$B_{2,1} = -\left\{(b-1)(b-2) - (a-b-2c+5)(a-b-c+3)\right\};$$
$$B_{2,2} = -2(a-c+2)(a-2b-c+5);$$
$$B_{3,0} = -a + 3b^2 - (a+3)^2 + (2a-3b+5)(b+c)$$

Table of C_i

$$C_0 = 1 = -C_1;\ C_2 = 1 + a - b;\ C_3 = 3b - 2a - 5;$$
$$C_4 = 2(a-b+3)(1+a-b) - (b-1)(b-4);$$
$$C_5 = -4(6+a-b)^2 + 2(b+11)(6+a-b) + b^2 - 13b - 20.$$

Table of D_i

$$D_0 = 0;\ D_1 = 1;\ D_2 = -2;$$
$$D_3 = 2a - b + 1;\ D_4 = -4(a-b+2);$$
$$D_5 = 4(6+a-b)^2 + 2(b-17)(6+a-b) - b^2 - b + 62.$$

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