Stability Analysis of Beck's Column with a Tip Mass Restrained by a Spring

스프링으로 지지된 자유단에 집중질량을 갖는 Beck 기둥의 안정성 해석

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(Received Auguest 23, 2005 : Accepted October 25, 2005)

Key Words : Beck's Column(Beck 기둥), Divergence Critical Load(발산임계하중), Flutter Critical Load(동요임계하중), Flutter Frequency(동요진동수), Follower Force(종동력), Tangential Follower Force(접선종동력), Subtangential Follower Force(경사종동력)

ABSTRACT

The purpose of this paper is to investigate free vibrations and critical loads of the Beck's columns with a tip spring, which carry a tip mass. The ordinary differential equation governing free vibrations of Beck's column subjected to a follower force is derived based on the Bernoulli-Euler beam theory. Both the divergence and flutter critical loads are calculated from the load-frequency curves that are obtained by solving the differential equation numerically. The critical loads are presented in the figures as functions of various non-dimensional system parameters such as the subtangential parameter, mass ratio and spring parameter.

요 약

이 논문은 자유단이 스프링으로 지지되고 자유단 집중질량을 갖는 Beck 기둥의 안정성 해석에 관한 연구이다. Bernoulli-Euler보 이론을 이용하여 경사종동력을 받는 Beck 기둥의 자유진동을 지배하는 미분방정식과 경계조건을 유도하였다. 이 미분방정식을 수치해석하여 하중-고유진동수 곡선을 얻고 이로부터 발산임계하중 및 동요임계하중을 산출하였다. 수치해석의 결과로부터 경사변수, 집중질량 및 스프링 강성이 임계하중에 미치는 영향을 고찰하였다.

1. 서 론

Since columns are one of the most important structural units as well as the beams and plates, free vibrations and stabilities of the columns have been studied by many researchers. In those studies, columns are modeled as the continuous or discrete systems, in which the subjected loads are considered as a conservative or non-conservative forces as shown in Fig. 1(a), (b) and (c).[1,2]

The unreality on structures subjected to the follower forces as a non-conservative force, i.e. Beck's columns had been discussed and argued.[3] However, the load directions can be changed

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systematically due to the advanced control techniques recently and the structures subjected to a follower force can be realistic in the modern engineering fields. From this viewpoint, stability problems of the structures that carry the non-conservative loads such as Beck's columns become to be very important in various engineering fields.[4]

Since Beck[5] had calculated critical loads of the cantilever columns subjected to a tangential follower force in 1952, both free vibrations and stabilities of the cantilever columns subjected to a non-conservative force have been investigated by many researchers. In 1976, Kounadis and Katsikadelis[6] had investigated the effects of the shear deformation and rotatory inertia on the behavior of Beck's column, and Sankaran and Rao[7] had calculated flutter critical loads of the tapered cantilever columns. In 1977, Pedersen[8] had researched stabilities of the uniform cantilever columns restrained by an elastic spring at free end, carrying a tip mass and a subtangential follower force as shown in Fig. 1(b). In 1984, Yoon and Kim[9] had studied the effect of the inertia moment of tip mass on the stability of Beck's column. In 1985, Yoon and Kim[10] had investigated the stability of Beck's columns with a spring at the clamped end. In 1992, Chen and Ku[11] had proposed the finite element model for calculating the natural frequencies and critical loads of Beck's columns carrying a tip mass. In 1994, Kuo and Yang[12] had calculated critical loads of the undamped non-conservative structural systems. In 1996, Sato[13] had studied instabilities of the cantilever Timoshenko columns with a tip mass, carrying a follower force. In 1997, Yoon et al[14] had studied the stability of Beck's columns restrained by a spring at the free end. Also in 1997, Ryu et al[15] had investigated dynamic stabilities of the cantilever columns restrained by the intermediate elastic spring carrying a subtangential follower force. In 1999, Langthjem and Sugiyama[16] had obtained optimal shapes for the dynamic stabilities of cantilever columns subjected to a follower force. In 2002, Andersen and Thomsen[17] had studied the post-critical behavior of Beck's column with a tip mass. In 2003, Dentiko[18] had investigated the lumped damping and stability of Beck's column with a tip mass. And in 2004, Rao and Rao[19] had studied the post-critical behaviour of Euler and Beck columns resting on an elastic foundation.

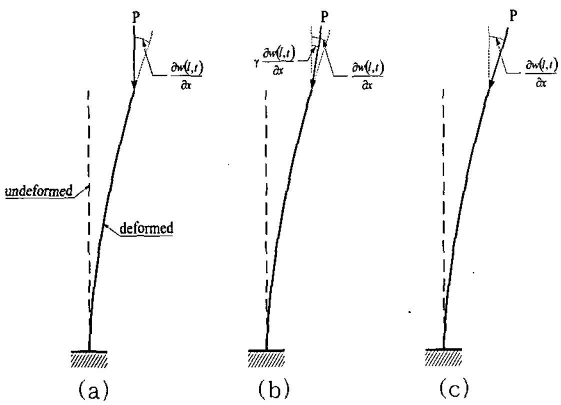

Fig. 1 Columns subjected to (a) conservative and (b), (c) non-conservative forces

It is well known that structures subjected to a conservative force always have the divergence critical loads. However, structures subjected to a non- conservative force have the divergence or flutter critical forces according to the geometry of structure, boundary conditions and characteristics of non-conservative force. For analysing stabilities of such structures, it is not possible to adopt the static concept but possible to adopt the dynamic concept which includes rotatory inertia effect in the theory.

This study deals with natural frequencies and critical loads of Beck's columns with an elastic spring at free end, which carry a tip mass and a subtangential follower force. The ordinary differential equation accompanying with the boundary conditions which governs the free vibrations of such Beck's columns is derived and solved numerically. From the load versus frequency curves, the divergence and flutter critical loads are calculated. The effects of the column geometry on natural frequencies are investigated.

2. Mathematical Model

2.1 Governing Differential Equation

As shown in Fig. 2, the dashed line depicts Beck's column with span length l, which is elastically restrained by a tip spring. The tip mass is attached at free end. For a while the solid line depicts a typical mode shape of Beck's column subjected to a follower force P.

The magnitude of mass is depicted as M whose moment of inertia of mass is J. The spring constant is expressed as K. The angle α is the rotation at free end which is obtained from the analysis result but not an input value. The γ ranging $0 \leq \gamma \leq 1$ is the subtangential parameter which presents the inclined rate of the follower force P from the axis x. It is noted that the column subjected to P with $\gamma=0$ is the well-known Euler's column, P with $0<\gamma<1$ is a subtangential follower force and P with $\gamma=1$ is a follower force. The $w(x,t)$ is the dynamic

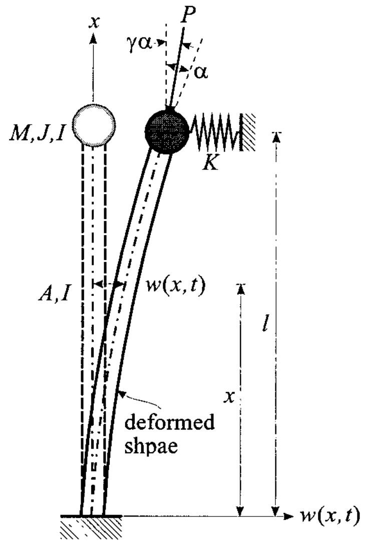

Fig. 2 Typical mode shape and its variables

displacement and t is time.

The partial differential equation governing free vibration of the uniform column subjected to an axial compressive load P is given in Eq. (1), in which the effects of rotatory inertia and shear deformation in the theory are excluded.

$$EI\frac{\partial^4 w(x,t)}{\partial x^4} + P\frac{\partial^2 w(x,t)}{\partial x^2} + \rho A\frac{\partial^2 w(x,t)}{\partial t^2} = 0 \tag{1}$$

where EI is flexural rigidity and ρ is mass density of the column material.

It is assumed that free vibration of the column is a harmonic motion and then its dynamic displacement is presented as follows.

$$w(x,t) = w_x \sin(\omega_i t) \tag{2}$$

where w_x is amplitude of the harmonic motion, which is function of only x, and ω_i is angular frequency and i is mode number.

In order to derive the governing equation as a non-dimensional form, the following non-dimensional variables are introduced.

$$\xi = x/l \tag{3}$$

$$\eta = w_x/l \tag{4}$$

$$p = Pl^2/(EI) \tag{5}$$

$$\mu = M/(\rho A l) \tag{6}$$

$$j = J/(\rho A l^3) \tag{7}$$

$$k = Kl^3/(EI) \tag{8}$$

$$C_i = \omega_i l^2 \sqrt{\rho A/(EI)} \tag{9}$$

where Eqs. (3) and (4) are non-dimensional coordinates (ξ, η) in which the coordinates (x, w_x) are nomalized by span length l and p is the load parameter. Also, u and j are the mass rate and non-dimensional moment of inertia of mass, respectively, and k is spring parameter and C_i is frequency parameter.

Substitution of Eq. (2) into Eq. (1), using Eqs. (3)~(9), yields the non-dimensional ordinary differential equation as follows.

$$\frac{d^4\eta}{d\xi^4} = -p\frac{d^2\eta}{d\xi^2} + C_i^2\eta \tag{10}$$

2.2 Boundary Condition

At clamped end, the amplitude (w_x) is zero, i.e mathematically $w_x=0$ and the angular of rotation (dw_x/dx) is zero, i.e. $dw_x/dx=0$. These two equations can be transformed as the non-dimensional forms as follows.

$$\eta = 0 \text{ at } \xi = 0 \tag{11}$$

$$\frac{d\eta}{d\xi} = 0 \text{ at } \xi = 0 \tag{12}$$

At free end, the mass M accompanying with moment of inertia of mass J is attached and the follower force P with subtangential parameter γ is carried. The free end is restrained by an elastic spring whose spring constant is K. Therefor, the bending moment and shear force loaded at free end are presented as follows.

$$EI\frac{d^2w_x}{dx^2} = \omega_i^2 J\frac{dw_x}{dx} \tag{13}$$

$$EI\frac{d^3w_x}{dx^3} = -P(1-\gamma)\frac{dw_x}{dx} + (K - \omega_i^2 M)w_x \tag{14}$$

Above two equations can be transformed by using Eqs. (3) ~ (9) as follows.

$$\frac{d^2\eta}{d\xi^2} - jC_i^2\frac{d\eta}{d\xi} = 0 \text{ at } \xi = 1 \tag{15}$$

$$\frac{d^3\eta}{d\xi^3} + p(1-\gamma)\frac{d\eta}{d\xi} + (uC_i^2 - k)\eta = 0 \text{ at } \xi = 1 \tag{16}$$

2.3 Numerical Method

It is possible to obtain the frequency parameter C_i and mode shape $\eta(\xi)$ when the ordinary differential Eq. (10) accompanying with the equations of boundary condition (11), (12), (15) and (16) is solved by using the adequate numerical methods. The Runge-Kutta method was used for integrating the differential equation and the Determinant Search Method combined with the Regula-Falsi method was used for obtaining the eigenvalue C_i in the differential Eq. (10).[20] Note that the mode shape $\eta(\xi)$ is obtained from the results of numerical integration.

The lowest two C_i($i=1, 2$) are calculated in this study, from which the load versus frequency curves by load steps can be obtained. The typical load versus frequency curves are shown in Fig. 3.

Firstly, the divergence critical load of Euler's column is explained. As shown in Fig. 3, C_i is decreased as p increases and finally, C_1($i=1$) reaches zero at p marked □ which is the divergence critical load parameter depicted as p_d. The column buckles under the load p_d. The Euler's column subjected p with $\gamma=0$ shown in Fig. 1(a) has always p_d.

Secondly, the flutter critical load of Beck's column is explained. Other two curves in Fig. 3, solid($i=1$) and dashed($i=2$) ones, meet at p marked ■ for being $C_1 = C_2$. The load p which becomes to be $C_1 = C_2$ as mentioned above is the flutter critical load parameter depicted as p_f and its corresponding frequency parameter at p_f is the flutter frequency parameter defined as $C_f(=C_1=C_2)$. This kind of column is so called

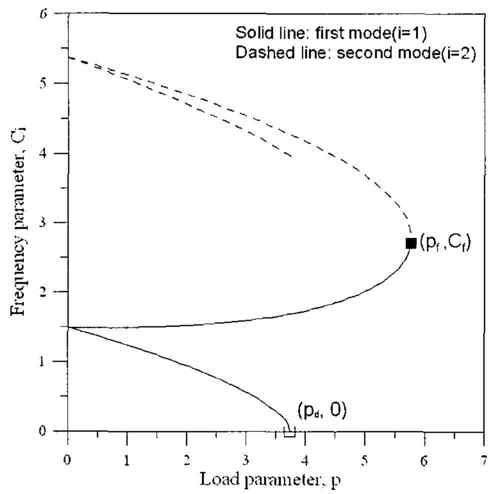

Fig. 3 Typical example of p versus C_i curves

as the Beck's column. The column under the load p_f becomes to be flutter and becomes to be instable since two mode shapes of C_1 and C_2 are same each other perfectly. The columns subjected to p with $\gamma=1$, namely tangential follower force, shown in Fig. 1(c) have always p_f. For a while, the columns under the load p with $0<\gamma<1$, namely subtangential follower force, shown in Fig. 1(b) have the divergence or flutter critical loads depending on the value of γ.

The load versus frequency curves are obtained by using numerical methods mentioned above and then from these curves, p_d or p_f is calculated. In processing of the numerical methods, it is not possible to obtain the values of exact p_d or p_f so that p_d or p_f is calculated under the tolerances of $C_1 \leq 1\times10^{-5}$ or $C_2-C_1 \leq 1\times10^{-5}$, respectively. And here, C_f is determined as the flutter frequency parameter $C_f=(C_1+C_2)/2$.

When the spring parameter k, subtangential parameter γ, mass ratio u and non-dimensional moment of inertia of mass j are given, two FORTRAN computer programs were written which can calculate lowest two C_i and the lowest p_d or p_f with the corresponding C_f.

3. Numerical Examples and Discussion

In order to verify the numerical results of this study, values of p_f of this study and reference[13] are compared in Table 1. From this table, the theories and numerical methods developed herein are validated.

Table 1 Comparison between this study and reference

Geometry of column	Flutter critical load, p_f	
	This study	Reference[13]
$u=1.0,\ j=0.1,\ k=0,\ \gamma=0.5$	14.20	14.19
$u=1.0,\ j=0.1,\ k=10,\ \gamma=1.0$	6.766	6.767

Figure 4 shows the load-frequency curves for which $u=0.5$, $j=0.1$, $k=5$ and $\gamma=0.2$, 0.31, 0.33 and 0.8. In both cases of $\gamma=0.2$ and 0.31, C_i are decreased as p increases and finally, C_1 reach zero, i.e. $C_1=0$, at p marked □. Such load p are the divergence critical load parameters $p_d=8.85$ for $\gamma=0.2$ and $p_d=12.4$ for $\gamma=0.31$. When the column is subjected to $p_d=8.85$ with $\gamma=0.2$ or $p_d=12.4$ with $\gamma=0.31$, the column buckles statically. For the columns with $\gamma=0.33$ and 0.8, the columns have $p_f=15.5$ and 7.27, marked ■, corresponding with $C_f=1.61$ and 4.09, respectively. The columns become to be instable dynamically soon when the columns are subjected to the load p_f. Therefor, it is very important to calculate p_d and p_f for analysing the stability of Beck's columns. Note that the critical loads are transformed from the divergence to flutter one in the range of $0.31<\gamma<0.33$.

The mode shapes of the columns for which $u=0.5$, $j=0.1$ and $k=5$ are shown in Fig. 5(a) and (b). The column in Fig. 5(a) has the divergence critical load p_d as shown in Fig. 4 and the first and second mode shapes have not the nodal points because of the coupling effects of tip spring and subtangential follower force. In

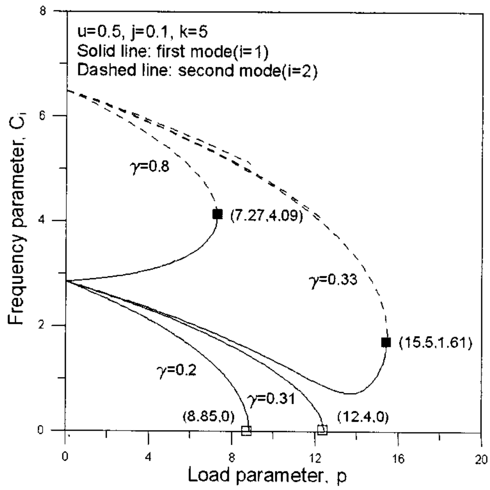

Fig. 4 p versus C_i curves

general case, the second mode has one nodal point. The column in Fig. 5(b) has the flutter critical load p_f and also, both mode shapes have not the nodal points. Nevertheless the free end is restrained by an elastic spring, the maximum amplitudes are arose in the free end. For interested reader's reference, values of C_1 and C_2 are presented in this figure.

The relationships between the subtangential parameter γ and critical loads p_d and p_f for the column with u=0.5, j=0.1 and k=0/5/10 are shown in Fig. 6. It is noted that critical loads are transformed from p_d to p_f at each value of γ marked by ●. For example, the column with k =5 have the divergence critical loads $p_d$with γ value less than 0.321 and the flutter critical loads p_f with γ value greater than 0.321. As γ increases, p_d is increased and p_f is decreased. with γ value less than 0.321 and the flutter critical loads p_f with γ value greater than 0.321. As γ increases, p_d is increased and p_f is decreased.

Figure 7 shows the mass ratio u versus flutter critical load parameter p_f curves for the column

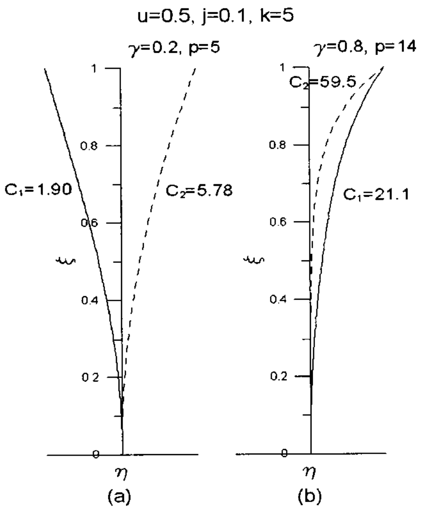

Fig. 5 Mode shapes

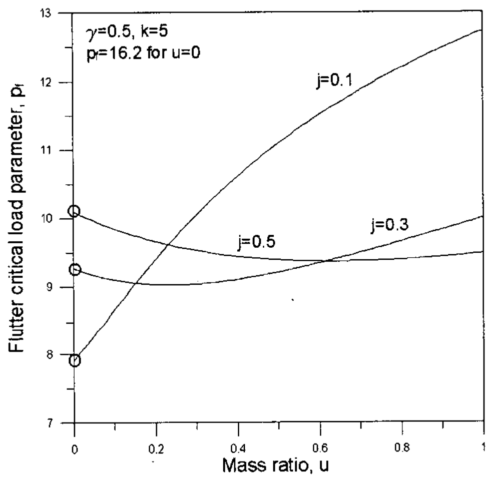

Fig. 7 u versus p_f curves

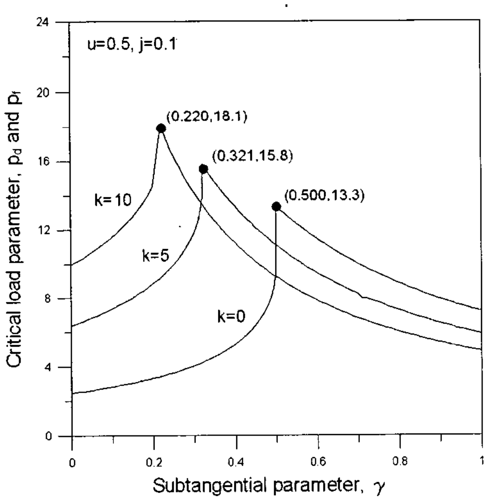

Fig. 6 γ versus p_d and p_f curves

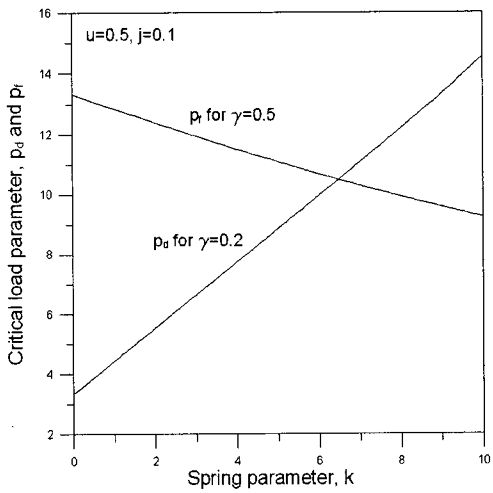

Fig. 8 k versus p_d and p_f curves

with $\gamma=0.5$ and $k=5$. It is noted that p_f is 16.2 for $u=0$, namely without the tip mass, as shown in the legend of Fig. 7. It means that the flutter critical load p_f is suddenly decreased from $p_f=16.2$ to each value of p_f marked by ○ due to the tip mass and its moment of inertia. As u increases, p_f is increased for $j=0.1$. For $j=0.3$ and 0.5, p_f decreases, reaches lowest point and increases again.

Figure 8 shows the relationships between spring parameter k and critical loads p_d, p_f for which $u=0.5$ and $j=0.1$. As k increases, p_d for $\gamma=0.2$ is increased but p_f for $\gamma=0.5$ is decreased. Two critical loads curves are almost linear but not perfect linear.

4. Concluding Remarks

This paper deals with stabilities of Beck's column with a tip mass, subjected to a subtangential follower force. The ordinary differential equation with boundary conditions of Beck's column was derived and solved numerically for calculating natural frequencies and its corresponding mode shapes. Both the divergence and flutter critical loads were calculated by using the load versus frequency curves obtained herein. As the numerical results, effects of subtangential parameter, mass ratio and spring parameter on natural frequencies, mode shapes and both the divergence and flutter critical loads are presented and discussed extensively. It is expected that results obtained herein should be utilized for the static and dynamic stabilities of the structures subjected to a subtangential follower force such as Beck's columns.

Acknowledgement

The authors appreciate the reviewers who gave the valuable comments on this paper.

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