INTUITIONISTIC FUZZY IDEALS OF A RING

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ABSTRACT. We introduce the notions of intuitionistic fuzzy prime ideals, intuitionistic fuzzy completely prime ideals and intuitionistic fuzzy weakly completely prime ideals. And we give a characterization of intuitionistic fuzzy ideals and establish relationships between intuitionistic fuzzy completely prime ideals and intuitionistic fuzzy weakly completely prime ideals.

0. INTRODUCTION

Zadeh [17] introduced the notion of a fuzzy set in a set X as a mapping from X into the closed unit interval $[0, 1]$. Rosenfeld [16] applied this concept to grpup theory. After that time, Das [7], Kumar [11], Liu [13] and Mukherjee & Sen [14, 15] applied this notion to group and ring theory.

In 1986, Atanassov [1] introduced the concept of intuitionistic fuzzy sets as the generalization of fuzzy sets. Recently, Çoker [5], Çoker & Eş [6], Gurcay, Çoker & Eş [8] and S. J. Lee & E. P. Lee [12] introduced the concept of intuitionistic fuzzy topological spaces using intuitionistic fuzzy sets and investigated some of their properties. In 1989, Biswas [3] introduced the concept of intuitionistic fuzzy subgroups and studied some of it's properties. In 2003, Banerjee & Basnet [2] investigated intuitionistic fuzzy subrings and intuitionistic fuzzy ideals using intuitionistic fuzzy sets. Also, Hur, Jang & Kang [9] and Hur, Kang & Song [10] studied various properties of intuitionistic fuzzy subgroupoids, intuitionistic fuzzy subgroups and intuitionistic fuzzy subrings.

In this paper, we introduce the notions of intuitionistic fuzzy prime ideals, intuitionistic fuzzy completely prime ideals and intuitionistic fuzzy weakly completely

prime ideals. And we give a characterization of intuitionistic fuzzy ideals and establish relationships between intuitionistic fuzzy completely prime ideals and intuitionistic fuzzy weakly completely prime ideals.

1. PRELIMINARIES

We will list some concepts and one result needed in the later sections.

For sets X, Y and Z, $f = (f_1, f_2) : X \to Y \times Z$ is called a *complex mapping* if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

Throughout this paper, we will denote the unit interval $[0, 1]$ as I.

Definition 1.1 (Atanassov [1]). Let X be a nonempty set. A complex mapping $A = (\mu_A, \nu_A) : X \to I \times I$ is called an *intuitionistic fuzzy set* (in short, IFS) in X if $\mu_A + \nu_A \leq 1$, where the mapping $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each $x \in X$ to A, respectively. In particular, $0_\sim$ and $1_\sim$ denote the *intuitionistic fuzzy empty set* and *intuitionistic fuzzy whole set* in X defined by $0_\sim(x) = (0, 1)$ and $1_\sim(x) = (1, 0)$, respectively.

We will denote the set of all IFSs in X as IFS(X).

Definition 1.2 (Atanassov [1]). Let X be a nonempty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFSs in X. Then

(1) $A \subset B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.

(2) $A = B$ iff $A \subset B$ and $B \subset A$.

(3) $A^c = (\nu_A, \mu_A)$.

(4) $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.

(5) $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.

(6) $[\,]A = (\mu_A, 1 - \mu_A)$, $\langle\,\rangle A = (1 - \nu_A, \nu_A)$.

Definition 1.3 (Çoker [5]). Let $\{A_i\}_{i \in J}$ be an arbitrary family of IFSs in X, where $A_i = (\mu_{A_1}, \nu_{A_i})$ for each $i \in J$. Then

(a) $\bigcap A_i = (\wedge \mu_{A_i}, \vee \nu_{A_i})$.

(b) $\bigcup A_i = (\vee \mu_{A_i}, \wedge \nu_{A_i})$.

Definition 1.4 (Hur, Jang & Kang [9]). Let A be an IFS in a set X and let $(\lambda, \mu) \in I \times I$ with $\lambda + \mu \leq 1$. Then the set $A^{(\lambda, \mu)} = \{x \in X : \mu_A(x) \geq \lambda \text{ and } \nu_A(x) \leq \mu\}$ is called a (λ, μ)-*level subset* of A.

Definition 1.5 (S. J. Lee & E. P. Lee [12]). Let $(\lambda, \mu) \in I \times I$ with $\lambda + \mu \leq 1$. Then an *intuitionistic fuzzy point* (in short, IFP) $x_{(\lambda,\mu)}$ of X is the IFS in X defined as follows : for each $y \in X$,

$$x_{(\lambda,\mu)}(y) = \begin{cases} (\lambda, \mu) & \text{if } y = x, \\ (0, 1) & \text{if } y \neq x. \end{cases}$$

In this case, x is called the *support* of $x_{(\lambda,\mu)}$ and λ and μ are called the *value* and *nonvalue* of $x_{(\lambda,\mu)}$, respectively. An IFP $x_{(\lambda,\mu)}$ is said to *belong to* an IFS $A = (\mu_A, \nu_A)$ in X, denoted by $x_{(\lambda,\mu)} \in A$ if $\lambda \leq \mu_A(x)$ and $\nu_A(x) \geq \mu$.

It is clear that an intuitionistic fuzzy point $x_{(\lambda,\mu)}$ can be represented by an ordered pair of fuzzy points as follows:

$$x_{(\lambda,\mu)} = (x_\lambda, 1 - x_{1-\mu})$$

We will denote the set of all IFPs in X as $\mathrm{IFp}(X)$.

Result 1.1 (S. J. Lee & E. P. Lee [12, Theorem 2.3]). Let $A, B \in \mathrm{IFS}(X)$. Then $A \subset B$ if and only if for each $x_{(\lambda,\mu)} \in \mathrm{IFp}(X)$, $x_{(\lambda,\mu)} \in A$ implies $x_{(\lambda,\mu)} \in B$.

Definition 1.6 (Hur, Jang & Kang [9]). Let $(X, \cdot)$ be a groupoid and let $A, B \in \mathrm{IFS}(X)$. Then the *intuitionistic fuzzy product* of A and B, $A \circ B$ is defined as follows: for each $x \in X$,

$$A \circ B(x) = \begin{cases} (\bigvee_{x=yz}[\mu_A(y) \wedge \mu_B(z)], \bigwedge_{x=yz}[\nu_A(y) \vee \nu_B(z)]) & \text{if } x = yz, \\ (0, 1) & \text{otherwise.} \end{cases}$$

Result 1.2 (Hur, Jang & Kang [9, Proposition 2.2]). Let $(X, \cdot)$ be a groupoid, let $x_{(\lambda,\mu)}, y_{(t,s)} \in \mathrm{IFp}(X)$ and let $A, B \in \mathrm{IFS}(X)$. Then

(1) $x_{(\lambda,\mu)} \circ y_{(t,s)} = (xy)_{(\lambda \wedge t, \mu \vee s)}$.

(2) $A \circ B = \bigcup_{x_{(\lambda,\mu)} \in A, y_{(t,s)} \in B} x_{(\lambda,\mu)} \circ y_{(t,s)}$.

Definition 1.7 (Hur, Kang & Song [10]). Let G be a group and let $A \in \mathrm{IFS}(G)$. Then A is call an *intuitionistic fuzzy subgroup* (in short, IFG) of G if it satisfies the following conditions:

(i) $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$ for any $x, y \in G$.

(ii) $\mu_A(x^{-1}) \geq \mu_A(x)$ and $\nu_A(x^{-1}) \leq \nu_A(x)$ for each $x \in G$.

We will denote the set of all IFGs as $\mathrm{IFG}(G)$.

Result 1.3 (Hur, Kang & Song [10, Proposition 2.6]). Let A be an IFG of a group G with identity e. Then $A(x^{-1}) = A(x)$ and $\mu_A(x) \leq \mu_A(e), \nu_A(x) \geq \nu_A(e)$ for each $x \in G$.

Definition 1.8 (Hur, Kang & Song [10]). Let $(R, +, \cdot)$ be a ring and let $0_{\sim} \neq A \in$ IFS(R). Then A is called an *intuitionistic fuzzy subring* (in short, IFSR) of R if it satisfies following conditions:

(i) A is an IFG with respect to the operation "$+$".

(ii) $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$ for any $x, y \in R$.

We will denote the set of all IFSRs as IFSR(R).

2. INTUITIONISTIC FUZZY IDEALS

Definition 2.1 (Hur, Kang & Song [10]). Let A be a non-empty IFSR of a ring R. Then the fuzzy subring A is called

(1) an *intuitionistic fuzzy left ideal* (in short, IFLI) of R if $\mu_A(xy) \geq \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(y)$ for any $x, y \in R$.
(2) an *intuitionistic fuzzy right ideal* (in short, IFRI) of R if $\mu_A(xy) \geq \mu_A(x)$ and $\nu_A(xy) \leq \nu_A(x)$ for any $x, y \in R$.
(3) an *intuitionistic fuzzy ideal* (in short, IFRI) of R if it is an IFLI and an IFRI of R .

We will denote the set of all IFRIs (*resp.* IFLIs and IFIs) of R as IFRI(R) (*resp.* IFLI(R) and IFI(R)).

Result 2.1 (Hur, Kang & Song [10, Proposition 4.6]). Let R be a ring. Then A is an ideal (*resp.* a left ideal and a right ideal) of R if and any of $(\chi_A, \chi_{A^c}) \in$ IFI(R) (*resp.* IFLI(R) and IFRI(R)).

Lemma 2.2. *Let R be a ring and let $A, B \in IFS(R)$.*

(1) *If $A, B \in$ IFLI(R) (resp. IFRI(R) and IFI(R)), then $A \cap B \in$ IFLI(R) (resp. IFRI(R) and IFI(R)).*
(2) *If $A \in$ IFRI(R) and $B \in$ IFLI(R), then $A \circ B \subset A \cap B$.*

Proof. (1) Suppose $A, B \in \text{IFLI}(R)$ and let $x, y \in R$. Then

$$\mu_{A \cap B}(x - y) = \mu_A(x - y) \wedge \mu_B(x - y)$$
$$\geq [\mu_A(x) \wedge \mu_A(y)] \wedge [\mu_B(x) \wedge \mu_B(y)]$$
$$= \mu_{A \cap B}(x) \wedge \mu_{A \cap B}(y)$$

and

$$\nu_{A \cap B}(x - y) = \nu_A(x - y) \vee \nu_B(x - y)$$
$$\leq [\nu_A(x) \vee \nu_A(y)] \vee [\nu_B(x) \vee \nu_B(y)]$$
$$= \nu_{A \cap B}(x) \vee \nu_{A \cap B}(y).$$

Also

$$\mu_{A \cap B}(xy) = \mu_A(xy) \wedge \mu_B(xy)$$
$$\geq \mu_A(y) \wedge \mu_{AB}(y) \qquad \qquad (\text{Since } A, B \in \text{IFLI}(R))$$
$$= \mu_{A \cap B}(y)$$

and

$$\nu_{A \cap B}(xy) = \nu_A(xy) \vee \nu_B(xy) \leq \nu_A(y) \vee \nu_A(y) = \nu_{A \cap B}(y).$$

Hence $A \cap B \in \text{IFLI}(R)$. Similarly, we can easily see the rest.

(2) Let $x \in G$ and suppose $A \circ B(x) = (0, 1)$. Then there is nothing to show. Suppose $A \circ B(x) \neq (0, 1)$. Then

$$A \circ B(x) = (\bigvee_{x = yz} [\mu_A(y) \wedge \mu_B(z)], \bigwedge_{x = yz} [\nu_A(y) \vee \nu_B(z)]).$$

Since $A \in \text{IFRI}(R)$ and $B \in \text{IFLI}(R)$,

$$\mu_A(y) \leq \mu_A(yz) = \mu_A(x), \nu_A(y) \geq \nu_A(yz) = \nu_A(x)$$

and

$$\mu_B(z) \leq \mu_B(yz) = \mu_B(x), \nu_B(z) \geq \nu_B(yz) = \nu_B(x).$$

Thus

$$\mu_{A \circ B}(x) = \bigvee_{x = yz} [\mu_A(y) \wedge \mu_B(z)] \leq \mu_A(x) \wedge \mu_B(x) = \mu_{A \cap B}(x)$$

and

$$\nu_{A \circ B}(x) = \bigwedge_{x = yz} [\nu_A(y) \vee \nu_B(z)] \geq \nu_A(x) \vee \nu_B(x) = \nu_{A \cap B}(x).$$

Hence $A \circ B \subset A \cap B$. This completes the proof. $\qquad \square$

A ring R is said to be *regular* if for each $a \in R$ there exists an $x \in R$ such that $a = axa$.

Result 2.2 (Burton [10, Theorem 9.4]). *A ring R is regular if and only if $JM = J \cap M$ for each right ideal J and let ideal M of R.*

Proposition 2.3. *A ring R is regular if and only if for each $A \in \mathrm{IFRI}(R)$ and each $B \in \mathrm{IFLI}(R)$, $A \circ B = A \cap B$.*

Proof. ($\Rightarrow$) Suppose R is regular. From Lemma 2.2, $A \circ B \subset A \cap B$. Thus it is sufficient to show that $A \cap B \subset A \circ B$. Let $a \in R$. Then, by the hypothesis, there exists an $x \in R$ such that $a = axa$. Thus

$$\mu_A(a) = \mu_A(axa) \geq \mu_A(ax) \geq \mu_A(a)$$

and

$$\nu_A(a) = \nu_A(axa) \leq \nu_A(ax) \leq \nu_A(a).$$

So $A(ax) = A(a)$. On the other hold,

$$\begin{aligned}
\mu_{A \circ B}(a) &= \bigvee_{a=yz} [\mu_A(y) \wedge \mu_B(z)] \\
&\geq \mu_A(ax) \wedge \mu_B(a) \qquad\qquad \text{(Since } a = axa\text{)} \\
&= \mu_A(a) \wedge \mu_B(a) = \mu_{A \cap B}(a)
\end{aligned}$$

and

$$\begin{aligned}
\nu_{A \circ B}(a) &= \bigwedge_{a=yz} [\nu_A(y) \vee \nu_B(z) \leq \nu_A(ax) \vee \nu_B(a) \\
&= \nu_A(a) \vee \nu_B(a) = \nu_{A \cap B}(a).
\end{aligned}$$

Thus $A \cap B \subset A \circ B$. Hence $A \circ B = A \cap B$.

($\Leftarrow$) Suppose the necessary condition holds. Let J and M be right and left ideals of R, respectively. Then, by Result 2.1,

$$(\chi_J, \chi_{J^c}) \in \mathrm{IFRI}(R) \quad \text{and} \quad (\chi_M, \chi_{M^c}) \in \mathrm{IFLI}(R).$$

Let $a \in J \cap M$ and let $A = (\chi_J, \chi_{J^c})$, $B = (\chi_M, \chi_{M^c})$. Then, by the hypothesis, $(A \circ B)(a) = (A \cap B)(a) = (1, 0)$. Thus

$$\mu_{A \circ B}(a) = \bigvee_{a=a_1 a_2} [\mu_A(a_1) \wedge \mu_B(a_2)] = \bigvee_{a=a_1 a_2} [\chi_J(a_1) \wedge \chi_M(a_2)] = 1$$

and

$$\nu_{A \circ B}(a) = \bigwedge_{a=a_1 a_2} [\nu_A(a_1) \vee \nu_B(a_2)] = \bigwedge_{a=a_1 a_2} [\chi_{J^c}(a_1) \vee \chi_{M^c}(a_2)] = 0.$$

So there exist $b_1, b_2 \in R$ such that $\chi_J(b_1) = 1$, $\chi_{J^c}(b_1) = 0$ and $\chi_M(b_2) = 1$, $\chi_{M^c}(b_2) = 0$ with $a = b_1 b_2$. Thus $a \in JM$, i.e., $J \cap M \subset JM$. Since $JM \subset J \cap M$, $JM = J \cap M$. Hence, by Result 2.2, R is regular. This completes the proof. $\square$

3. Intuitionistic Fuzzy Prime Ideals

Definition 3.1. Let P be an IFI of a ring R. Then P is said to be *prime* if P is not a constant mapping and for any $A, B \in \text{IFI}(R)$, $A \circ B \subset P$ implies either $A \subset P$ or $B \subset P$.

We will denote the set of all intuitionistic fuzzy prime ideals of R as $\text{IFPI}(R)$.

Proposition 3.2. *Let J be an ideal of a ring R such that $J \neq R$. Then J is a prime ideal of R if and only if $(\chi_J, \chi_{J^c}) \in \text{IFPI}(R)$.*

Proof. ($\Rightarrow$) Suppose J is a prime ideal of R and let $P = (\chi_J, \chi_{J^c})$. Since $J \neq R$, P is not a constant mapping on R. Assume that there exist $A, B \in \text{IFI}(R)$ such that $A \circ B \subset P$ and $A \not\subset P$ and $B \not\subset P$. Then there exist $x, y \in R$ such that

$$\mu_A(x) > \mu_P(x) = \chi_J(x), \quad \nu_A(x) < \nu_P(x) = \chi_{J^c}(x)$$

and

$$\mu_B(y) > \mu_P(y) = \chi_J(y), \quad \nu_B(y) < \nu_P(y) = \chi_{J^c}(y).$$

Thus $\mu_A(x) \neq 0$, $\nu_A(x) \neq 1$ and $\mu_B(y) \neq 0$, $\nu_B(y) \neq 1$. But $\chi_J(x) = 0$ and $\chi_J(y) = 0$. So $x \notin J$ and $y \notin J$. Since J is a prime ideal of R, by the process of the proof of Theorem 2 in Mukherjee & Sen [14], there exist an $r \in R$ such that $xry \notin J$. Let $a = xry$. Then clearly, $\chi_J(a) = 0$ and $\chi_{J^c}(a) = 1$. Thus

$$A \circ B(a) = (0, 1). \tag{$*$}$$

On the other hold,

$$\mu_{A \circ B}(a) = \bigvee_{a=cd} [\mu_A(c) \wedge \mu_B(d)] \geq \mu_A(x) \wedge \mu_B(ry) \qquad \text{(Since } a = xry\text{)}$$

$$= \mu_A(x) \wedge \mu_B(y) \qquad \text{(Since } B \in \text{IFI}(R)\text{)}$$

$$> 0 \qquad \text{(Since } \mu_A(x) \neq 0 \text{ and } \mu_B(y) \neq 0\text{)}$$

and

$$\nu_{A \circ B}(a) = \bigwedge_{a=cd} [\nu_A(c) \vee \nu_B(d)] \leq \nu_A(x) \vee \nu_B(ry) \leq \nu_A(x) \vee \nu_B(y)$$

$$< 1. \qquad \text{(Since } \nu_A(x) \neq 1 \text{ and } \nu_B(y) \neq 1)$$

Then $A \circ B(a) \neq 0_\sim$. This contradicts $(*)$. So P satisfies the second condition of Definition 3.1. Hence $P = (\chi_J, \chi_{J^c}) \in \text{IFPI}(R)$.

$(\Leftarrow)$ Suppose $P = (\chi_J, \chi_{J^c}) \in \text{IFPI}(R)$. Since P is not a constant mapping on R, $J \neq R$. Let A and B be two ideals of R such that $AB \subset J$. Let $\widetilde{A}, \widetilde{B} \in \text{IFI}(R)$. Consider the product $\widetilde{A} \circ \widetilde{B}$. Let $x \in R$.

Suppose $\widetilde{A} \circ \widetilde{B}(x) = (0,1)$. Then clearly $\widetilde{A} \circ \widetilde{B} \subset P$.

Suppose $\widetilde{A} \circ \widetilde{B}(x) \neq (0,1)$. Then

$$\mu_{\widetilde{A} \circ \widetilde{B}}(x) = \bigvee_{x=yz} [\chi_A(y) \wedge \chi_B(z)] \neq 0$$

and

$$\nu_{\widetilde{A} \circ \widetilde{B}}(x) = \bigwedge_{x=yz} [\chi_{A^c}(y) \vee \chi_{B^c}(z)] \neq 1.$$

Thus there exist $y, z \in R$ with $x = yz$ such that

$$\chi_A(y) \neq 0, \ \chi_{A^c}(y) \neq 1 \text{ and } \chi_B(z) \neq 0, \ \chi_{B^c}(z) \neq 1.$$

So $\chi_A(y) = 1$, $\chi_{A^c}(y) = 0$ and $\chi_B(z) = 1$, $\chi_{B^c}(z) = 0$. This implies $y \in A$ and $z \in B$. Thus $x = yz \in AB \subset J$. So $\chi_J = 1$ and $\chi_{J^c}(x) = 0$. It follows that $\widetilde{A} \circ \widetilde{B} \subset P$. Since $P \in \text{IFPI}(R)$, either $\widetilde{A} \subset P$ or $\widetilde{B} \subset P$. Thus either $A \subset J$ or $B \subset J$. Hence J is a prime ideal of R. This completes the proof. $\qquad \square$

Proposition 3.3. *Let P be an intuitionistic fuzzy prime ideals of a ring R and let $R_P = \{x \in R : P(x) = P(0)\}$. Then R_P is a prime ideal of R.*

Proof. Let $x, y \in R_P$. Then $P(x) = P(0)$ and $P(y) = P(0)$. Thus

$$\mu_P(x - y) \geq \mu_P(x) \wedge \mu_P(y) = \mu_P(0)$$

and

$$\nu_P(x - y) \leq \nu_P(x) \vee \nu_P(y) = \nu_P(0).$$

Since $P \in \text{IFI}(R)$,

$$\mu_P(0) = \mu_P(0(x - y)) \geq \mu_P(x - y)$$

and

$$\nu_P(0) = \nu_P(0(x - y)) \leq \nu_P(x - y).$$

So $x - y \in R_P$. Now let $r \in R$ and let $x \in R_P$. Then

$$\mu_P(rx) \geq \mu_P(x) = \mu_P(0) \text{ and } \nu_P(rx) \leq \nu_P(x) = \nu_P(0).$$

By Result 1.3, $P(rx) = P(0)$. So $rx \in R_P$. Similarly we have $xr \in R_P$. Hence R_P is an ideal of R. Let J and M be two ideals of R such that $JM \subset R_P$. We define two complex mappings

$$A = (\mu_A, \nu_A) : R \to I \times I \text{ and } B = (\mu_B, \nu_B) : R \to I \times I \quad (resp.)$$

by

$$A = P(0)(\chi_J, \chi_{J^c}) \text{ and } B = P(0)(\chi_M, \chi_{M^c}), \quad (resp.)$$

where $P(0)(\chi_J, \chi_{J^c}) = (\mu_P(0)\chi_J, \nu_P(0)\chi_{J^c})$. Then we can easily prove that $A, B \in$ IFI(R). Let $x \in R$.

  Suppose $A \circ B(x) = (0, 1)$. Then $A \circ B \subset P$.
  Suppose $A \circ B(x) \neq (0, 1)$. Then

$$\mu_{A \circ B}(x) = \bigvee_{x=yz} [\mu_A(y) \wedge \mu_B(z)] = \bigvee_{x=yz} [\mu_P(0)\chi_J(y) \wedge \mu_P(0)\chi_M(z)] \neq 0$$

and

$$\nu_{A \circ B}(x) = \bigwedge_{x=yz} [\nu_A(y) \vee \nu_B(z)] = \bigwedge_{x=yz} [\nu_P(0)\chi_{J^c}(y) \vee \nu_P(0)\chi_{M^c}(z)] \neq 1.$$

Thus there exist $y, z \in R$ with $x = yz$ such that

$$\mu_P(0)\chi_J(y) \wedge \mu_P(0)\chi_M(z) \neq 0 \text{ and } \nu_P(0)\chi_{J^c}(y) \vee \nu_P(0)\chi_{M^c}(z) \neq 1.$$

So $\chi_J(y) = 1$, $\chi_{J^c}(y) = 0$ and $\chi_M(z) = 1$, $\chi_{M^c}(z) = 0$. Thus $y \in J$ and $z \in M$, i. e., $x = yz \in JM \subset R_P$. So $P(x) = P(0)$, i. e., $A \circ B \subset P$. Since $P \in$ IFPI(R) and A, $B \in$ IFI(R), either $A \subset P$ or $B \subset P$.

  Suppose $A \subset P$. Then $P(0)(\chi_J, \chi_{J^c}) \subset P$. Assume that $J \subset R_P$. Then there exists an $a \in J$ such that $a \notin R_P$. Thus $P(a) \neq P(0)$. By Result 1.3, $\mu_P(a) < \mu_P(0)$ and $\nu_P(a) > \nu_P(0)$. Then

$$\mu_A(a) = \mu_P(0)\chi_J(a) = \mu_P(0) > \mu_P(a)$$

and

$$\nu_A(a) = \nu_P(0)\chi_{J^c}(a) = 0 \leq \nu_P(0) < \nu_P(a).$$

This contradicts the assumption that $A \subset P$. So $J \subset R_P$. By the similar arguments, we can show that if $B \subset P$, then $M \subset R_P$. Hence R_P is a prime ideal of R. This completes the proof. $\qquad \square$

Remark 3.4. Let $P \in \mathrm{IFI}(\mathbb{Z})$. Then, by Proposition 3.3, R_P is an ideal of $\mathbb{Z}$. Hence there exists an integer $n \geq 0$ such that $R_P = n\mathbb{Z}$.

Proposition 3.5. *Let $P \in \mathrm{IFI}(\mathbb{Z})$ with $R_P = n\mathbb{Z} \neq (0)$. Then P can take at most r values, where r is the number of distinct positive divisors of n.*

Proof. Let $a \in \mathbb{Z}$ and let $d = (a, n)$. Then there exist $r, s \in \mathbb{Z}$ such that $d = ar + ns$. Thus

$$\mu_P(d) = \mu_P(ar + ns) \geq \mu_P(ar) \wedge \mu_P(ns) \geq \mu_P(a) \wedge \mu_P(n)$$

and

$$\nu_P(d) = \nu_P(ar + ns) \leq \nu_P(ar) \vee \nu_P(ns) \leq \nu_P(a) \vee \nu_P(n).$$

Since $n \in R_P = n\mathbb{Z}$, by Result 1.3,

$$\mu_P(n) = \mu_P(0) \geq \mu_P(a) \text{ and } \nu_P(n) = \nu_P(0) \leq \nu_P(a).$$

Thus $\mu_P(d) \geq \mu_P(a)$ and $\nu_P \leq \nu_P(a)$. Since d is a divisor of a, there exists a $t \in \mathbb{Z}$ such that $a = dt$. Then

$$\mu_P(a) = \mu_P(dt) \geq \mu_P(d) \text{ and } \nu_P(a) = \nu_P(dt) \leq \nu_P(d).$$

So $P(a) = P(d)$. Moreover, by Result 1.3, $P(x) = P(-x)$ for each $x \in R$. Hence for each $a \in \mathbb{Z}$ there exists a positive divisor d of n such that $P(a) = P(d)$. This completes the proof. $\qquad \square$

The following result gives a complete characterization of intuitionistic fuzzy prime ideals of $\mathbb{Z}$:

Theorem 3.6. *Let $P \in \mathrm{IFPI}(\mathbb{Z})$ with $\mathbb{Z}_p \neq (0)$. Then P has two distinct values. Conversely, if $P \in \mathrm{IFS}(\mathbb{Z})$ such that $P(n) = (\lambda_1, \mu_1)$ when $p \mid n$ and $P(n) = (\lambda_2, \mu_2)$ when $p \nmid n$, where p is a fixed prime, $\lambda_1 > \lambda_2$ and $\mu_1 < \mu_2$, then $P \in \mathrm{IFPI}(\mathbb{Z})$ with $\mathbb{Z}_p \neq (0)$.*

Proof. Suppose $P \in \mathrm{IFPI}(\mathbb{Z})$ with $\mathbb{Z}_p = n\mathbb{Z} \neq (0)$. Then, by Proposition 3.3, $\mathbb{Z}_p$ is a prime ideal of $\mathbb{Z}$. Thus n is a prime integer. Since n has two distinct positive integers, by Proposition 3.5, P has at most two distinct values. On the other hand, an intuitionistic fuzzy prime ideals cannot be a constant mapping. Hence P has two distinct values.

Conversely, let P be an IFS in $\mathbb{Z}$ satisfying the given conditions. Let $a, b \in \mathbb{Z}$.

(i) Suppose $p \mid (a - b)$. Then $P(a - b) = (\lambda_1, \mu_1)$. Thus

$$\lambda_1 = \mu_P(a - b) \geq \mu_P(a) \wedge \mu_P(b) \qquad \text{(Since } \lambda_1 > \lambda_2)$$

and

$$\mu_1 = \nu_P(a - b) \leq \nu_P(a) \vee \nu_P(b). \qquad \text{(Since } \mu_1 < \mu_2)$$

(ii) Suppose $p \nmid (a - b)$. Then $p \nmid a$ or $p \nmid b$. Thus either $P(a) = (\lambda_2, \mu_2)$ or $P(b) = (\lambda_2, \mu_2)$. So

$$\lambda_2 = \mu_P(a - b) \geq \mu_P(a) \wedge \mu_P(b) \quad \text{and} \quad \mu_2 = \nu_P(a - b) \leq \nu_P(a) \vee \nu_P(b).$$

(iii) Suppose $p \mid ab$. Then clearly $\mu_P(ab) \geq \mu_P(b)$ and $\nu_P(ab) \leq \nu_P(b)$.

(iv) Suppose $p \nmid ab$. Then $p \nmid a$ and $p \nmid b$. Thus

$$\mu_P(ab) \geq \mu_P(b) \text{ and } \nu_P(ab) \leq \nu_P(b).$$

Consequently, by Result 1.3, $P \in \text{IFI}(\mathbb{Z})$ with $\mathbb{Z}_P = p\mathbb{Z} \neq (0)$. Moreover, by the similar argments of the proof of Proposition 3.2, we can see that $P \in \text{IFPI}(\mathbb{Z})$. This completes the proof. $\qquad\square$

Proposition 3.7. *Let R be a ring with 1. If every* IFI *of R has finite values, then R is a Noetherian ring.*

Proof. Let $\{J_i\}_{i \in \mathbb{Z}^+}$ be a sequence of ideals of R such that $J_1 \subset J_2 \subset J_3 \subset \cdots$ and let $J = \bigcup_{i \in \mathbb{Z}^+} J_i$. Then clearly J is an ideal of R. We define a complex mapping $P = (\mu_P, \nu_P) : R \to I \times I$ as follows:

For each $x \in R$,

$$P(x) = \begin{cases} (0, 1) & \text{if } x \notin J, \\ (\frac{1}{i_1}, 1 - \frac{1}{i_1}) & \text{if } x \in J, \end{cases}$$

where $i_1 = \min\{i : x \in J_i\}$. Then it is clear that $P \in \text{IFI}(R)$ from the definition of P. Moreover, we can easily see that $P \in \text{IFI}(R)$. If the chain dose not terminate, then P takes infinitely many values. This contradicts the hypothesis. Thus the chain terminates. Hence R is a Noetherian ring. This completes the proof. $\qquad\square$

Proposition 3.8. *Let $A : \mathbb{Z} \to I \times I$ be the complex mapping such that*

(i) $A(x) = A(-x)$ for each $x \in \mathbb{Z}$.

(ii) $\mu_A(x + y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x + y) \leq \nu_A(x) \vee \nu_A(y)$ for any $x, y \in \mathbb{Z}$.

If there exists a non-zero integer m such that $A(m) = A(0)$, then A can take at most finitely many values.

Proof. It is clear that $A \in \text{IFS}(\mathbb{Z})$ from the definition of A. Moreover, we can easily show that $A \in \text{IFI}(\mathbb{Z})$ such that $\mathbb{Z}_A \neq (0)$. Hence, by Proposition 3.5, the mapping A can take at most finitely many values. $\qquad \square$

4. INTUITIONISTIC FUZZY COMPLETELY PRIME IDEALS

Definition 4.1. Let P be an IFI of a ring R and let $(\lambda, \mu) \in I \times I$ with $\lambda + \mu \leq 1$. Then P is called an *intuitionistic fuzzy completely prime ideals* (in short, IFCPI) of R if it satisfies the following conditions:

(i) P is not a constant mapping.

(ii) For any $x_{(\lambda,\mu)}, y_{(t,s)} \in \text{IFp}(R), x_{(\lambda,\mu)} \circ y_{(t,s)} \in P$ implies either $x_{(\lambda,\mu)} \in P$ or $y_{(t,s)} \in P$.

We will denote the set of all IFCPIs of R as IFCPI(G).

Proposition 4.2.

(1) *Let R be a ring. Then* $\text{IFCPI}(R) \subset \text{IFPI}(R)$.

(2) *Let R be a commutative ring. Then* $\text{IFPI}(R) \subset \text{IFCPI}(R)$. *Hence* $\text{IFCPI}(R) = \text{IFPI}(R)$.

Proof. (1) Let $P \in \text{IFCPI}(R)$ and let $A, B \in \text{IFI}(R)$ such that $A \circ B \subset P$. Suppose $A \not\subset P$. Then, by Result 1.1, there exists an $x_{(\lambda,\mu)} \in \text{IFp}(R)$ such that $x_{(\lambda,\mu)} \in P$ but $x_{(\lambda,\mu)} \notin P$. Let $y_{(t,s)} \in B$. Then, by Result 1.2 (1), $x_{(\lambda,\mu)} \circ y_{(t,s)} = (xy)_{(\lambda \wedge t, \mu \vee s)}$. On the other hand,

$$\mu_P(xy) \geq \mu_{A \circ B}(xy) \geq \mu_A(x) \wedge \mu_B(y) = \lambda \wedge t = \mu_{x_{(\lambda,\mu)} \circ y_{(t,s)}}(xy)$$

and

$$\nu_P(xy) \leq \nu_{A \circ B}(xy) \leq \nu_A(x) \vee \nu_B(y) = \mu \vee s = \nu_{x_{(\lambda,\mu)} \circ y_{(t,s)}}(xy).$$

Let $z \in R$ such that $x \neq xy$. Then clearly $[x_{(\lambda,\mu)} \circ y_{(t,s)}](z) = (0,1)$. Thus $x_{(\lambda,\mu)} \circ y_{(t,s)} \in P$. Since $P \in \text{IFCPI}(R)$, $x_{(\lambda,\mu)} \in P$ or $y_{(t,s)} \in P$. Since $x_{(\lambda,\mu)} \notin P$, $y_{(t,s)} \in P$. So, by Result 1.1, $B \subset P$. Hence $P \in \text{IFPI}(R)$.

(2) Let $P \in \text{IFPI}(R)$ and let $x_{(\lambda,\mu)}, y_{(t,s)} \in IFp(R)$ such that $x_{(\lambda,\mu)} \circ y_{(t,s)} \in P$. Then

$$\mu_{x_{(\lambda,\mu)} \circ y_{(t,s)}}(xy) \leq \mu_P(xy) \text{ and } \nu_{x_{(\lambda,\mu)} \circ y_{(t,s)}}(xy) \geq \nu_P(xy).$$

Thus, by Result 1.2 (1),

$$\lambda \wedge t \leq \mu_P(xy) \text{ and } \mu \vee s \geq \nu_P(xy). \tag{$**$}$$

We define two complex mappings

$$A = (\mu_A, \nu_A) : R \to I \times I \text{ and } B = (\mu_B, \nu_B) : R \to I \times I$$

as follows: For each $z \in R$,

$$A(z) = \begin{cases} (\lambda, \mu) & \text{if } z \in (x) \\ (0, 1) & \text{otherwise} \end{cases} \text{ and } B(z) = \begin{cases} (t, s) & \text{if } z \in (y) \\ (0, 1) & \text{otherwise} \end{cases}$$

where (x) is the ideal generated by x. Then clearly $A, B \in \text{IFS(R)}$ from the definitions of A and B. It is easily seen that if z is not expressible in the form $z = uv$ for some $u \in (x)$ and $v \in (y)$, then $A \circ B(z) = (0, 1)$. Suppose there exist $u \in (x)$ and $v \in (y)$ such that $z = uv$. Then

$$\mu_{A \circ B}(z) = \bigvee_{z = uv, u \in (x), v \in (y)} [\mu_A(u) \wedge \mu_B(v)] = \lambda \wedge t$$

and

$$\nu_{A \circ B}(z) = \bigwedge_{z = uv, u \in (x), v \in (y)} [\nu_A(u) \vee \nu_B(v)] = \mu \vee s.$$

Since R is commutative and $u \in (x)$, there exist $n \in \mathbb{Z}$ and $b \in R$ such that $u = nx + xb$. Since $v \in (y)$, there exist $m \in \mathbb{Z}$ and $c \in R$ such that $v = my + yc$. Since R is commutative, for some $d \in R$,

$$uv = (nx + xb)(my + yc) = xyd + mnxy.$$

Then

$$\mu_P(uv) \geq \mu_P(xy) \qquad\qquad\qquad \text{(Since } P \in \text{IFI}(R)\text{)}$$

$$\geq \lambda \wedge t \qquad\qquad\qquad\qquad\qquad \text{(By (}**\text{))}$$

and

$$\nu_P(uv) \leq \nu_P(xy) \leq \mu \vee s.$$

Thus $z_{(\lambda \wedge t, \mu \vee s)} = u_{(\lambda, \mu)} \circ v_{(t, s)} \in P$. So, in all, $A \circ B \subset P$. On the other hand, from the definitions of A and B, we can easily prove that $A, B \in \text{IFI}(R)$. Since $P \in \text{IFPI}(R)$, either $A \subset P$ or $B \subset P$. Thus either $x_{(\lambda, \mu)} \in P$ or $y_{(t, s)} \in P$. Hence $P \in \text{IFCPI}(R)$. This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad \square$

Proposition 4.3. *Let P be a non-constant IFI of a ring R.*

(1) *If P is an IFPI (resp. IFCPI) of R, then*

    (i) *R_P is a prime ideal (resp. a completely prime ideal) of R.*

    (ii) *$\text{Im } P$ consists of exactly two points of $I \times I$.*

(2) *If* $P(0) = (1,0)$ *and* P *satisfies the conditions* (i) *and* (ii) *above, then* $P \in$ IFPI(R) *(resp.* IFCPI(R)*)*.

Proof. (1) We shall confine our proof to the case of intuitionistic fuzzy prime ideals. An anologous proof can be given by for intuitionistic fuzzy completely prime ideals. Suppose $P \in$ IFPI(R). Then, by Proposition 3.3, R_P is a prime ideal of R. Assume that Im P contains more than two values. Then there exist $x, y \in R \smallsetminus R_P$ such that $P(x) \neq P(y)$. Suppose without loss of generality that $\mu_P(x) < \mu_P(y)$ and $\nu_P(x) > \nu_P(y)$. Since $P \in$ IFI(R) and $A(y) \neq A(0)$, by Result 1.3, $\mu_P(x) < \mu_P(y) < \mu_P(0)$ and $\nu_P(x) > \nu_P(y) > \nu_P(0)$. Let $(\lambda, \mu), (t, s) \in I \times I$ be chosen such that

$$\mu_P(x) < \lambda < \mu_P(y) < t < \mu_P(0)$$

and

$$\nu_P(x) > \mu < \nu_P(y) > s > \nu_P(0). \qquad (* * *)$$

Let (x) and (y) denote respectively the ideals generated by x and y. We define two complex mappings

$$A = (\mu_A, \nu_A) : R \to I \times I \quad \text{and} \quad B = (\mu_B, \nu_B) : R \to I \times I \quad (\textit{resp. })$$

as follows:

$$A = (\lambda\chi_{(x)}, \mu\chi_{(x)^c}) \quad \text{and} \quad B = (t\chi_{(y)}, s\chi_{(y)^c}) \quad (\textit{resp. }).$$

Then it is easily seen that $A, B \in$ IFI(R) from the definitions of A and B. Let $z \in R$ which cannot be expressed in the from $z = uv$ for $u \in (x)$ and $v \in (y)$. Then $A \circ B(z) = (0,1)$. Thus $A \circ B \subset P$. Now let $z \in R$. Suppose there exist $u \in (x)$ and $v \in (y)$ such that $z = uv$ for some $u \in (x)$ and $v \in (y)$. Then

$$\mu_{A \circ B}(z) = \bigvee_{z=uv, u\in(x), v\in(y)} [\mu_A(u) \wedge \mu_B(v)] = \lambda \wedge t = \lambda$$

and

$$\nu_{A \circ B}(z) = \bigwedge_{z=uv, u\in(x), v\in(y)} [\nu_A(u) \vee \nu_B(v)] = \mu \vee s = \mu.$$

Since $u \in (x)$, there exist $m \in \mathbb{Z}$ and $r_i \in R$ ($i = 1, 2, 3, 4$) such that $u = mx + r_1 x + x r_2 + r_3 x r_4$. Similarly, there exist $n \in \mathbb{Z}$ and $s_i \in R$ ($i = 1, 2, 3, 4$) such that $v = ny + s_1 y + y s_2 + s_3 y s_4$. Since $P \in$ IFI(R), by Result 1.3,

$$\mu_P(z) = \mu_P(uv) \geq \mu_P(x) \vee \mu_P(y) > \lambda$$

and

$$\nu_P(z) = \nu_P(uv) \leq \nu_P(x) \wedge \nu_P(y) < \mu.$$

Thus $\mu_{A \circ B}(z) \leq \mu_P(z)$ and $\nu_{A \circ B}(z) \geq \nu_P(z)$ in this case also. So $A \circ B \subset P$. Since $P \in \text{IFPI}(R)$, either $A \subset P$ or $B \subset P$. Then either $\mu_A(x) = \lambda \leq \mu_P(x)$, $\nu_A(x) = \mu \geq \nu_P(x)$ or $\mu_B(y) = t \leq \mu_P(y)$, $\nu_B(y) = s \geq \nu_P(y)$. This contradicts $(***)$. Hence $\text{Im } P$ consists of exactly two points of $I \times I$.

(2) Suppose $P(0) = (1,0)$ and P satisfies the conditions (i) and (ii). Then, by the similar arguments of proof of Proposition 3.2, we can see that $P \in \text{IFPI}(R)$. This completes the proof. $\square$

Corollary 4.3. Let P be an intuitionistic fuzzy completely prime ideal of a ring R. Then for any $x, y \in R$,

$$P(xy) = (\mu_P(x) \vee \mu_P(y), \nu_P(x) \wedge \nu_P(y)).$$

Remark 4.4. Proposition 4.3 generalizes Proposition 3.5.

Definition 4.5. Let A be a non-constant IFI of a ring R. Then A is called an *intuitionistic fuzzy weakly completely prime ideal* of R if for any $x, y \in R, A(xy) = (\mu_A(x) \vee \mu_A(y), \nu_A(x) \wedge \nu_A(y))$.

The following is the immediate result of Definitions 4.1 and 4.5.

Proposition 4.6. *Let A be an intuitionistic fuzzy weakly completely prime ideal of a ring R. Then for each $(\lambda, \mu) \in I \times I$ with $\lambda + \mu \leq 1$, $x_{(\lambda,\mu)} \circ y_{(t,s)} \in A$ implies that either $x_{(\lambda,\mu)} \in A$ or $y_{(t,s)} \in A$. Furthermore, for each $(\lambda, \mu) \in I \times I$ such that $\lambda + \mu \leq 1, \lambda < \mu_A(0)$ and $\mu > \nu_A(0)$, $A^{(\lambda,\mu)}$ is a completely prime ideal of R. In particular, $A^{(0,1)}$ is a completely prime ideal of R.*

Conversely if for each $(\lambda, \mu) \in I \times I$ with $\lambda + \mu \leq 1$, $A^{(\lambda,\mu)}$ is a completely prime ideal then A is an intuitionistic fuzzy weakly completely prime ideal.

The following is the example that an intuitionistic fuzzy weakly completely prime ideal need not be an intuitionistic fuzzy completely prime ideal.

Example 4.7. Let $R = \mathbb{Z} \times \mathbb{Z}$, let $S = \{0\} \times \mathbb{Z}$ and let $T = (2) \times \mathbb{Z}$. We define a complex mapping $A = (\mu_A, \nu_A) : R \to I \times I$ as follows : for each $x \in R$,

$$A(x) = \begin{cases} (1,0) & \text{if } x \in S, \\ (\frac{1}{2}, \frac{1}{3}) & \text{if } x \in T \smallsetminus S, \\ (0,1) & \text{if } x \in R \smallsetminus T. \end{cases}$$

Then clearly $A \in$ IFS(R) from the definition of A. Moreover, we can easily show that A is an intuitionistic fuzzy weakly completely prime ideal but, by Proposition 4.2, A is not an intuitionistic fuzzy weakly completely prime ideal.

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