J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math.
Volume 12, Number 3 (August 2005), Pages 161–167 ISSN 1226-0657

THE SPECTRAL PROPERTY $\sigma(\mathrm{Re}(T)) = \mathrm{Re}(\sigma(T))$ OF p-HYPONORMAL OPERATORS

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ABSTRACT. In this note we consider the projective property σ Re(T) = Re $\sigma(T)$ of p-hyponormal operators and log-hyponormal operators.

1. INTRODUCTION

Let $\mathcal{H}$ be complex Hilbert spaces and let $B(\mathcal{H})$ denote the set of all bounded linear operators on $\mathcal{H}$. For $T \in B(\mathcal{H})$, we denote the spectrum and the approximate point spectrum of T by $\sigma(T)$ and $\sigma_a(T)$, respectively.

An operator $T \in B(\mathcal{H})$ is called *p-hyponormal* if $(T^*T)^p - (TT^*)^p \geq 0$ for some $p \in (0, \infty)$. If $p = 1$, T is *hyponormal* and if $p = \frac{1}{2}$, T is *semi-hyponormal.* By the consequence of Löwner's inequality (*cf.* Löwner [10]) if T is p-hyponomal for some $p \in (0, \infty)$, then T is also q-hyponormal for every $q \in (0, p]$. Thus we assume, without loss of generality, that $p \in (0, \frac{1}{2})$. An operator $T \in B(\mathcal{H})$ is called *log-hyponormal* if T is invertible and satisfies log $(T^*T) \geq$ log (TT^*). Since

$$\log : (0, \infty) \longrightarrow (-\infty, \infty)$$

is a monotone function, every invertible p-hyponormal operator is log-hyponormal. But there exists a log-hyponormal operator which is not p-hyponormal (*cf.* Tanahashi [12, Example 12]). An operator $T \in B(\mathcal{H})$ is said to be *paranormal* if $\|T^2x\| \geq \|Tx\|^2$ for every unit vector $x \in \mathcal{H}$. It is well known (*cf.* Ando [1], Chō & Jin [5], Fujii, Himeji & Matsumoto [8]) that

$$p\text{-hyponormal} \Longrightarrow \text{log-hyponormal} \Longrightarrow \text{paranormal}.$$

Received by the editors April 20, 2005 and, in revised form, July 21, 2005.
2000 *Mathematics Subject Classification.* 47B20.
Key words and phrases. p-hyponormal operators, log-hyponormal operators.

For the study of spectral theory of operators, spectral mapping theorems are important. It is familiar that if T is normal then for every polynomial $p(\lambda, \lambda^*)$ the spectral mapping theorem holds;

$$\sigma\big(p(T)\big) = p(\sigma(T)) = \big\{p(\lambda, \lambda^*); \lambda \in \sigma(T)\big\}.$$

In particular, if $p(\lambda, \lambda^*) := \frac{1}{2}\big(\lambda + \lambda^*\big) = \mathrm{Re}(\lambda)$, then we have that

$$\sigma\big(\mathrm{Re}(T)\big) = \mathrm{Re}\left(\sigma(T)\right) \tag{1}$$

for any normal operator T. We called the equality (1) the *projective* property.

The projective property for semi-normal operators was shown by Putnam [11] and the projective property for Toeplitz operators was showed by Berberian [2]. Chō, Hwang & Lee [4] has showed the *subprojective* property for p-hyponormal or log-hyponormal operators. There are two typical examples which are semi-hyponormal but not hyponormal (*cf.* Chō & Jin [5], Xia [13]). Chō, Huruya, Kim & Lee [3] has showed that the two typical examples satisfy the projective property. In this paper, we will show the projective property for p-hyponormal operators and log-hyponormal operators with some constraints.

2. THE PROJECTIVE PROPERTY

For an operator $T \in B(\mathcal{H})$, a point z is in the normal approximate point spectrum $\sigma_{na}(T)$ of T if there exists a sequence $\{x_n\}$ of unit vectors such that

$$(T - z)x_n \to 0 \quad \text{and} \quad (T - z)^* x_n \to 0 \quad \text{as} \quad n \to \infty.$$

Theorem A (Chō, Hwang & Lee [4, Lemma 1.1]). *If $T \in B(\mathcal{H})$ and $\sigma_a(T) = \sigma_{na}(T)$, then*

$$\mathrm{Re}(\sigma(T)) \subset \sigma(\mathrm{Re}\, T) \quad \textit{and} \quad \mathrm{Im}\, \sigma(T) \subset \sigma(\mathrm{Im}\, T).$$

From this theorem, we can show the "subprojective" property, $\mathrm{Re}(\sigma(T)) \subset \sigma(\mathrm{Re}\, T)$, for the spectra of p-hyponormal operators and log-hyponormal operators.

Let $T \in B(\mathcal{H})$ and let *iso* $\sigma(T)$ be the set of all isolated points of the spectrum $\sigma(T)$ of T. If $\lambda \in$ *iso* $\sigma(T)$, the Riesz idempotent E_λ of T with respect to λ is defined by

$$E_\lambda = \frac{1}{2\pi i} \int_{\partial D} (T - z)^{-1} dz,$$

where D is an open disk which is far from the rest of $\sigma(T)$.

Theorem B (Chō & Tanahashi [6, Theorem 7]). *Let $T \in B(\mathcal{H})$ be p-hyponormal (or log-hyponormal). Let $\lambda \in \sigma(T)$ be an isolated point of $\sigma(T)$ and let E_λ be the Riesz projection for λ. Then E_λ is self-adjoint and*

$$E_\lambda \mathcal{H} = \ker(T-\lambda) = \ker(T-\lambda)^*.$$

An operator $T \in B(\mathcal{H})$ is called *convexoid* if *conv* $\sigma(T) = \operatorname{cl} W(T)$ where $W(T) = \{(Tx,x); \|x\|=1\}$ is the numerical range of T and *conv* denotes the convex hull. An operator T is *restriction-convexoid* if the restriction of T to every invariant subspace is covexoid.

Theorem 1. *Let T be a p-hyponormal (or a log-hyponormal) operator. If T has the following conditions;*

(i) *T is restriction convexoid, and*

(ii) *$\operatorname{Re}\sigma(T)$ consists of a finite number of connected components,*

then $\sigma(\operatorname{Re} T) \subset \operatorname{Re}\sigma(T)$.

Proof. Let P be the projection on the complex plane C such that $P(\lambda) = \operatorname{Re}\lambda$. Write $\operatorname{Re}\sigma(T) = \bigcup_{i=1}^n S_i$, where S_i are components of $\operatorname{Re}\sigma(T)$ and $\sigma_i := P^{-1}(S_i) \cap \sigma(T)$. Using the Riesz idempotent, by Theorem B, $\mathcal{H}$ can be decomposed by $\mathcal{H} = \bigoplus_{i=1}^n \mathcal{H}_i$, where $T_i = T|_{\mathcal{H}_i}$ and $\sigma(T_i) = \sigma_i$ and hence $\operatorname{Re}\sigma(T_i) = S_i$, *i. e.*,

$$T = \begin{pmatrix} T_1 & 0 & \cdots & 0 \\ 0 & T_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & T_n \end{pmatrix} \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \vdots \\ \mathcal{H}_n \end{pmatrix}$$

Since T is restriction convexoid, T_i $(i = 1, \ldots, n)$ is convexoid. Moreover,

$$\operatorname{Re}\sigma(T) = \bigcup_{i=1}^n \operatorname{Re}\sigma(T_i).$$

Let $[\alpha_i, \beta_i]$ is the smallest interval containing $\sigma(\operatorname{Re} T_i), i = 1, \ldots, n$. Since $\sigma(\operatorname{Re} T) = \bigcup_{i=1}^n \sigma(\operatorname{Re} T_i)$, we have $\sigma(\operatorname{Re} T) \subset \bigcup_{i=1}^n [\alpha_i, \beta_i]$. We want to show that $[\alpha_i, \beta_i] \subset \operatorname{Re}\sigma(T_i)$, $i = 1, \ldots, n$. Since $\operatorname{Re}\sigma(T_i)$ is connected, it will suffice to show that $\alpha_i, \beta_i \in \operatorname{Re}\sigma(T_i)$, $i = 1, \ldots, n$. Assume to the contrary that $\alpha_i \notin \operatorname{Re}\sigma(T_i)$. If L is the vertical line $\operatorname{Re}\lambda = \alpha_i$, then $\sigma(T_i)$ must lie in a side with respect to L.

Suppose that $\sigma(T_i)$ lies in the right-side of L. Thus there exists $\varepsilon > 0$ such that $\operatorname{Re}\sigma(T_i) \geq \alpha_i + \varepsilon$. Since T_i is convexoid, it follows that $\inf\{\operatorname{Re} W(T_i)\} \geq \alpha_i + \varepsilon$. Thus $\inf\{W(\operatorname{Re} T_i)\} \geq \alpha_i + \varepsilon$, *i. e.*, $\operatorname{Re} T_i \geq (\alpha_i + \varepsilon)I$ and hence $\inf\{\sigma(\operatorname{Re} T_i)\} \geq \alpha_i + \varepsilon$.

Therefore $\alpha_i \geq \alpha_i + \varepsilon$, which leads to a contradiction. If $\sigma(T_i)$ lies in the left-side of L, applying the preceding argument with $\alpha - \varepsilon$ in place of $\alpha + \varepsilon$ gives a contradiction. Therefore $[\alpha_i, \beta_i] \subset \operatorname{Re}\sigma(T_i)$ for $i = 1, \ldots, n$, and therefore $\sigma(\operatorname{Re} T) \subset \operatorname{Re}\sigma(T)$. □

In Theorem 1, the condition (ii) can be easily applied to the case that $\sigma(T)$ does not consist of connected components: For example, if $T = T_1 \oplus D$, where T_1 is a p-hyponormal operator and convexoid whose spectrum is the unit disk and D is a diagonal operator whose diagonals are $\{\frac{1}{n} + i\}_{n=1}^{\infty}$ then $\operatorname{Re}\sigma(T) = [-1, 1]$, which is connected.

Corollary 2. *Let T be a p-hyponormal (or a log-hyponormal) operator. If T is restriction convexoid and* $\operatorname{Re}\sigma(T)$ *consists of a finite number of connected components then* $\sigma(\operatorname{Re} T) = \operatorname{Re}\sigma(T)$.

Proof. This follows from Theorem A and Theorem 1. □

3. AN EXAMPLE

It is well known (*cf.* Hildebrandt [9]) that T is convexoid if $T - \lambda$ is normaloid for every complex numbers. Therefore, any normaloid opertor which is translation-invariant is convexoid.

Let $\ell^2(\mathcal{Z})$ be the Hilbert space of all doubly-infinite sequences $a = \{a_k\}$ of complex numbers such that $\|a\|^2 = \sum_{k=-\infty}^{\infty} |a_k|^2 < \infty$ and let V be the bilateral shift:

$$(Va)_k = a_{k-1}.$$

Let $\mathcal{K}$ be a Hilbert space and let $\mathcal{H}$ be the Hilbert space of all doubly-infinite sequences $x = \{x_k\}$ of elements of $\mathcal{K}$ such that $\|x\| = \sum_{k=-\infty}^{\infty} \|x_k\|^2 < \infty$. Then we have $\mathcal{H} = \ell^2(\mathcal{Z}) \otimes \mathcal{K}$. Let $e_m = \{a_k\} \in \ell^2(\mathcal{Z})$ such that $a_m = 1$ and 0's elsewhere. Every $x = \{x_k\} \in \mathcal{H}$ has the representation $\sum_{k=-\infty}^{\infty} e_k \otimes x_k$. Let $\{A_k\}_{k=-\infty}^{\infty}$ be a doubly-infinite sequence of positive operators on $\mathcal{K}$ such that $\{\|A_k\|\}_{k=-\infty}^{\infty}$ is bounded. We define bounded operators A and U (*resp.*) on $\mathcal{H}$ by

$$Ae_k \otimes x_k = e_k \otimes A_k x_k \quad \text{and} \quad Ue_k \otimes x_k = e_{k+1} \otimes x_k \ \ (\textit{resp.})$$

where $k = 0, \pm 1, \pm 2, \ldots$. Then U has the form $V \otimes id_{\mathcal{K}}$. Put $T = UA$. Such an operator is called an *operator valued bilateral weighted shift* (*cf.* Clancey [7]) and provides an interesting example (*cf.* Chō & Jin [5]). If positive operators $\{A_k\}$

satisfy that $A_{k+1} \geq A_k$ for every k and there exists j such that $A_{j+1}^2 \not\geq A_j^2$, then T is semi-hyponormal but not hyponormal.

Theorem C (Chō, Huruya, Kim & Lee, Theorem 2). *Let A be an operator valued bilateral weighted shift such that $r(A) = \|A\|$. Then A is convexoid and*

$$\sigma(\operatorname{Re} A) = \operatorname{Re} \sigma(A).$$

Theorem 3. *There exists an operator valued weighted shift T such that T is semi-hyponormal and $T - \lambda$ is not p-hyponomral for some λ, which p is any $p \in \left(0, \frac{1}{2}\right)$.*

Proof. Let $\mathcal{K}$ be a two-dimensional complex vector space and let $\mathcal{H}$ be the direct sum of countably many copieis of $\mathcal{K}$. Explicitly, $\mathcal{H}$ is the set of all sequences

$$x = \langle \ldots, x_{-1}, \boxed{x_0}, x_1, \ldots \rangle$$

of vectors in $\mathcal{K}$ such that $\|x\| = \sum_{k=-\infty}^{\infty} \|x_k\|^2 < \infty$. Let $T = UA$ be an operator valude bilateral weighted shift. Let positive 2×2 matrices C and D be given by

$$C = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}.$$

Let $\{A_n\}$ be a sequence of positive 2×2 matrices defined by

$$A_n = \begin{cases} C, & n \leq 0 \\ D, & n \geq 1 \end{cases}$$

Then operator $T = UA$ has the matrix form as follows:

$$T = \begin{pmatrix} \ddots & & & & & & \\ \ddots & 0 & & & & O & \\ & C & 0 & & & & \\ & & \ddots & \ddots & & & \\ & & & C & \boxed{0} & & \\ & & & & D & 0 & \\ & O & & & & D & 0 \\ & & & & & \ddots & \ddots \end{pmatrix}$$

We now claim that

(i) T is semi-hyponormal, and
(ii) $T - 4$ is not p-hyponormal for any $p > 0$.

(i) By a direct calculation, we have

$$T^*T = \begin{pmatrix} \ddots & & & & & & \\ & C^2 & & & & & \\ & & C^2 & & & & \\ & & & \boxed{D^2} & & & \\ & & & & D^2 & & \\ & & & & & D^2 & \\ & & & & & & \ddots \end{pmatrix}$$

and

$$TT^* = \begin{pmatrix} \ddots & & & & & & \\ & C^2 & & & & & \\ & & C^2 & & & & \\ & & & \boxed{C^2} & & & \\ & & & & D^2 & & \\ & & & & & D^2 & \\ & & & & & & \ddots \end{pmatrix}$$

Thus $T^*T - TT^* = D^2 - C^2 < 0$, but $(T^*T)^{\frac{1}{2}} - (TT^*)^{\frac{1}{2}} = D - C \geq 0$. Hence T is not hyponormal but semi-hyponormal.

(ii) To show that $T - 4$ is not p-hyponormal for any $p > 0$. It suffices to show that $T - 4$ is not paranormal. Let

$$\tilde{x} = \langle \dots, 0, x_{-1}, \boxed{0}, 0, 0, \dots \rangle \quad \text{and} \quad x_{-1} = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$$

be the unit vector in $\mathcal{H}$ such that $\|\tilde{x}\| = \|x_{-1}\| = 1$. Then

$$\begin{aligned} \|(T-4)^2\tilde{x}\| &= \|T^2\tilde{x} - 8T\tilde{x} + 16\tilde{x}\| \\ &= \left\| \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} + 16 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \right\| \\ &= \left\| \begin{pmatrix} \frac{14}{\sqrt{2}} \\ -\frac{11}{\sqrt{2}} \end{pmatrix} \right\| = \sqrt{\frac{317}{2}} \end{aligned}$$

and

$$\|(T-4)\tilde{x}\|^2 = \left\| \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} - 4 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} -\frac{2}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} \end{pmatrix} \right\|^2 = \frac{29}{2}$$

which shows that

$$\|(T-4)\tilde{x}\|^2 > \|(T-4)^2\tilde{x}\|,$$

which implies that $T - 4$ is not paranormal. □

The semi-hyponormal operator valued bilateral weighted shift T which is provided in Theorem 3 is convexoid by Theorem C. Therefore it is the example which is convexoid semi-hyponormal but not translation-invariant.

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