Zero-mean Gaussian을 이용한 소구간 사다리꼴공식의 오차

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요 약

이 논문에서는 정적분의 수치계산 방법 중에 하나인 사다리꼴 공식의 평균오차를 zero mean-Gaussian을 이용하여 연구한다. 구간 [0,1]에 n 개의 소구간을 잡고 계산의 단순화를 위하여 각 소구간의 길이가 같다고 하고 길이를 h라 하면, $r\leq 2$일 때, 상수 c_r을 직접 계산하여 연속된 두 개의 소구간 위에서 단순 사다리꼴공식과 복합 사다리꼴공식 사이의 평균오차가 $O(h^{2r+3})$임을 보인다.

키워드 : 평균오차, 수치구적법, 사다리꼴공식

An Error Bound of Trapezoidal Rule on Subintervals using Zero-mean Gaussian

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ABSTRACT

In this paper, we study the average case error of the Trapezoidal rule using zero mean-Gaussian. Assume that we have n subintervals (for simplicity equal length) partitioning [0,1] and that each subinterval has the length h. Then, for $r\leq 2$, we show that the average error between simple Trapezoidal rule and the composite Trapezoidal rule on two consecutive subintervals is bounded by h^{2r+3} through direct computation of constants c_r.

Key Words : Average Case Error, Numerical Integration, Trapezoidal Rule

1. Introduction

Many numerical computations in science and engineering can only be solved approximately since the available in formation is partial. For instance, for problems defined on a space of functions, information about f is typically provided by a few function values,

$$N(f)=[f(x_1),f(x_2),\ \cdots\ ,f(x_n)].$$

Knowing $N(f)$, the solution is approximated by a numerical method.

The error between the true solution and the approximation depends on a problem setting. In the **worst case setting**, the error of a numerical scheme is defined by its worst performance with respect to the given class of functions; see [1, 4] and [6]. In this paper, we concentrate on another setting, the **average case setting**. In this setting, we assume that the class F of input functions is equipped with a probability measure. Then the average case error of an algorithm is defined by its expectation, rather than by its worst case performance. The average case analysis is important and significant number of results have already been obtained (see, e.g., [6] and the references cited therein).

2. Definitions

It is well known that the average case setting requires the space of functions to be equipped with a probability measure. In this paper, we choose a probability measure μ_r which is a variant of an r**-fold Wiener measure** ω_r; see[3, 5] and [6]. The probability measure ω_r is a

※ This research was supported by the Kyung Hee University Research Fund in 2005 (KHU-20050358).
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논문접수 : 2005년 3월 28일, 심사완료 : 2005년 6월 14일

Gaussian measure with zero mean and correlation function given by

$$M_{\omega_r}(f(x)f(y)) = \int_F f(x)f(y)\,\omega_r(df) = \int_0^1 \frac{(x-t)_+^r}{r!}\frac{(y-t)_+^r}{r!}\,dt,$$

where $(z-t)_+^r = [\max\{0,\ z-t\}]^r$. Equivalently, f distributed according to ω_r can be viewed as a Gaussian stochastic process with zero mean and autocorrelation given above. However, since ω_r is concentrated on functions with boundary conditions $f(0)=f'(0)=\cdots=f^{(r)}(0)=0$, we choose to study a slightly modified measure μ_r that preserves basic properties of ω_r, yet does not require any boundary conditions. More precisely, we assume that a function f, as a stochastic process, is given by $f(x)=f_1(x)+f_2(1-x)$, $x\in[0,1]$, where f_1 and f_2 are independent and distributed according to ω_r.

In this paper, as the class F of input functions, we choose the space $F=C^r[0,1]$ that is equipped with a probability measure μ_r which is a variant of the r-**fold Wiener measure**. In order to define it, if we first recall basic properties of the classical r-**fold Wiener measure** ω_r, then according to [2-5], the probability measure μ_r defined on σ-field of the space $C^r[0,1]$ is zero mean Gaussian with the correlation function given by

$$M_{\mu_r}(f(x),f(y)) = \int_0^1 \frac{(x-t)_+^r(y-t)_+^r+(1-x-t)_+^r(1-y-t)_+^r}{r!\,r!}\,dt = \int_0^1 \frac{(x-t)_+^r(y-t)_+^r+(t-x)_+^r(t-y)_+^r}{r!\,r!}\,dt.$$

We study the problem of approximating an integral $I(f)=\int_0^1 f(x)dx$ for $f\in F=C^r[0,1]$, assuming that the class of integrands is equipped with the probability measure μ_r.

Assume that we have n subintervals (not necessarily equal length) partitioning $[0,1]$. Let $0=x_0<x_1<\cdots<x_{n-1}<x_n=1$. But for simplicity, we let $x_i=ih$ for $i=0,\cdots,n$ where $h=\frac{1}{n}$. With this indexing, we get

$$I_i(f)\equiv \int_{x_{i-1}}^{x_{i+1}} f(x)dx \quad\text{and}\quad T_i(f)=h\{f(x_{i-1})+f(x_{i+1})\}$$

while T_i is the basic Trapezoidal rule using $f(x_{i-1})$ and $f(x_{i+1})$. Let $\overline{T_i}$ be the composite Trapezoidal rule that uses $f(x_{i-1})$, $f(x_i)$ and $f(x_{i+1})$. i.e., $\overline{T_i}(f)=\frac{h}{2}\{f(x_{i-1})+2f(x_i)+f(x_{i+1})\}$. Also let $V_i(f)=\frac{1}{8}(\overline{T_i}(f)-T_i(f))$.

3. An error bound on two consecutive subintervals

In this section, we consider two consecutive subintervals. In order to find a new error bound for the subintervals, we need to compute the distributions of V_i. In fact, they are Gaussian with zero-mean and are given in next theorem. This is the main result of this paper.

Theorem. For $r\le 2$,

$$M_{\mu_r}(V_iV_j)=\delta_{ij}\cdot c_r\cdot h^{2r+3}, \quad \text{for } i\le j,$$

where δ_{ij} is the Kronecker delta and the constants C_r are independent of h's and equal respectively: $c_0=\frac{1}{16}$, $c_1=\frac{1}{192}$ and $c_2=\frac{581}{61440}$.

Proof. Let $V_{i1}=V_i(f_1)$ and $V_{i2}=V_i(f_2)$. Then $V_i(f)=V_{i1}+V_{i2}$, and due to the independence of f_1 and f_2, we have $M_{\mu_r}(V_iV_j)=M_{\omega_r}(V_{i1}V_{j1})+M_{\omega_r}(V_{i2}V_{j2})$. It is easy to see that

$$V_i(f)=\frac{h}{8}\nabla_i f=\nabla_i f_1+\nabla_i f_2,$$

where $h=(x_{i+1}-x_{i-1})/2$ and $\nabla_i f=f(x_{i+1})-2f(x_i)+f(x_{i+1})$. Now,

$$M_{\omega_r}(V_{i1}V_{j1}) = \int_0^1\Big[-\frac{h}{8}\nabla_i\Big(\frac{(\cdot-t)_+^r}{r!}\Big)\Big]\Big[-\frac{h}{8}\nabla_j\Big(\frac{(\cdot-t)_+^r}{r!}\Big)\Big]dt = \int_0^1 L_{i1}(t)\cdot L_{j1}(t)\,dt,$$

where L_{i1} is the first term and L_{j1} is the second term in above integral. Since $L_{j1}(t)=0$ for $t\le x_{i+1}$,

$$M_{\omega_r}(V_{i1}V_{j1})=\int_0^{x_{i+1}} L_{i1}(t)\cdot L_{j1}(t)\,dt.$$

Similarly,

$$M_{\omega_r}(V_{i2}V_{j2}) = \int_{x_{j-1}}^{1} L_{i2}(t)\cdot L_{j2}(t)\,dt.$$

Since we can easily show that the operator ∇_i is exact for polynomials of degree ≤ 2, $L_{j1}=0$ for $t\le x_{i+1}$ and $L_{i2}=0$ for $x\ge x_j$. Therefore for $i<j$, $M_{\mu_r}(V_iV_j)=0$. We now compute the case of $i=j$ for $M_{\mu_r}(V_iV_i)=M_{\omega_r}(V_{i1}V_{i1})+M_{\omega_r}(V_{i2}V_{i2})$.

$$\begin{aligned} M_{\omega_r}(V_{i1}V_{i1}) &= M_{\omega_r}([V_{i1}]^2) \qquad (1)\\ &= \int_{x_{i-1}}^{x_{i+1}}\left[-\frac{h}{8}\nabla_i\left(\frac{(\cdot-t)_+^r}{r!}\right)\right]^2 dt. \end{aligned}$$

We now compute $\left[-\frac{h}{8}\nabla_i\left(\frac{(\cdot-t)_+^r}{r!}\right)\right]^2$ on $[x_{i-1},x_{i+1}]$ $[x_{i-1},x_{i+1}]$. By the definition of ∇_i,

$$\begin{aligned} &\left[-\frac{h}{8}\nabla_i\left(\frac{(\cdot-t)_+^r}{r!}\right)\right]^2 \qquad (2)\\ &= \left[-\frac{h}{8}\left(\frac{(x_{i-1}-t)_+^r}{r!}-2\,(x_i-t)_+^r\ r!+\frac{(x_{i+1}-t)_+^r}{r!}\right)\right]^2. \end{aligned}$$

If we apply the equation (1) to (2) by setting $u=\frac{t-x_{i-1}}{2h}$, then we have

$$\begin{aligned} &\int_{x_{i-1}}^{x_{i+1}}\left[-\frac{h}{8}\nabla_i\left(\frac{(\cdot-t)_+^r}{r!}\right)\right]^2 dt\\ &=\int_{x_{i-1}}^{x_{i+1}}\left[-\frac{h}{8}\left(\frac{(x_{i-1}-t)_+^r}{r!}-2\frac{(x_i-t)_+^r}{r!}+\frac{(x_{i+1}-t)_+^r}{r!}\right)\right]^2 dt\\ &=\int_0^1\left[-\frac{h}{8}\left(\frac{(2h)^r(0-u)_+^r}{r!}-2\frac{(2h)^r(\frac{1}{2}-u)_+^r}{r!}+\frac{(2h)^r(1-u)_+^r}{r!}\right)\right]^2 2h\,du\\ &=\frac{2^{2r}h^{2r+3}}{32(r!)^2}\int_0^1\left[(0-u)_+^r-2(\tfrac{1}{2}-u)_+^r+(1-u)_+^r\right]^2 du\\ &= c_{r1}h^{2r+3}, \end{aligned}$$

where

$$c_{r1}=\frac{2^{2r}}{32(r!)^2}\int_0^1[(0-u)_+^r-2(1\,2-u)_+^r+(1-u)_+^r]^2 du.$$

Similarly, we compute $M_{\omega_r}(V_{i2}V_{i2})$ using $u=\frac{t-x_{i-1}}{2h}$.

$$\begin{aligned} &M_{\omega_r}(V_{i2}V_{i2})=M_{\omega_r}([V_{i2}]^2)\\ &=\int_{x_{i-1}}^{x_{i+1}}\left[-\frac{h}{8}\nabla_i\left(\frac{(t-\cdot)_+^r}{r!}\right)\right]^2 dt\\ &=\int_{x_{i-1}}^{x_{i+1}}\left[-\frac{h}{8}\left(\frac{(t-x_{i-1})_+^r}{r!}-2\frac{(t-x_i)_+^r}{r!}+\frac{(t-x_{i+1})_+^r}{r!}\right)\right]^2 dt\\ &=\int_0^1\left[-\frac{h}{8}\left(\frac{(2h)^{r(u-0)_+^r}}{r!}-2\,(2h)^r(u-\frac{1}{2})_+^r\ r!+\frac{(2h)^r(u-1)_+^r}{r!}\right)\right]^2 2h\,du\\ &=\frac{2^{2r}h^{2r+3}}{32(r!)^2}\int_0^1\left[(u-0)_+^r-2(u-\tfrac{1}{2})_+^r+(u-1)_+^r\right]^2 du\\ &= c_{r2}h^{2r+3}, \end{aligned}$$

where

$$c_{r2}=\frac{2^{2r}}{32(r!)^2}\int_0^1\left[(u-0)_+^r-2(u-\tfrac{1}{2})_+^r+(u-1)_+^r\right]^2 du.$$

Since

$$\begin{aligned} &\int_0^1\left[(0-u)_+^r-2(\tfrac{1}{2}-u)_+^r+(1-u)_+^r\right]^2 du\\ &=\int_0^1\left[(u-0)_+^r-2(u-\tfrac{1}{2})_+^r+(u-1)_+^r\right]^2 du, \end{aligned}$$

we have $c_{r1}=c_{r2}$. Therefore, $c_r=c_{r1}+c_{r2}=2c_{r1}$.

For $r=0$,

$$\begin{aligned} c_0 &= 2c_{01}\\ &= 2\cdot\frac{1}{32}\int_0^1\left[(0-u)_+^0-2(\tfrac{1}{2}-u)_+^0+(1-u)_+^0\right]^2 du\\ &= \frac{1}{16}\int_0^{\frac{1}{2}}\left[(0-u)_+^0-2(\tfrac{1}{2}-u)_+^0+(1-u)_+^0\right]^2 du\\ &\quad+\frac{1}{16}\int_{\frac{1}{2}}^1\left[(0-u)_+^0-2(\tfrac{1}{2}-u)_+^0+(1-u)_+^0\right]^2 du\\ &= \frac{1}{16}\int_0^{\frac{1}{2}}[-2+1]^2 du+\frac{1}{16}\int_{\frac{1}{2}}^1 du\\ &= \frac{1}{16}. \end{aligned}$$

For $r=1$,

$$
\begin{aligned}
c_1 &= 2c_{11} \\
&= 2\cdot\frac{1}{32}\int_0^1\Big[(0-u)_+^1-2(\frac{1}{2}-u)_+^1+(1-u)_+^1\Big]^2du \\
&= \frac{1}{16}\int_0^{\frac{1}{2}}\Big[-2(\frac{1}{2}-u)_++(1-u)_+\Big]^2du \\
&\quad +\frac{1}{16}\int_{\frac{1}{2}}^1\Big[(0-u)_+-2(\frac{1}{2}-u)_++(1-u)_+\Big]^2du \\
&= \frac{1}{16}\int_0^{\frac{1}{2}}\Big[-2(\frac{1}{2}-u)+(1-u)\Big]^2du+\frac{1}{16}\int_{\frac{1}{2}}^1(1-u)^2du \\
&= \frac{1}{192}.
\end{aligned}
$$

Finally, for $r=2$,

$$
\begin{aligned}
c_2 &= 2c_{21} \\
&= 2\cdot\frac{1}{8}\int_0^1\Big[(0-u)_+^2-2(\frac{1}{2}-u)_+^2+(1-u)_+^2\Big]^2du \\
&= \frac{1}{4}\int_0^{\frac{1}{2}}\Big[(0-u)_+^2-2(\frac{1}{2}-u)_+^2+(1-u)_+^2\Big]^2du \\
&\quad +\frac{1}{2}\int_{\frac{1}{2}}^1\Big[(0-u)_+^2-2(\frac{1}{2}-u)_+^2+(1-u)_+^2\Big]^2du \\
&= \frac{1}{4}\int_0^{\frac{1}{2}}\Big[-2(\frac{1}{2}-u)^2+(1-u)^2\Big]^2du \\
&\quad +\frac{1}{16}\int_{\frac{1}{2}}^1[(1-u)^2]^2du \\
&= \frac{581}{61440}.
\end{aligned}
$$

This completes the proof.

3. Conclusion

For Simplicity, we have chosen n n equal length subintervals partitioning $[0,1]$. However it is not necessary because if we take h as the largest length of subintervals, then it is straightforward to have the same result. Moreover, if we can compute an error bound for $r \geq 3$ which is more complicated of course, then it will lead us to compute covariance of average error. We will explore this in the later paper.

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